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PREFACE

MODELING OF MICRO DIFFRACTIVE OPTICS

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Consider a time-harmonic electromagnetic plane wave incident on a periodic surface. The diffraction problem is to predict energy distributions of the propagating waves away from the structure. The process is governed by time-harmonic Maxwell's equations. In this paper, direct, inverse, and optimal design problems in the mathematical modeling of diffractive optics are studied. Particular attentions are paid to a variational approach. For the direct problem, a variational formulation and the well-posedness of the model problem are introduced. For the inverse problem, results on uniqueness and stability are presented. Related topics will also be briefly discussed.

1 Introduction

Consider scattering of electromagnetic waves by a biperiodic structure. The structure separates the whole space into three regions: Above and below the structure the medium is assumed to be homogeneous. However, inside the structure, the medium can be very general. In fact, the dielectric coefficient only needs to be bounded measurable. The medium is assumed to be nonmagnetic with a constant magnetic permeability throughout. Given the structure and a time-harmonic electromagnetic plane wave incident on the structure, the scattering (diffraction) problem is to predict the field distributions away from the structure. Scattering of electromagnetic waves in a biperiodic structure has recently received considerable attention. We refer to [19], [20], [1], [17], [7], [12], [13] for results and additional references on existence, uniqueness, and numerical approximations of solutions.

The scattering theory in periodic structures has many applications in micro diffractive optics, where doubly periodic structures are often called *crossed diffraction gratings*[24]. Micro diffractive optics is a fundamental and vigorously growing technology with many applications. Significant recent technology developments of high precision micromachining techniques have permitted the creation of gratings (periodic structures) and other diffractive structures with tiny features. The practical applications of diffractive optics technology have driven the need for mathematical models and numerical algorithms.

2 Direct modeling

The electromagnetic fields are governed by the time harmonic Maxwell equations (time dependence $e^{-i\omega t}$):

$$\nabla \times E - i\omega\mu H = 0 , \tag{1}$$

$$\nabla \times H + i\omega\varepsilon E = 0 , \tag{2}$$

where E and H denote the electric and magnetic fields in $\mathbf{R}^3$, respectively. The magnetic permeability μ is assumed to be one everywhere. There are two constants Λ_1 and Λ_2, such that the dielectric coefficient ε satisfies, for any n_1, $n_2 \in Z = \{0, \pm 1, \pm 2, \ldots\}$,

$$\varepsilon(x_1 + n_1\Lambda_1, x_2 + n_2\Lambda_2, x_3) = \varepsilon(x_1, x_2, x_3) .$$

Further, it is assumed that, for some fixed positive constant b and sufficiently small $\delta > 0$, $\varepsilon(x_1, x_2, x_3) = \varepsilon_1$, for $x_3 > b-\delta$, $\varepsilon(x_1, x_2, x_3) = \varepsilon_2$, for $x_3 < -b+\delta$, where $\varepsilon(x) \in L^\infty$, $Re(\varepsilon(x)) \geq \varepsilon_0$, $Im(\varepsilon(x)) \geq 0$, ε_0, ε_1 and ε_2 are constants, ε_0, ε_1 are real and positive, and $Re\,\varepsilon_2 > 0$, $Im\,\varepsilon_2 \geq 0$. The case $Im\,\varepsilon_2 > 0$ accounts for materials which absorb energy.

Let $\Omega_0 = \{x \in \mathbf{R}^3 : -b < x_3 < b\}$, $\Omega_1 = \{x \in \mathbf{R}^3 : x_3 > b\}$, $\Omega_2 = \{x \in \mathbf{R}^3 : x_3 < -b\}$.

Consider a plane wave in Ω_1, $(E_I, H_I) = (se^{iq \cdot x}, pe^{iq \cdot x})$, incident on Ω_0. Here $q = (\alpha_1, \alpha_2, -\beta) = \omega\sqrt{\varepsilon_1}(cos\theta_1 cos\theta_2, cos\theta_1 sin\theta_2, -sin\theta_1)$ is the incident wave vector whose direction is specified by θ_1 and θ_2, with $0 < \theta_1 < \pi$ and $0 < \theta_2 \leq 2\pi$. The vectors s and p satisfy $s = \frac{1}{\omega\varepsilon_1}(p \times q)$, $q \cdot q = \omega^2\varepsilon_1$, $p \cdot q = 0$.

We are interested in quasiperiodic solutions, i.e., solutions E and H such that the fields E_α, H_α defined by, for $\alpha = (\alpha_1, \alpha_2, 0)$, $E_\alpha = e^{-i\alpha \cdot x}E(x_1, x_2, x_3)$, $H_\alpha = e^{-i\alpha \cdot x}H(x_1, x_2, x_3)$, are periodic in the x_1 direction of period Λ_1 and in the x_2 direction of period Λ_2.

Denote

$$\nabla_\alpha = \nabla + i\alpha = \nabla + i(\alpha_1, \alpha_2, 0) .$$

It is easy to see from (2.2) and (2) that E_α and H_α satisfy

$$\nabla_\alpha \times E_\alpha - i\omega H_\alpha = 0 , \tag{3}$$

$$\nabla_\alpha \times H_\alpha + i\omega\varepsilon E_\alpha = 0 . \tag{4}$$

Due to a consideration for coercivity, it turns out to be natural to solve the following problem:

$$\nabla_\alpha \times (\frac{1}{\varepsilon}\nabla_\alpha \times H_\alpha) - \nabla_\alpha(\frac{1}{\varepsilon_C}\nabla_\alpha \cdot H_\alpha) - \omega^2 H_\alpha = 0 ,$$

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where ε_C is a fixed positive constant which satisfies $inf_{x \in \Omega_0} Re \frac{1}{\varepsilon(x)} \geq \frac{3}{4\varepsilon_C}$.

We also need boundary conditions in the x_3 direction. These conditions may be derived by the radiation condition, the periodicity of the structure, and the Green functions.

Denote

$$\Gamma_1 = \{x \in \mathbf{R}^3 : x_3 = b\} \text{ and } \Gamma_2 = \{x_3 = -b\}.$$

Define for $j = 1, 2$ the coefficients

$$\beta_j^{(n)}(\alpha) = e^{i\gamma_j^n/2}|\omega^2 \varepsilon_j - |\alpha_n + \alpha|^2|^{1/2}, \quad n \in Z,$$

where $\gamma_j^n = arg(\omega^2 \varepsilon_j - |\alpha_n + \alpha|^2)$, $0 \leq \gamma_j^n < 2\pi$. We assume that $\omega^2 \varepsilon_j \neq |\alpha_n + \alpha|^2$ for all $n \in Z$, $j = 1, 2$. This condition excludes "resonance".

For functions $f \in \mathcal{H}^{\frac{1}{2}}(\Gamma_j)^3$, define the operator T_j^α by

$$(T_j^\alpha f)(x_1, x_2) = \sum_{n \in \Lambda} i\beta_j^{(n)} f^{(n)} e^{i\alpha_n \cdot x},$$

where $f^{(n)} = \frac{1}{\Lambda_1 \Lambda_2} \int_0^{\Lambda_1} \int_0^{\Lambda_2} f(x) e^{-i\alpha_n \cdot x}$.

Denote the operator B_j by

$$B_j f = -i \sum_{n \in Z^2} \frac{1}{\beta_j^{(n)}} \{(\beta_j^{(n)})^2 (f_1^{(n)}, f_2^{(n)}, 0) + ((\alpha + \alpha_n) \cdot f^{(n)})(\alpha + \alpha_n)\} e^{i\alpha_n \cdot x}.$$

Therefore, the scattering problem can be formulated as follows[12]:

$$\nabla_\alpha \times (\frac{1}{\varepsilon} \nabla_\alpha \times H_\alpha) - \nabla_\alpha(\frac{1}{\varepsilon_C} \nabla_\alpha \cdot H_\alpha) - \omega^2 H_\alpha = 0 \text{ in } \Omega_0, \tag{5}$$

$$\nu_1 \times (\nabla_\alpha \times (H_\alpha - H_{I,\alpha})) = B_1(P(H_\alpha - H_{I,\alpha})) \text{ on } \Gamma_1 \tag{6}$$

$$\nu_2 \times (\nabla_\alpha \times H_\alpha) = B_2(P(H_\alpha)) \text{ on } \Gamma_2, \tag{7}$$

$$(T_1^\alpha - \frac{\partial}{\partial \nu_1})H_{\alpha,3} = 2i\beta_1 p_3 e^{-i\beta_1 b}, \text{ on } \Gamma_1, \tag{8}$$

$$(T_2^\alpha - \frac{\partial}{\partial \nu_2})H_{\alpha,3} = 0, \text{ on } \Gamma_2, \tag{9}$$

where ν_j is the outward normal to the surface Γ_j and P is the projection onto the plane orthogonal to ν_1.

Theorem 1 *([12]) For all but possibly a discrete set of ω, the above scattering problem attains a unique weak solution $H \in \mathcal{H}^1(\Omega_0)^3$.*

In fact, one can further establish an equivalence of the current variational formulation and the original scattering problem.

3 Inverse problems

An inverse diffraction problem is to determine the periodic structure or the shape of the interface from the measured scattered fields.

Let the scattering profile be described by the periodic surface $S = \{(x_1, x_2, x_3) : x_3 = f(x_1, x_2)\}$ of period $\Lambda = (\Lambda_1, \Lambda_2)$, that is, $f(x_1 + n_1\Lambda_1, x_2 + n_2\Lambda_2) = f(x_1, x_2)$ for integers n_1, n_2, and some positive constants Λ_1, Λ_2. The function f is supposed to be sufficiently smooth, for example of C^2. The space below S is filled with some perfectly reflecting material (a conductor). Let $\Omega = \{(x \in \mathbf{R}^3 : x_3 > f(x_1, x_2)\}$ be filled with a material whose dielectric coefficient is a fixed constant $\varepsilon = \varepsilon_0 > 0$. Consider a plane wave in Ω of the same form as in Section 1, $E_I = se^{iq \cdot x}$, $H_I = pe^{iq \cdot x}$, incident on S.

From the Maxwell equations, it is easily seen that

$$(\triangle + \omega^2 \varepsilon_0)E = 0 \quad \text{in } \Omega. \tag{10}$$

Since the region below S is a perfect conductor, only reflected waves exist:

$$\nu \times E = 0 \quad \text{on } S , \tag{11}$$

where ν is the outward normal to the surface. The following boundary condition may be derived from the radiation condition:

$$e_3 \times (\nabla \times (E - E_I)) = B(P(E - E_I)) \quad \text{on } \Gamma, \tag{12}$$

where $e_3 = (0, 0, 1)$, B is a pseudo-differential operator, and P is the projection onto the plane orthogonal to e_3.

Therefore, the direct scattering problem can be formulated as follows: To find a quasiperiodic solution that solves the problem (2.4), (3.2), and (3.8). The inverse problem can be stated as follows: For a given incident plane wave E_I, determine $f(x_1, x_2)$ from the knowledge of $e_3 \times E|_\Gamma$.

Suppose that $E_{f_j}(x)$ $(j = 1, 2)$ are Λ-quasiperiodic and solve the scattering problem (2.4), (3.2), and (3.8) with respect to the profiles $f_j(x_1, x_2)$, where the functions f_j are Λ-periodic. Let $b > max\{f_1(x_1, x_2), f_2(x_1, x_2)\}$ be a fixed constant. Denote $D_j = \{f_j < x_3 < b\}$.

Two profiles Γ_1 and Γ_2 are said to satisfy Property (A) if there is a simply connected bounded domain U such that the following two conditions are satisfied: U is convex; $\partial U = \partial U_1 \cup \partial U_2$, $\partial U_1 \subset \Gamma_1$ and $\partial U_2 \subset \Gamma_2$; and ∂U is of C^2.

Theorem 2 $\binom{16}{}$ *Assume that f_1, f_2 are Λ-periodic C^2 function and that the profiles $S_1 = \{x_3 = f_1(x_1, x_2)\}$ and $S_2 = \{x_3 = f_2(x_1, x_2)\}$ satisfy Property (A). Then there is a constant $\delta(k) > 0$ such that if the radius of $U \leq \delta(k)$ then $e_3 \times E_{f_1}|_{x_3=b} = e_3 \times E_{f_2}|_{x_3=b}$ implies $f_1(x_1, x_2) = f_2(x_1, x_2)$.*

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The result extends earlier 2-D uniqueness results in [22],[6]. In applications, it is impossible to make exact measurements. Thus stability results are crucial in the reconstruction of profiles. Regarding the stability of the inverse diffraction problem, only partial stability results are available in [14] (2-D) and [16] (3-D). Another direction is to study the inverse transmission problem or the non-conductor problem. In this case, one determines the structure from information on reflected and on transmitted waves. The problem becomes much more difficult. The only available results are some stability results proved in [14] for singly periodic structures.

4 Related topics

In this section, we briefly describe some related on-going research projects. Additional information may be found in the references.

Computation. We have recently developed interface least-squares finite element methods for solving the diffraction problem [9],[15]. The idea is to formulate the problem as an interface problem with the grating surface as the interface. The model problem can then be solved by a new least-squares finite element method that incorporates the jump conditions at interfaces into the objective functional. The method allows the use of different finite element spaces on either side of the interface and the jump conditions are enforced through the least-squares functional. As with general least-squares finite element methods, the resulting discrete system is symmetric, positive definite, and so is easily treated by various existing preconditioning techniques, e.g., multigrid methods. Both electric and magnetic fields can be determined simultaneously, which avoids the unstable numerical differentiation process. With sufficiently smooth interfaces, significantly better estimates than that for the standard finite element methods can be expected.

Chiral gratings. Chiral gratings provide an exciting combination of the medium and structure, which gives rise to new features and applications. For instance, chiral gratings are capable of converting a linearly polarized incident field into two nearly circularly polarized diffracted modes in different directions. Other potential applications include antennas, microwave devices, waveguides, and many other fields. Mathematically, in chiral media, the electric and magnetic fields are no longer decoupled in the constitution equations unlike in the standard Maxwell's system. Therefore, the model system is always in vector form. Recently, a variational approach has been developed in [3],[4]. Results on the well-posedness of the model problem and finite element methods have been established.

Surface enhanced nonlinear optical effects. A remarkable application of nonlinear diffractive optics is to generate coherent radiation at a frequency that is twice that of available lasers, so called *second harmonic generation* (SHG). Recently, it has been found experimentally that diffraction gratings can greatly enhance nonlinear optical effects. The model for SHG is the system of nonlinear Maxwell's equations with quadratic nonlinear terms. Little is known mathematically on nonlinear optics in periodic structures. The model problem in the 2-D setting has been solved in [10],[11]. Currently, we are investigating the model problem in 3-D. The problem is difficult since it involves a complicated nonlinear P.D.E. in vector form. We plan to conduct a perturbation analysis of the solutions by initially examing the first order term. A crucial step will be to obtain refined regularity results on the solutions. We plan to do this by using the L^p theory together with a version of Gehring's lemma (the reversed Hölder inequality). A related optimal design problem for nonlinear coatings is also of interest [5].

Design and homogenization. Numerical computation of the inverse and design problems in biperiodic structures is completely open. The problem can be posed as a nonlinear least-squares problem. Difficulties arise since the scattering pattern depends on the interface in a very implicit fashion and in general the set over which the function is minimized is neither convex nor closed. The formulation of the design problem is very close to similar problems in elasticity, for which fast and efficient algorithms have recently been developed. Initial progress on the design problem has been made via weak convergence analysis methods [2],[18],[21] and the homogenization theory [8] along with "relaxation" technique [23]. The main idea is to allow the grating profiles to be highly oscillating and to use relaxed formulation of the optimization problem. The critical step is to determine the relaxed formulation which involves materials and the effective dielectric properties [8].

Acknowledgments

1. T. Abboud, *Formulation variationnelle des équations de Maxwell dans un réseau bipéiodique de* R^3, C. R. Acad. Sci. Paris, t. 317, Série I (1993), 245-248.
2. Y. Achdou and O. Pironneau, *Optimization of a photocell,* Optimal Control Appl. Meth. **12** (1991), 221-246.

6

3. H. Ammari and G. Bao, *Analysis of the diffraction by periodic chiral structures*, C.R. Acad. Sci., Paris, Série I, t. **326** (1998), 1371-1376.

4. H. Ammari and G. Bao, *Coupling of finite element and boundary element methods for the electromagnetic diffraction by a periodic chiral structure*, Research Notes in Mathematics, to appear.

5. H. Ammari, G. Bao, and K. Hamdache, *The effect of thin coatings on second harmonic generation*, Electronic J. Diff. Eq., to appear.

6. G. Bao, *A uniqueness theorem for an inverse problem in periodic diffractive optics*, Inverse Problems **10** (1994), 335-340.

7. G. Bao, *Variational approximation of Maxwell's equations in biperiodic structures*, SIAM J. Appl. Math. **57** (1997), 364-381.

8. G. Bao and E. Bonnetier, *Optimal design of periodic diffractive structures in TM polarization*, Appl. Math. Opt., to appear.

9. G. Bao, Y. Cao, and H. Yang, *Least-squares finite element computation of diffraction problems*, Math. Meth. in the Appl. Sci., to appear.

10. G. Bao and Y. Chen, *A nonlinear grating problem in diffractive optics*, SIAM J. Math. Anal., Vol. 28, No. 2 (1997), 322-337.

11. G. Bao and D. Dobson, *Diffractive optics in nonlinear media with periodic structure*, Euro. J. Appl. Math. **6** (1995), 573-590.

12. G. Bao and Dobson, *On the scattering by biperiodic structures*, Proc. Am. Math. Soc., to appear.

13. G. Bao, D. Dobson, and J. A. Cox, *Mathematical studies of rigorous grating theory*, J. Opt. Soc. Am. **A 12**(1995), 1029–1042.

14. G. Bao and A. Friedman, *Inverse problems for scattering by periodic structures*, Arch. Rat. Mech. Anal. **132** (1995), 49-72.

15. G. Bao and H. Yang, *A least-squares finite element analysis for diffraction problems*, SIAM J. Numer. Anal., to appear.

16. G. Bao and Z. Zhou, *An inverse problem for scattering by a doubly periodic structure*, Trans. Ameri. Math. Soc. **350** (1998), 4089-4103.

17. O. Bruno and F. Reitich, *Numerical solution of diffraction problems: A method of variation of boundaries III. Doubly-periodic gratings*, J. Optical Soc. Amer., 10 (1993), 2551–2562.

18. D. Dobson, *Optimal design of periodic antireflective structures for the Helmholtz equation*, Euro. J. Appl. Math. **4** (1993), 321-340.

19. D. Dobson, *A variational method for electromagnetic diffraction in biperiodic structures*, Modél. Math. Anal. Numér. **28** (1994), 419-439.

20. D. Dobson and A. Friedman, *The time-harmonic Maxwell equations in a doubly periodic structure*, J. Math. Anal. Appl. **166** (1992), 507-528.

21. J. Elschner and G. Schmidt, *Numerical solution of optimal design problems for binary gratings*, J. Comput. Phys. **146** (1998), 603-626.

22. A. Kirsch, *Uniqueness theorems in inverse scattering theory for periodic structures*, Inverse Problems **10** (1994), 145-152.

23. R. Kohn and G. Strang, *Optimal design and relaxation of variational problems I, II, III*, Comm. Pure Appl. Math. **39**(1986), 113-137, 139-182, 353-377.

24. *Electromagnetic Theory of Gratings,* Topics in Current Physics, Vol. 22, edited by R. Petit, Springer-Verlag, Heidelberg, 1980.

ON THE OSCILLATION OF DELAY DIFFERENTIAL EQUATION WITH DISTRIBUTED DELAY

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In this paper, the oscillatory behaviour of a class of nonlinear differential equations with distributed delay has been discussed and a sharp condition has been established.

1 Introduction

The oscillatory behaviour of first order differential equations has been broadly investigated. Of particular importance has been the investigation of oscillations which caused by delay effects (see [2] and [3]).

Our purpose in this paper is to establish the oscillatory conditions for a certain class of first order nonlinear differential equations with ditributed delay

$$\dot{x}(t) + \int_0^{\tau(t)} f(x(\tau - \theta))d\alpha(t,\theta) = 0, t \geq t_0 > 0, \qquad (1)$$

where $\alpha(t,\theta)$ is continuous function in t for every fixed $\theta \in [0, \tau(t)]$, and nondecreasing function in θ for every fixed $t \in [t_0, +\infty), \tau(t) \in C([t_0, \infty), (0, \infty)),$ $t \geq \tau(t)$ and

$$\lim_{t \to \infty} (t - \tau(t)) = \infty; \qquad (2)$$

$f \in C(\mathbf{R}, \mathbf{R}), uf(u) > 0$ for $u \neq 0$, and increasing in $[-A, A]$ for large enough $A > 0$.

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Denote $V(t) = \int_0^{\tau(t)} d\alpha(t,\theta)$ and $\delta(t) = \max_{0 \leq s \leq t}[s - \tau(s)]$, hence we have $\delta(\delta(t)) \leq \delta(t) < t$.

Suppose that

$$\int_{\delta(t)}^{t} V(s)ds \geq \frac{1}{e}, \text{ for } t \geq t_0 \qquad (3)$$

and

$$\int_a^{\infty} V(s)ds = \infty, \ a \geq 0. \qquad (4)$$

In recent papers [1] and [4], Bingtuan Li and Yang Kuang studied the equation

$$\dot{x}(t) + p(t)x(t - \tau) = 0$$

and

$$\dot{x}(t) + p(t)f(x(t - \tau(t))) = 0,$$

and obtained an infinite integral condition in [1] and a sharp condition in [4]. In this paper, we will extend the results in [4] to the more generalized equation (1). One can see that the results in this paper are more generalized and the results that we obtained include the results in [4].

2   Main Results

Lemma 1 *Suppose that (4) holds for any $a \geq 0$. Then the nonoscillatory solution of Eq.(1) tends to zero monotonically as $t \longrightarrow +\infty$.*

Lemma 2 *Suppose that (4) and*

$$|f(u) - u| \leq M|u|^{1+r} \text{ for } u \in (-\epsilon, \epsilon), \ r > 0, M \geq 0 \qquad (5)$$

hold for some $0 < \epsilon \leq A$. Then nonoscillatory solution $x(t)$ of Eq.(1) satisfies that

$$|x(t)| \leq B\exp(-\frac{1}{2}\int_T^t V(s)ds), \ t \geq T$$

for some $B \geq 0$ and $T > 0$.

Lemma 3 *Suppose that (4),(5) hold. If Eq.(1) has nonoscillatory solution, then*

$$\int_{\delta(t)}^{t} V(s)ds \leq 2, \ t \geq T_0$$

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for sufficiently large T_0.

Lemma 4 *Assume that (3) and (5) hold. If $x(t)$ is a nonoscillatory solution of Eq.(1), then $\frac{x(\delta(t))}{x(t)}$ is bounded for large enough t.*

Lemma 5 *Assume that (3),(5) and*

$$\int_{t_0}^{\infty} V(t)[\exp(\int_{\delta(t)}^{t} V(s)ds - \frac{1}{e}) - 1]dt = \infty \qquad (6)$$

hold. If $x(t)$ is a nonoscillatory solution of Eq.(1), then there exists $\psi(t)$ and T such that

$$\int_{T}^{\infty} V(t)[W(t) - e \int_{\psi(t)}^{t} V(s)W(s)ds]dt = \infty$$

where $W(t) = \frac{x(\delta(t))}{x(t)}$.

Theorem *Assume that (3),(5) and (6) hold. Then every solution of Eq.(1) oscillates.*

References

1. Li Bingtuan, *J.Math.Anal.Appl.*, **192**, 312 (1995).
2. G.S.Ladde, V.Lakshmikantham and B.G.Zhang, *Oscillation Theory of Differential Equations with Deviating argument*, (Marcel Dekker, New York, 1987).
3. I.Györi and G.Ladas, *Oscillation Theory of Delay Differential Equation with Applications*, (Clarendon Press,Oxford, 1991).
4. Li Bingtuan and Yang Kuang, *Nonlinear Analysis,Theory,Methods and Applications*, **25**, 1265 (1997).

A NUMERICAL SCHEME FOR THE MULLINS-SEKERKA EVOLUTION IN THREE SPACE DIMENSIONS

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We give a semi-implicit numerical scheme for solving a free boundary value problem
in three space dimensions. The algorithm is implemented and some computational
experiments are performed

1   Introduction

Consider a binary material at a temperature such that two different concentration phases are stable. Away from equilibrium one may find several different regimes or stages of evolution. For instance, an initially nearly homogeneous state may separate into a two phase state through nucleation, where small islands of one phase appear at several random locations and times, or through spinodal decomposition, in which a fine-grained mixture is found throughout the sample. Following either of these routes of initial phase separation, one observes a coarsening process in which small particles dissolve or coalesce, while larger particles grow, all the time preserving the total amount of each species. The final or equilibrium state has a rather simple structure with very few, typically only two, regions of different concentration.

Here, we are interested in the coarsening stage of the evolution, where some particles dissolve while others grow. We are especially interested in the evolution of the shapes of particles.

There is some choice as to how one might model this evolution but all models fall into one of two general categories; diffuse interfaces or sharp interfaces. The foremost diffuse-interface model was introduced many years ago by J. Cahn and J. Hilliard (see [3] and [4]).

Among the possible sharp-interface descriptions, the Mullins-Sekerka model [8] takes the quasi-static approach. In this case, material diffuses quickly to equilibrium in each of the two phases but material is lost or gained at the interface according to its mean curvature and the harmonic fields in the two phases. This occurs in such a way that, as the interface evolves, the enclosed volume remains constant, consistent with conservation of species.

Formal asymptotics [9], involving matched inner and outer expansions,

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show that intermediate level sets of solutions to the Cahn-Hilliard equation evolve, asymptotically as interaction length tends to zero, according to the Mullins-Sekerka flow. This can also be proved rigorously (see [1]) and therefore morphology described by one model may be duplicated by the other. On the other hand, the Mullins-Sekerka model is a free boundary problem and stable numerical methods for solutions to free boundary problems are notoriously hard to find. In [2] an algorithm for the two-dimensional Mullins-Sekerka problem was introduced and several computational experiments were performed (see also [5]). That algorithm, like an earlier treatment (see [7]) was based on a boundary integral formulation but involved a semi-implicit time-stepping, greatly improving stability.

In this paper we present some initial steps to numerically treat the Mullins-Sekerka problem in three space dimensions by extending the ideas in [2]. Our results are far from optimal but the approach shows promise.

Implementation of the scheme involves discretizing the surfaces and estimating the mean curvature and outward normal at each point on the surface. We then discretize the boundary integrals to form a system of linear equations whose solution gives the normal velocity at each point. Because the explicit scheme must use an impractically small time step to remain stable, we use a semi-implicit scheme based on a linearization of the map which evolves the surface according to the normal velocity. With the resulting modified normal velocities and previously calculated outward normals, we advance each point of the surface.

2   Mathematical Formulation

Let Ω denote a bounded and simply connected domain in three-dimensional space. Let Γ_0 denote a finite collection of smooth simple closed surfaces in the domain. We want to find a function $u(x,t)$ and a free boundary $\Gamma(t)$ which for all $x \in \Omega$ and $t \geq 0$ satisfies

i) $\Delta u(\cdot,t) = 0$ in $\Omega\backslash\Gamma(t)$
ii) $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$
iii) $u = K$ on $\Gamma(t)$
iv) $\left[\frac{\partial u}{\partial n}\right]_{\Gamma(t)} = V$ on $\Gamma(t)$
v) $\Gamma(0) = \Gamma_0,$

where n is the outward unit normal to either $\partial\Omega$ or to $\Gamma(t)$, $\left[\frac{\partial u}{\partial n}\right]_{\Gamma(t)}$ is the jump in the normal derivative of u across $\Gamma(t)$, K is the mean curvature, with the convention that convex bodies have positive mean curvature, and V is the outward normal velocity of $\Gamma(t)$. In this article we consider only $\Omega = \mathbb{R}^3$, and

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therefore replace ii) with

$$\nabla u(x,t) = \mathcal{O}\left(\frac{1}{|x|^3}\right) \qquad\qquad \text{as } |x| \to \infty.$$

We label this system as (2.1).

We now show that there exists an integral representation of the solution to system (2.1).

Lemma 2.1 *Let Γ be the union of finitely many disjoint simple closed surfaces such that Γ separates $\mathbb{R}^3$ into finitely many bounded regions and one unbounded region. Let n denote the outward unit normal to Γ. For each $g \in L^2(\Gamma)$ define*

$$W_g(x) = \frac{-1}{4\pi} \int_\Gamma \frac{1}{|x-y|} g(y) dS_y \qquad\qquad x \in \mathbb{R}^3 \backslash \Gamma.$$

Then the following hold:

1. $\Delta W_g = 0$ *in* $\mathbb{R}^3 \backslash \Gamma$.

2. $-\left[\frac{\partial W_g}{\partial n}\right]_\Gamma \equiv \frac{\partial W_g^{out}}{\partial n} - \frac{\partial W_g^{in}}{\partial n} = g$ *on* Γ.

3. *If we also assume that $\int_\Gamma g(y) dS_y = 0$, then $|\nabla W_g| = \mathcal{O}\left(\frac{1}{|x|^3}\right)$ and $W_g = \mathcal{O}\left(\frac{1}{|x|}\right)$ as $|x| \to \infty$.*

Proof. Statement 1 holds since $\Delta x\left(\frac{1}{|x-y|}\right) = 0$ for $x \neq y$. Statement 2 is proved in detail in [6] section 3D, while statement 3 may be verified by straightforward computation.

We have just found a function which is harmonic on $\mathbb{R}^3 \backslash \Gamma$. This function plus any constant is also harmonic on $\mathbb{R}^3 \backslash \Gamma$. We now show that if a harmonic function on $\mathbb{R}^3 \backslash \Gamma$ has sufficiently regular trace on Γ, then this trace is realized by the harmonic function found previously from the jump in the normal derivative, up to a constant.

Lemma 2.2 *Let Γ be as in the previous lemma. Suppose u defined on $\mathbb{R}^3$, $f \in H^1(\Gamma)$, and $g \in L^2(\Gamma)$ satisfy*

$$\begin{array}{lll} i) & \Delta u = 0 & \text{in } \mathbb{R}^3 \backslash \Gamma \\ ii) & |\nabla u| = \mathcal{O}\left(\frac{1}{|x|^3}\right) & \text{as } |x| \to \infty. \\ iii) & u = f & \text{on } \Gamma \\ iv) & -\left[\frac{\partial u}{\partial n}\right]_\Gamma = g & \text{on } \Gamma. \end{array}$$

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Then there exists a constant c such that

    a) $f(x) = -\frac{1}{4\pi} \int_\Gamma \frac{1}{|x-y|} g(y) dS_y + c$ *for $x \in \Gamma$.*

Furthermore,

    b) $\int_\Gamma g(y) dS_y = 0$.

The proof is straightforward. We leave it to the reader.

    We now combine Lemmas 2.1 and 2.2 with $f = K$ and $g = V$, the mean curvature and normal velocity of Γ, respectively. Thus, we have

Theorem 2.1 *If $\bigcup\limits_{T \geq t \geq 0} \Gamma(t)$ is a continuous family of C^3 surfaces which satisfy*

    i) $K(x,t) = -\frac{1}{4\pi} \int_{\Gamma(t)} \frac{1}{|x-y|} V(x,t) dS_y + c(t)$ *for $x \in \Gamma$.*

    ii) $\int_{\Gamma(t)} V(y,t) dS_y = 0$.

where T is some finite time, $V(x,t)$ is the normal velocity of $\Gamma(t)$ and $K(x,t)$ is the mean curvature of $\Gamma(t)$, then $\Gamma(t)$ is the interface associated with the solution of (2.1).

    Conversely, if $(u, \Gamma(t))$ is a solution to (2.1), then i) and ii) hold.

    We now solve an inverse problem.

Lemma 2.3 *Given $f \in H^1(\Gamma)$, with Γ as in Lemma 2.1, there exists a unique $g \in L^2(\Gamma)$ and constant c such that a) and b) of Lemma 2.2 hold.*

Again, we leave the proof to the reader.

    We can now solve (2.1) in the following way: Given $\Gamma(t)$ we calculate $K(x,t)$ on $\Gamma(t)$. We solve the system

$$\begin{cases} K(x,t) = -\frac{1}{4\pi} \int_{\Gamma(t)} \frac{1}{|x-y|} V(y,t) dS_y + c(t) \\ \int_{\Gamma(t)} V(y,t) dS_y = 0 \end{cases}$$

for $V(\cdot, t)$ and $c(t)$. We can then advance the surface by

$$x(t+h) \approx x(t) + Vnh$$

where h is the time step and n is the outward unit normal. However, this explicit scheme is unstable unless we use an extremely small time step. Instead we use a semi-implicit scheme. We solve the system

$$\begin{cases} K(x,t) - h\frac{\partial K(x,t)}{\partial h} = -\frac{1}{4\pi} \int_{\Gamma(t)} \frac{1}{|x-y|} V(y,t) dS_y + c(t) \\ \int_{\Gamma(t)} V(y,t) dS_y = 0 \end{cases}$$

for $V(\cdot, t)$ and $c(t)$ and advance the surface as before.

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3   Numerical Scheme

3.1   Discretization of Surfaces

Our purpose is to discretize any number of simple, smooth closed surfaces. Here we demonstrate the algorithm by discretizing the torus and the sphere and then adding distortions to produce an initial surface. Our grid corresponds to an $N \times M$ array Surface$[N][M]$.

For the sphere, the discretization is based on the parameterization of the surface using spherical coordinates, namely

$$x = R \cos\theta \cos\phi, \ y = R \sin\theta \cos\phi, \ z = R \sin\phi,$$

where R is the radius.

Letting n go from 0 to $N-1$ and m go from 0 to $M-1$ we let

$$\theta = \frac{2\pi n}{N}, \ \phi = -\frac{\pi}{2} + \frac{\left(m + \frac{1}{2}\right)\pi}{M}.$$

In effect we are wrapping the grid into a cylinder and then pinching the top and bottom to form a sphere.

Note that $\theta = 0$ when $n = 0$ and θ is almost 2π when $n = N - 1$. Similarly, when $m = 0$, ϕ is just slightly larger than $-\frac{\pi}{2}$ and when $m = M-1$, ϕ is slightly smaller than $\frac{\pi}{2}$. The top view and side view of appropriate hemispheres are shown below for $N = 20$ and $M = 20$.
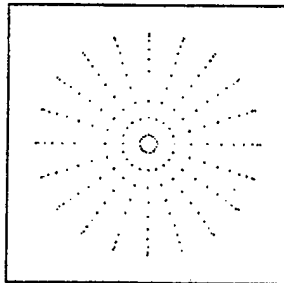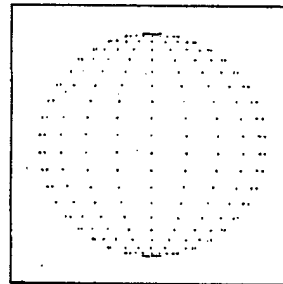
Top View   Side View

Note that longitudinally the points are equally spaced but that latitudinally they are not. This is a weakness of this discretization.

For the torus, let R denote the large radius of the torus and r the small radius. Then

$$x = (R + r\cos\phi)\cos\theta, \ y = (R + r\cos\phi)\sin\theta, \ z = r\sin\phi,$$

where θ is the angle between the positive x axis and the projection of the line segment l, from the origin to the point P on the surface, onto the xy plane. Also, ϕ is found by intersecting the torus with the half-plane H containing l and with edge the z-axis, to get a circle. Then ϕ is the angle between the ray from the origin through the center of the circle and the radius of the circle through P.

Again letting n go from 0 to $N-1$ and m go from 0 to $M-1$ we let

$$\theta = \frac{2\pi n}{N}, \ \phi = \frac{2\pi m}{M}.$$

This, in effect, wraps the grid into a cylinder and then bends the cylinder around to join the ends. Pictures of the torus with large radius 5, small radius 1, $N = 30$, and $M = 20$ are shown below.
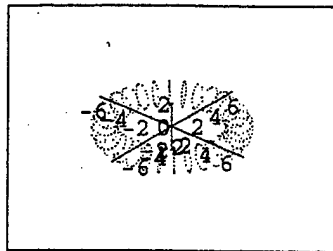
Entire Torus
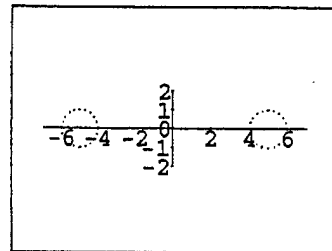
Cross Section

3.2 Approximation of Curvature

Once we have discretized a surface, we must approximate the mean curvature of the surface at each point P. We do this by choosing four nearby points, the "nearest neighbors," and changing to a coordinate system centered at P in which those neighbors satisfy $w = au^2 + buv + cv^2$ for some a, b and c. This latter quadratic surface is desirable since the mean curvature is simply $(a+c)$ and the normal to the surface at the origin is the unit vector $-\vec{k}$.

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Choosing the nearest neighbors is the first step. For the torus, the neighbors are the points on the grid directly to the right, left, top, and bottom of the point P, allowing for the wrap-around on the edges of the grid. However, the sphere must use a different neighbor scheme on the top and bottom edges of the grid, which correspond to the points surrounding the poles of the sphere. We therefore choose the point directly across from P to be the neighbor in question. In terms of the grid, for most points of the sphere, the neighbors of Surface[n][m] are

$$\text{Surface}[n+1][m], \ \text{Surface}[n][m+1], \ \text{Surface}[n-1][m], \ \text{Surface}[n][m-1].$$

If $n = 0$, we exchange Surface[$n-1$][m] for Surface[$N-1$][m]. If $n = N-1$, we exchange Surface[$n+1$][m] for Surface[0][m]. If $m = 0$ or $m = M-1$, we exchange the appropriate neighbor for Surface$[\frac{N}{2}+n, m]$ if $n < \frac{N}{2}$ or Surface$[n - \frac{N}{2}, m]$ if $n \geq \frac{N}{2}$. For this reason, N must be even. We label the point P as $X[0]$ and its neighbors $X[1]$ through $X[4]$, going counter-clockwise.

We now must choose a new coordinate system in which the points will satisfy $w = au^2 + buv + cv$. We first translate the points so that $X[0]$ becomes the origin, and label the new points as trans$X[0]$ through trans$X[4]$. Next we choose as the uw plane the one passing through the origin, trans$X[1]$, and trans$X[3]$. These three points must lie on a parabola in that plane, in fact, it will be $w = au^2$. Solving for a and the vectors to be the u and w axis involves rotating the uw plane so that trans$X[1]$ is on an axis in a temporary coordinate system, solving for the tangent of the angle the plane needs to rotate for the three points to lie on a parabola, and rotating the plane by that angle. Then a and the u and w axis can be determined. Taking the cross product of the u and w vectors gives us the v axis. To find b and c we plug in the points trans$X[2]$ and trans$X[4]$ in their new coordinate system and solve the resulting system of equations. If these points both lie on the vw pane, we let $b = 0$ and choose the least squares fit for c. Hence, we find the mean curvature, $(a + c)$, and the normal to the surface, the unit vector $-\vec{k}$.

3.3 Approximation of Partial Derivatives

For each point we must calculate $\frac{\partial K}{\partial h}$, that is, the linearization of the map which takes the curvature of the surface at time t to the curvature of the surface at time $t + h$. This is a lengthy process which is facilitated greatly by Maple. First note that K is a function of a and c. In turn, a and c depend on other variables, and the chain descends eventually to the initial point and its four neighbors.

3.4   Discretization of Integrals

We have the following system of equations to discretize:

(i) $K(x) - \frac{\partial K(x)}{\partial h} = -\frac{1}{4\pi} \int_\Gamma \frac{1}{|x-y|} V(y) dS_y + c.$

(ii) $0 = \int_\Gamma V(y) dS_y.$

We first divide the surface into "patches" where each patch is centered at a point on the surface. This is done by taking the nearest neighbors and connecting their "midpoints" with each other and the center point as shown below.
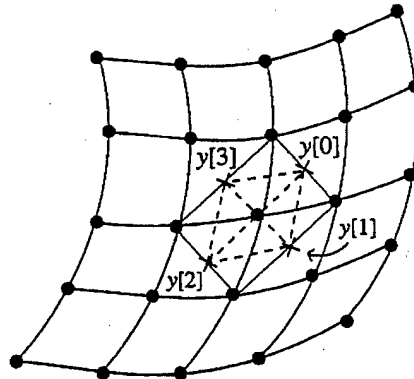
Patch is bounded by the dotted lines

The "midpoints," $y[0]$ through $y[3]$, are found by taking the average of the u and v coordinates and then calculating w from the equation $w = au^2 + buv + v^2$. If the center point is Surface$[n][m]$, we label the four triangular patches as Patch$[n][m][k]$ where k goes from 0 to 3, 0 corresponding to the patch nearest $X[1]$ and then proceed counter-clockwise. We denote their sum as Patch$[n][m]$.

We want the total area of the patches to be close to the actual surface area of the surface. The following table shows this is achieved with a large number of subdivisions for the torus, but not for the sphere due to the ill-fitting discretization. In the table, the torus has $R = 5, r = 1$; the sphere has radius 1.

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Comparing Area Estimates of Surfaces with Actual Area

Surface	N	M	Estimated Area	Actual Area
Torus	20	10	185.237	197.392
Torus	40	20	194.188	197.392
Torus	80	60	197.060	197.392
Sphere	10	10	13.515	12.566
Sphere	20	20	14.119	12.566
Sphere	20	40	13.438	12.566

We now discretize the integrals. We have for $x = \text{Surface}[n][m]$ two integrals to estimate. For a patch on the surface centered at $y \neq x$, we have

$$\int_{\text{Patch}_{i,j}} \frac{1}{|x - z|} V(z) dS_z \approx \sum_{k=0}^{3} |\text{Patch}\,[i][j][k]| \cdot \frac{1}{3} \left(\frac{1}{|x - y|} + \frac{1}{|x - y_{k-1}|} + \frac{1}{|x - y_k|} \right) \cdot V(y).$$

The summand uses the average value of the kernel at vertices of the kth triangular patch. If $y = x$, we derive an integral for each triangular patch in the following way: Translate the triangle so that we have the following:
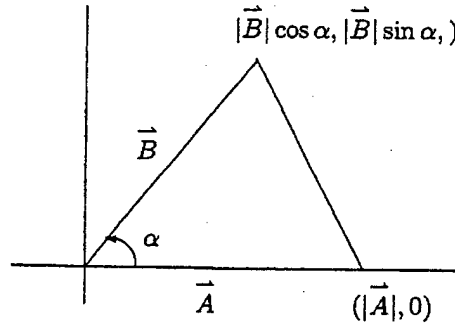
where $\cos \alpha = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$ and $\sin \alpha = \sqrt{1 - \cos^2 \alpha}$. Then

$$\int_{\text{Patch}[n][m][k]} \frac{1}{|x - z|} dS_z = \int_0^\alpha f(\theta) d\theta$$

where $f(\theta)$ describes the line opposite α.

If the line is vertical, its equation is $x = |\vec{A}|$, which in polar coordinates is $r = \frac{|\vec{A}|}{\cos \theta}$. Then

$$\int_0^\alpha f(\theta) d\theta = |\vec{A}| \int_0^\alpha \sec \theta d\theta = |\vec{A}| \ln|\sec \alpha + \tan \alpha|.$$

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If the line is not vertical, its equation in polar coordinates is $r = f(\theta) = \frac{m|\vec{A}|}{m\cos\theta - \sin\theta}$, where $m = \frac{|\vec{B}|\sin\alpha}{|\vec{B}|\cos\alpha - |\vec{A}|}$ is the slope. The integral $m|\vec{A}|\int_0^\alpha \frac{1}{m\cos\theta - \sin\theta}\,d\theta$ is found by rationalizing the denominator as follows:

$$\int \frac{1}{m\cos\theta - \sin\theta}\,d\theta = m\int \frac{\cos\theta}{m^2 - (m^2+1)\sin^2\theta}\,d\theta + \int \frac{\sin\theta}{(m^2+1)\cos^2\theta - 1}\,d\theta.$$

We get

$$m|\vec{A}|\int_0^\alpha \frac{1}{m\cos\theta - \sin\theta}\,d\theta = \frac{|\vec{A}|m}{2\sqrt{m^2+1}}\ln\left|\frac{\left(\sqrt{m^2+1}\cos\alpha + 1\right)^2\left(\sqrt{m^2+1} - 1\right)}{\left(\sqrt{m^2+1}\sin\alpha - m\right)^2\left(\sqrt{m^2+1} + 1\right)}\right|.$$

We denote the value of $\int \frac{1}{|x-y|}\,dS_y$ with $x = $ Surface$[n][m]$ over the Patch$[i][j]$ as $I_{n,m,i,j}$. Then we get equation (i) in the following discretized form

$$-4\pi\left(K(x) - \frac{\partial K(x)}{\partial n}\right) \approx \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} I_{n,m,i,j}V_{i,j} + c$$

where $V_{i,j}$ is the velocity of Surface$[i][j]$.

For (ii) we get

$$0 \approx \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} V_{i,j} \cdot |\,\text{Patch}[i][j]|.$$

Since we have $N \cdot M$ choices of $x = $ Surface$[n][m]$, we have $N \cdot M + 1$ linear equations with each $V_{i,j}$ and c as unknowns.

3.5   Advancing the Surface

Once we have solved for V we advanced the surface. For each point on the surface, we add the normal at that point multiplied by the time step and velocity.

4   Results

The following pages contain graphics and data pertaining to a sphere, distorted spheres, and a torus. While the scheme is still quite unstable, results are promising. Most of the surfaces exhibited a decrease in surface area and we believe that a dynamic redistribution of surface data points to prevent bunching would help stabilize the scheme.

Each surface is pictured using Maple graphics. Also included is a table listing surface areas at certain iterations. As the surface breaks down, the surface areas increase. Noticeably high mean curvatures are marked by an asterisk. Usually high mean curvatures correspond to large surface areas, but not always. Again, we believe that bunching of surface data points is at the root of the breakdown. This could be alleviated by including a component of the velocity tangential to the surface in such a way that
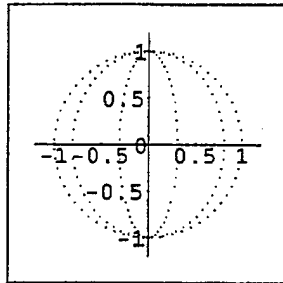
$t = 0$
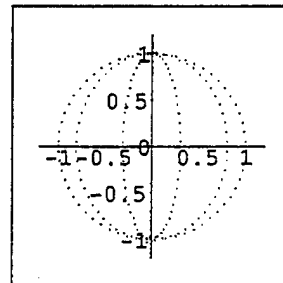
13th iteration
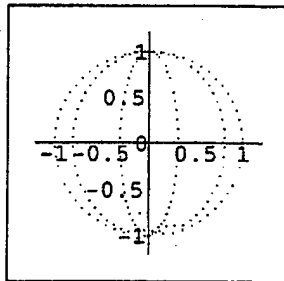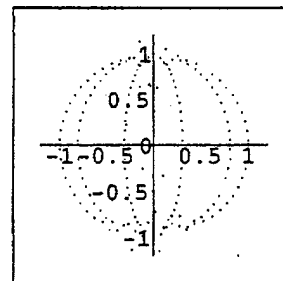
14th iteration 16th iteration

Sphere with radius 1, $N = 10; M = 30;$ Time step $= 0.0001$

A sphere should ideally have no diffusion of particles. This indeed was

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the case until the surface became unstable. As shown in the table below, the surface areas generated in the first 12 iterations are within 0.0001 of their previous values. However, after the 13th iteration, mean curvatures of over 50 started appearing. Surface area also increased. As seen in the figures above, bunching and then eruptions occurred at the poles.

SPHERE

Iteration	Surface Area
0	13.959657
1	13.959610
3	13.959571
5	13.959553
7	13.959543
9	13.959535
11	13.959530
12	13.959686
13*	13.988053
14	15.452335

*Mean curvatures of over 50 after this iteration.
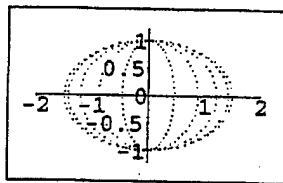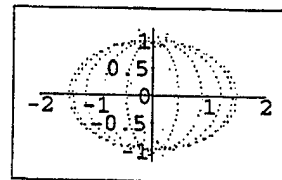
$t = 0$ 8th iteration

Sphere widened at equator, $N = 20, M = 30$, time step $= 0.001$

This surface was quite unstable at the poles. However, surface area did decrease until the 7th iteration.

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SPHERE WIDENED AT EQUATOR

Iteration	Surface Area
0	23.929643
1	23.924496
3	23.912062
5	23.900737
6	23.892807
7*	26.023065
8	87.655583

*Mean curvatures of over 50 after this iteration.
The next experiment starts with an elongated sphere.
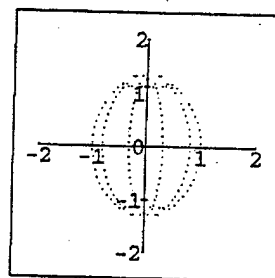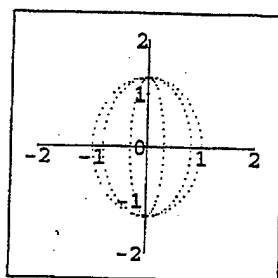
$t = 0$	5th iteration

Sphere lengthened at poles, $N = 10, M = 30$, time step $= 0.001$
This oval surface behaved very unusually. After a jump in mean curvature in the 5th iteration, the surface area tripled. Yet the surface, although rather oddly shaped, was stable and decreased in area for the next 8 iterations.

SPHERE LENGTHENED AT POLES

Iteration	Surface Area
0	16.271194
1	16.271876
2	16.259973
3	16.304395
4	16.210148
5*	16.237525
6	48.538159

*Mean curvatures of 56 reached in this iteration.

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We now take our initial surface to be a torus. The evolution here is more stable, probably because the gridding is more uniform.
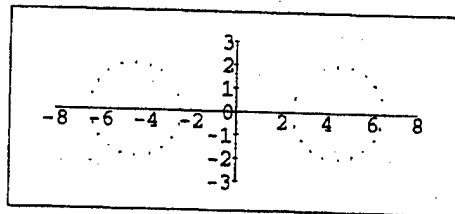
$t = 0$
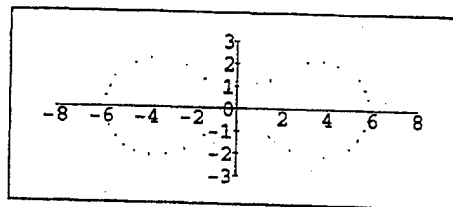
668th iteration
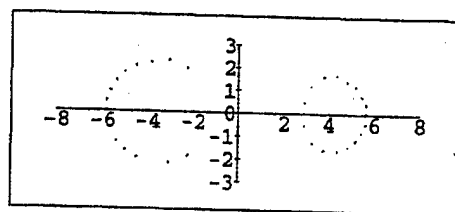
669th iteration

Torus with large radius 4.5 and small radius 2, $N = 20$; $M = 20$; Time step $= 0.01$
The torus was the most stable surface tested; problems did not occur until the 668th iteration. The torus should eventually evolve to a sphere,

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and indeed the cross section shown previously migrates towards the origin. However, we have not yet overcome the problem of how to change topological type numerically and in a way which is consistent with the original formulation of the problem.

TORUS

Iteration	Surface Area
0	347.85
100	343.56
300	332.89
500	321.43
600	315.07
660	311.12
667*	310.67
668	314.18

*High mean curvatures after this iteration.

References

1. Alikakos, N.D., Bates, P.W., and Chen, X., *Convergence of the Cahn-Hilliard Equation scheme to the Hele-Shaw Model*, Arch. Rat. Mech. Anal., **128** (1994), 165–205.

2. Bates, P.W., Chen, X., and Deng, X., *A Numerical Scheme for the Two Phase Mullins-Skerka Problem*, Electr. J. Diff. Equs., **1995**, No. 11 (1995), 1–28.

3. Cahn, J.W., *On spinodal decomposition*, Acta Metallurgica 9 (1961), 795–801.

4. Cahn, J.W., and Hilliard, J.E. *Free energy of a non-uniform system I: Interfacial energy*, J. Chem. Phys. 28 (1958), 258–266

5. Deng, X., *A Numerical Analysis Approach for the Hele-Shaw Problem*, M. S. Thesis, Brigham Young University, April 1994.

6. Folland, G. B., *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, NJ, 1976.

7. McFadden, G.B., Voorhees, P.W., Boisvert, R.F., and Meiron, D.I., *A Boundary Integral Method for the Simulation of Two-Dimensional Particle Coarsening*, J. Sci. Comp. 1 (1986), 117–143.

8. Mullins, W. W. and Sekerka, R. F., *Morphological Stability of a Particle Growing by Diffusion and Heat Flow*, J. Appl. Phys. 34 (1963), 323–329.

9. Pego, R.L., *Front Migration in the Nonlinear Cahn-Hilliard Equation*, Proc. Roy Soc. London 422 (1989), 261–278.

APPROXIMATE NORMALLY HYPERBOLIC INVARIANT MANIFOLDS FOR SEMIFLOW

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In this paper, we establish the existence of normally hyperbolic invariant manifolds for semiflows.

We consider a C^1 semiflow defined on a Banach space X, that is, it is continuous on $[0, \infty) \times X$, and for each $t \geq 0, T^t : X \to X$ is C^1, and

$$T^t \circ T^s(x) = T^{t+s}(x)$$

for all $t, s \geq 0$ and $x \in X$. A typical example is the solution operator for a differential equation.

In [5] we proved that a compact normally hyperbolic invariant manifold M persists under small C^1 perturbations in the semiflow. We also showed that in a neighborhood of M, there exist a center-stable manifold and a center-unstable manifold which intersect in the manifold M. In [5], the compactness and invariance of the manifold M were important assumptions.

In the present paper, we establish the existence of normally hyperbolic invariant manifolds for semiflows. We prove that if T^t has a "good" approximate normally hyperbolic "invariant" manifold M for the semiflow T^t, then T^t has a normally hyperbolic invariant manifold $\tilde{M}$ near M.

In many singular perturbation problems for evolutionary partial differential equations, one is interested in stationary solutions which have certain qualitative features, such as interior or boundary layers or localized spikes. The stability of these states is also important. The canonical shape of such solutions, in a neighborhood of the abrupt spatial disturbance, can usually be determined by a rescaling or blow-up procedure. Thus, a reasonable approximation to the shape of a solution is found quite easily. A strategy to find a true solution, or at least to prove existence of a solution of the type of interest, is to first create, in function space, a manifold M made up of these

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approximate solutions parameterized by the spatial locations of the singularities. The PDE is then written as a (semi)flow in function space decomposed into directions tangential and normal to M. These components are then estimated in sufficient detail to show the existence of stationary points in a small neighborhood of M. This approach was pioneered in the early papers of G. Fusco and J. Hale in [9] and by J. Carr and R. Pego in [8], who were interested in the Allen-Cahn equation, a bistable reaction-diffusion equation with very small diffusion coefficient, ϵ^2. In fact, the authors were more interested in the flow in a neighborhood of the manifold than in the stationary solutions and used the technique to show that solutions with interior layers evolved exponentially slowly, the layers moving with speed of the order $e^{-c/\epsilon}$ where c is a constant depending upon distance between layers.

The approach was also taken to obtain similar results for the one-dimensional Cahn-Hilliard equation in [1] and [7]. In that case, since the equation is fourth order, the analysis is somewhat harder but the geometric picture is the same.

More recently, N. Alikakos and G. Fusco have used the method to show the exponentially slow motion of "bubble"-like solutions to the Cahn-Hilliard equation in higher space dimensions [2]. [6] M. Kowalczyk uses the approximate invariant manifold approach to find spiked solutions to the shadow Gierer-Meinhardt system of biological pattern formation in multi-dimensional domains and [4] has similar results for the Cahn-Hilliard equation.

Stationary spiked solutions to many nonlinear elliptic equations have been found using essentially the same approach but viewing it as a Liapunov-Schmidt reduction idea, rather than dynamically. Some representative works along these lines include [3], [12], and the references therein. A good survey of spike solutions may be found in [11].

What has been lacking is a way to deduce the existence of a true invariant manifold in a small neighborhood of the approximately invariant manifold constructed by hand. Here we hope to give a general result along those lines, perhaps simplifying much of the analysis needed in the applications mentioned above, and at the same time giving stronger conclusions.

We first consider maps on a Banach space X. Consider $M \subset X$ a C^1 connected submanifold of X and $T \in C^{1,1}(B(M,r), X)$ for some $r > 0$, where $B(M,r) = \{x \in X : \text{dist}(x, M) < r\}$.

Definition 1 M *is said to be an approximate normally hyperbolic invariant manifold if the following conditions hold for some* $\delta > 0$

(H1) M *is approximately invariant, i.e.,* $T(M) \subset B(M, \delta)$.

(H1) For each $m \in M$ there is a decomposition

$$X = X_m^c \oplus X_m^u \oplus X_m^s$$

of closed subspaces with X_m^u, X_m^s being transversal to $T_m M$, the tangent space of M at m. There is an $\epsilon > 0$ such that $X^u(\epsilon) \oplus X^s(\epsilon) = \{m + x^u + x^s : x^\alpha \in X_m^\alpha$ and $|x| < \epsilon$ for $\alpha = u, s\}$ is a tubular neighborhood of M and $B(M, \delta) \subset X^u(\epsilon) \oplus X^s(\epsilon)$. For each $m \in M$, $T(M) \cap (X^u(\epsilon) \oplus X^s(\epsilon)) \neq \emptyset$.

(H3) For each $m_0 \in M$

$$\Pi_{m_1}^\alpha DT(m_0) : X_{m_0}^\alpha \to X_{m_1}^\alpha$$

are isomorphisms for $\alpha = c, u$, where $m_1 \in M$ is determined by using the tubular neighborhood, by writing $T(m_0) = m_1 + x^u + x^s$, and Π_m^α is the projection onto X_m^α with kernel being the sum of the other subspaces. Also there exists $\lambda \in (0, 1)$ such that

$$\|\Pi_{m_1}^s DT(m_0)|_{X_{m_0}^s}\| < \lambda \min\{1, \inf\{|\Pi_{m_1}^c DT(m_0)x^c| : x^c \in X_{m_0}^c, |x^c| = 1\}\},$$

$$\lambda \inf\{|\Pi_{m_1}^u DT(m_0)x^u| : x^u \in X_{m_0}^u, |x^u| = 1\} > \max\{1, \|\Pi_{m_1}^u DT(m_0)|_{X_{m_0}^c}\|\}.$$

Since we do not assume that M is compact or finite dimensional, for technical reasons, we need to assume that T, Π_m^c, Π_m^u, and Π_m^s have some uniform properties:

(H4) (1) There is an $r_1 > 0$ and $L_1 > 0$ such that for any $m_0 \in M$, $m_1, m_2 \in B(m_0, r)$, and $\alpha = c, u, s$,

$$\|\Pi_{m_1}^\alpha - \Pi_{m_2}^\alpha\| \leq L_1 |m_1 - m_2|.$$

(2) There is a constant $B > 0$ such that $\|\Pi_m^\alpha\| \leq B$ for all $m \in M$, and $\alpha = c, u, s$;

(3) There exists $\mu_0 > 0$ such that for any $m \in M$ and $\alpha = u, s$,

$$\|\Lambda_m^\alpha\| \leq \mu_0,$$

where $\Lambda_m^u \in L(X_m^c, X_m^u)$ and $\Lambda_m^s \in L(X_m^c, X_m^s)$ are determined by

$$T_m M = (I + \Lambda_m^u + \Lambda_m^s) X_m^c;$$

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(4) There is an $r_2 > 0$ and $L_2 > 0$ such that for any $m_0 \in M$, and $m_1, m_2 \in B(m_0, r)$,

$$\|DT(m_1) - DT(m_2)\| \leq L_2 |m_1 - m_2|.$$

(5) There are constants $a > 0$, $B_1 > 0$, and $b_1 > 0$ such that

$$\inf\{|\Pi^c_{m_1} DT(m_0) x^c| : x^c \in X^c_{m_0}, |x^c| = 1\} \geq a,$$

$$\|DT|_{B(M,r)}\| \leq B_1,$$

$$\|\Pi^\alpha_{m_1} DT(m_0)|_{X^u_{m_0}}\| \leq b_1,$$

for $\alpha = c, s$, and

$$\|\Pi^\alpha_{m_1} DT(m_0)|_{X^s_{m_0}}\| \leq b_1,$$

for $\alpha = c, u$.

Condition (2) implies that the space X^c_m is an approximation of the tangent space of M at m with an error bounded by μ_0. Condition (3) automatically holds when M is contained in a compact set. The reason for this assumption is that the graph transform is a global transform and some uniform estimates are needed.

Theorem 1. *Assume that $(H1) - (H4)$ hold. Then, there exist $\delta^* > 0$, $\mu_0^* > 0$, and $b_1^* > 0$ such that if $\delta \leq \delta^*$, $\mu_0 \leq \mu_0^*$, $b_1 < b_1^*$, then T has a unique C^1 connected normally hyperbolic invariant manifold $\tilde{M}$ near M.*

We now consider the manifolds for semiflows. Let $T \in C([0, +\infty) \times B(M, r), X)$ be a semiflow, i.e.,

$$T^0 = I, \qquad T^{t+s} = T^t \circ T^s, \quad \text{for } t, s \geq 0.$$

We assume that for all $t \geq 0$, $T^t \in C^1(B(M, r), X)$. Suppose there exists $t_0 > 0$ such that T^{t_0} satisfies (H1)-(H4). From Theorem A, there exists a C^1 normally hyperbolic invariant manifold $\tilde{M}$ for T^{t_0}.

Furthermore, if we assume

(H6) For any $\eta > 0$, there exists $\zeta > 0$, such that for any $x \in B(M, r)$, $t \in [0, \zeta]$, we have

$$|T^t(x) - x| < \eta.$$

Then we have **Theorem 2.** *The normally hyperbolic invariant manifold $\tilde{M}$ for T^{t_0} is the normally hyperbolic invariant manifold for the semiflow T^t.*

The basic ideas of the proof of Theorem 1 are the same as those we used in [5]. However, here we have to overcome the difficulties caused by the non-compactness of the manifold. The complete proof of these results will appear in [3].

References

1. N. D. Alikakos, P. W. Bates and G. Fusco, Slow motion for the Cahn-Hilliard equation in one space dimension, *J. Diff. Eqns.* **90** (1990), 81–135.
2. N. D. Alikakos and G. Fusco, Slow dynamics for the Cahn-Hilliard equation in higher space dimensions, Part II: The motion of bubbles, *Arch. Rational Mech. Anal.*, to appear.
3. P. W. Bates, E. N. Dancer, and J. Shi, Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability, *Adv. Diff. Eqns.*, (1999).
4. P. W. Bates and G. Fusco, Equilibria with many nuclei for the Cahn-Hilliard equation, *J. Diff. Eqs.*, (1999), to appear.
5. P. W. Bates, K. Lu and C. Zeng, Existence and persistence of invariant manifolds for semiflows in Banach spaces, *Memoirs of the AMS.* **645** (1998).
6. P. W. Bates, K. Lu and C. Zeng, Invariant Manifolds and Foliations for Infinite Dimensional Dynamical Systems, *in preparation*.
7. P.W. Bates and P.J. Xun, Metastable patterns for the Cahn-Hilliard equation: Part II, Layer dynamics and slow invariant manifold, *J. Diff. Eqns.* **116** (1995), 165-216.
8. J. Carr and R.L. Pego, Metastable patterns in solutions of $u_t + \epsilon^2 u_{xx} - f(u)$, Comm. Pure Appl. Math., **42**, (1989), 523–576.
9. G. Fusco and J.K. Hale, Slow motion manifolds, dormant instability and singular perturbations, Dynam. Diff. Eqns., **1** (1989), 75–94.
10. M. Kowalczyk, Multiple spike layers in the shadow Gierer-Meinhardt system: Existence of equilibria and approximate invariant manifold, *Duke Math. J.*, to appear.
11. W. M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, *Notices of the AMS*, (1998) **45**, 9-18
12. J. Wei and M. Winter, Stationary solutions for the Cahn-Hilliard equation, *Ann. Inst. Henri Poincare*, **15**, (1998), 459–492

SUFFICIENT CONDITIONS FOR OSCILLATION OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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In this paper,some sufficient conditions are established for oscillation of second order nonlinear differential equations and nonhomogeneous linear differential equations

1 INTRODUCTION

Consider the second order nonlinear differential equation

$$(p(t)y')' + q(t)y = H(t, y, y') \tag{1}$$

where $p(t), p'(t), q(t) \in C(R^+, R^+)$, $H \in C(R^+ \times R^2, R)$ guarantee that Eq. (1) with initial conditions has uniquely solution, $R^+ = (0, +\infty)$. When $H(t, y, y') \equiv h(t)$, Eq. (1) becomes a nonhomogeneous linear differential equations. The homogeneous linear differential equation corresponding to (1) is

$$(p(t)y')' + q(t)y = 0 \tag{2}$$

As it is customary, a nontrivial solution is called to be oscillatory if it has arbitrarily large zeros. Otherwise,the solution is called non-oscillatory. Eq. (1) or (2) is called to be oscillatory if all solutions of (1) or (2) are oscillatory.

It is much more difficult to deal with the oscillation of Eq. (1) than Eq. (2) [1,2,3]. This paper will give some sufficient conditions for the oscillation of Eq. (1) by discussing the relation between Eq. (1) and Eq. (2).

Let $x(t)$ and $y(t)$ are nontrivial solutions of Eq. (1) and Eq. (2), respectively. Multiplying both members of Eq. (1) by $x(t)$ subtracts multiplying both members of Eq. (2) by $y(t)$ to get

$$[p(t)(x(t)y'(t) - x'(t)y(t))]' = Hx(t) \tag{3}$$

Theorem 1. Let $x(t)$ be a oscillatory solution of Eq. (2). If there is a $T > 0$ such that for all $t \geq T$

$$Hx(t) \geq 0 \quad \text{or} \quad Hx(t) \leq 0,$$

then Eq. (1) is oscillatory.

Proof Assume, by way of contradiction, that (1) has an eventually positive (or negative) solution $x(t)$. Let $\{t_n\}$ is the zero point set of $x(t)$, i.e., $x(t_n) = 0$ and $\lim_{n\to\infty} x(t_n) = \infty$. From the uniqueness of solution of Eq. (1) with initial data, $x'(t_n) \neq 0$ for all $t_n \in \{t_n\}$. If $Hx(t) \geq 0$, there always exists $N \in \{t_n\}$ with $N > n$ such that

$$x'(t_n) < 0 \quad \text{and} \quad x'(t) < 0.$$

By integrating (3) over the intervals $[t_n, t_N]$, we get

$$-p(t_n)x'(t_n)y(t_n) + p(t_N)x'(t_N)y(t_N) = \int_{t_N}^{t_n} Hx(s)ds \qquad (4)$$

This leads to a contradiction since the left side of (4) is negative and the right side of (4) is nonnegative. The proof for $Hx(t) \leq 0$ is the same as the above and omitted it.

Theorem 2. Let $x(t)$ be a oscillatory solution of Eq. (2). If there is a $T > 0$ such that for all $t \geq T$

$$Hx(t) \geq h(t) \quad \text{and} \quad \sigma \triangleq \int_{t_0}^{t} Hx(s)ds \geq 0 \quad \forall T < t_0 \gg t.$$

then Eq. (1) is oscillatory.

Theorem 3. Let $x(t)$ be a oscillatory solution of Eq. (2). If there is a $T > 0$ such that for all $t \geq T$

$$Hx(t) \geq h(t) \quad \text{and} \quad \sigma \triangleq \int_{t_0}^{t} Hx(s)ds \leq 0 \quad \forall T < t_0 \gg t.$$

then Eq. (1) is oscillatory.

Theorem 4. Let $x(t)$ be a oscillatory solution of Eq. (2). If the following conditions are satisfied

(i) There is a $T > 0$ such that for all $t \geq T$

$$Hx(t) \geq h(t) \quad \text{or} \quad Hx(t) \leq h(t).$$

(ii) Let $F(t)$ be an indefinite integral of $h(t)$, and A and B are the zero point sets of $h(t)$ and $F(t)$, respectively. We have $A \subset B$.

Then Eq. (1) is oscillatory.

When $H(t, y, y') \equiv h(t)$, Eq. (1) becomes a nonhomogeneous linear differential equations:

$$(p(t)y')' + q(t)y = h(t) \qquad (5)$$

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Based on the above theorems, we can easily get the following corollaries

Corollary 1. Let $x(t)$ be a oscillatory solution of Eq. (2). If there is a $T > 0$ such that for all

$$\sigma \triangleq \int_{t_0}^{t} h(s)x(s)ds \geq 0 \quad \text{or} \quad (\leq 0) \quad \forall T < t_0 \gg t.$$

then Eq. (6) is oscillatory.

Corollary 2. Let $x(t)$ be a oscillatory solution of Eq. (2), $F(t)$ be an indefinite integral of $h(t)x(t)$, and A and B are the zero point sets of $h(t)$ and $F(t)$, respectively. If $A \subset B$, then Eq. (6) is oscillatory.

Example. The following nonhomogeneous linear differential equation

$$y'' + y' = -2\sin t + \cos(y^2 + t^2) \tag{6}$$

is oscillatory.

In fact, $x(t) = \sin t$ is a solution of the homogeneous linear differential equation corresponding to (7) and

$$\begin{aligned} H\sin t &= -2\sin^2 t + \sin t \cos(y^2 + t^2) \\ &= 1 - 2\sin^2 t + [\sin t \cos(y^2 + t^2) - 1] \\ &\leq \cos 2t. \end{aligned}$$

It is obvious that $\frac{1}{2}\sin 2t$ is an indefinite integral of $\cos 2t$ and its zero point set involves the zero point set of $\sin t$. From Theorem 4, Eq. (7) is oscillatory.

References

1. M. S. Keener *Applicable Anal.* **1**, 57-63 (1971).
2. S. M. Rankin, *J.Math.Anal.Appl.* **53**, 550-553 (1976).
3. S. M. Rankin, *Pacific J. Math.* **80**, 237-243 (1979).

THE LIMIT CYCLE OF TWO SPECIES PREDATOR-PREY MODEL WITH GENERAL FUNCTIONAL RESPONSE

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We study a system of ODE's modelling the interaction of one predator and one prey under ecologically natural regularity conditions. Using the technique of Lie'nard equation, the conditions for existence and uniqueness of limit cycle around the positive equilibrium point are obtained, respectively.

1 Introduction

In the present paper, we first consider two species predator-prey model with general functional response as following system:

$$\begin{cases} \frac{dx}{dt} = xg(x) - yp(x) \\ \frac{dy}{dt} = y(-g(x) + h(x)), \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ are the densities of the prey and predator, respectively. $g(x)$ is the growth rate of the prey in the absence of predator, $q(x)$ is the density-dependent death rate of the prey, and $p(x), h(x)$ represent respectively, the functional and numerical response of predator. The specific standard assumptions on $g(x), p(x)$ and $h(x)$ are:

(A_1) $g: R_+ \to R$ is of class $C^1, g(0) = \alpha > 0$, there exists
$\quad k > 0$ such that $g(k) = 0$ and $(x - k)g(x) < 0$ for $x \neq k$.

(A_2) $p: R_+ \to R_+$ is of class $C^1, p(0) = 0$ and $p'(x) > 0$ for $x \in R_+$.

(A_3) $q: R_+ \to R_+$ is of class $C^1, q(0) > 0$ and $q'(x) \leq 0$
$\quad$ for all $x \in R_+$, and $q(x) \to q_\infty > 0 \quad (x \to +\infty)$.

(A_4) $h: R_+ \to R_+$ is of class $C^1, h(0) = 0$ and $h'(x) > 0$ for all $x \in R_+$.

Some special cases of system (1) have been investigated by [1-6], from 1934 up to now.

In this paper, using the technique of Lie'nard equation, the conditions for the existence and uniqueness of the system(1) are obtained.

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2 Lemmas

In order to obtain our main results, we need the following lemmas.

Lemma 1 System (1) can be transformed into the form of Lie'nard's equation

$$\ddot{x} + f(x)\dot{x} + \tilde{g}(x) = 0 \tag{2}$$

or into the equivalent system:

$$\dot{x} = u - F(x), \quad \dot{u} = -\tilde{g}(x), \quad F(x) = \int_0^x f(x)dx \tag{3}$$

by a change of variable.

proof We first introduce the transformation

$$x = x, \quad \xi = xg(x) - yp(x) \tag{4}$$

i.e

$$y = \frac{xg(x) - \xi}{p(x)}.$$

Obviously, the transformation (4) is regular:

$$\det \begin{pmatrix} 1 & 0 \\ g(x) + xg(x) - yp'(x) & -p(x) \end{pmatrix} = -p(x) \neq 0 \quad (x > 0)$$

and one to one bicontinuous.

It is not hard to find that if we substitute the regular transformation (4) into (1), it follows that

$$\begin{cases} \dot{x} = \xi, \\ \dot{\xi} = xg(x)(q(x) - h(x)) + \left\lceil p(x)\left(\frac{xg(x)}{p(x)}\right)' - q(x) + h(x)\right\rceil \xi + \frac{p'(x)}{p(x)}\xi^2. \end{cases} \tag{5}$$

Letting
$$\begin{cases} f_0(x) = -xg(x)(q(x) - h(x)), \\ f_1(x) = -\left\lceil p(x)\left(\frac{xg(x)}{p(x)}\right)' - q(x) + h(x)\right\rceil \\ f_2(x) = -\frac{p'(x)}{p(x)}, \end{cases} \tag{6}$$

it follows that

$$\dot{x} = \xi, \quad \dot{\xi} = -f_0(x) - f_1(x)\xi - f_2(x)\xi^2. \tag{7}$$

We introduce the transformation again

$$u = \xi \exp \int_{x_0}^x f_2(x)dx, \quad \text{or } \xi = u \exp\left(-\int_{x_0}^x f_2(x)dx\right), \tag{8}$$

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it follows that

$$\dot{x} = u \ \exp\left(-\int_{x_0}^{x} f_2(x)dx\right)$$
$$\dot{u} = -f_0(x) \ \exp\left(\int_{x_0}^{x} f_2(x)dx\right) - f_1(x)u. \tag{9}$$

i.e

$$\dot{x} = u \ \exp\left(-\int_{x_0}^{x} f_2(x)dx\right)$$
$$\dot{u} = -f_0(x) \ \exp\left(\int_{x_0}^{x} f_2(x)dx\right) - f_1(x)u. \tag{10}$$

Using the time transformation

$$d\tau = \exp\left(-\int_{x_0}^{x} f_2(x)dx\right) dt$$

and still denoting τ by t, then system (11) can be transformed into the system:

$$\dot{x} = u$$
$$\dot{u} = -f_0(x) \ \exp\left(2\int_{x_0}^{x} f_2(x)dx\right) - f_1(x)\exp\left(\int_{x_0}^{x} f_2(x)dx\right) u. \tag{11}$$

Let

$$f(x) = f_1(x)\exp\left(\int_{x_0}^{x} f_2(x)dx\right),$$
$$\tilde{g}(x) = f_0(x)\exp\left(2\int_{x_0}^{x} f_2(x)dx\right), \tag{12}$$

then the system (11) can be transformed into

$$\dot{x} = u,$$
$$\dot{u} = -f(x)u - \tilde{g}(x), \tag{13}$$

i.e.

$$\ddot{x} + f(x)\dot{x} + \tilde{g}(x) = 0, \tag{14}$$

or into the equivalent system (Lie'nard system)

$$\begin{cases} \dot{x} = u - F(x), \\ \dot{u} = -\tilde{g}(x), \end{cases} \tag{15}$$

where

$$F(x) = \int_{x_0}^{x} f(x)dx,$$
$$\tilde{g}(x) = xg(x)(-q(x) + h(x))\exp\left(2\int_{x_0}^{x} f_2(x)dx\right),$$
$$f(x) = -\left\lceil p(x)\left(\frac{xg(x)}{p(x)}\right)' - q(x) + h(x)\right\rceil \exp\left(\int_{x_0}^{x} f_2(x)dx\right).$$

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This completes the proof of the lemma 1.

In order to prove our main results on the existence and uniqueness of a periodic solution for the system(1), we shall utilize the known results for a Lie'nard system in [7] which we state as the following three lemmas.

Lemma 2 Consider the following Lie'nard system:

$$\dot{x} = y - F(x), \quad \dot{y} = -\tilde{g}(x), \quad F(x) = \int_{x_0}^{x} f(x)dx. \tag{16}$$

Suppose the continuous $f, \tilde{g}$ satisfy assumptions:

(1) $x\tilde{g}(x) > 0$ for $x \neq 0$, $\tilde{G}(x) = \int_0^x \tilde{g}(x) \to +\infty$ as $|x| \to +\infty$,

(2) $xF(x) < 0$, for $|x| << 1$,

(3) there exist $M > 0$ and $k_1 > k_2$ such that

$$F(x) \geq k_1 \text{ for } x > M, \quad F(x) \leq k_2 \text{ for } x < -M$$

Then system(16) has at least one stable limit cycle.

Lemma 3 consider the following Lie'nard system:

$$\dot{x} = -y - \tilde{F}(x), \quad \dot{y} = \tilde{g}(x), \quad \tilde{F}(x) = \int_{x_0}^{x} \tilde{f}(x)dx \tag{17}$$

Suppose the continuous functions $\tilde{f}, \tilde{g}$ satisfy assumptions:

(1) $x\tilde{g}(x) > 0$ for $x \neq 0$, $\tilde{G}(x) = \int_0^x \tilde{g}(x) \to +\infty$ as $|x| \to +\infty$,

(2) $\tilde{f}(0) < 0, (resp. > 0)$,

(3) $\frac{\tilde{f}(x)}{\tilde{g}(x)}$ is monotone nondecreasing (resp.nonincreasing) for $x \neq 0$.

Then system (17) has at most one stable(resp. unstable) limit cycle.

Lemma 4 Consider the system

$$\frac{dx}{dt} = X(x,y), \quad \frac{dy}{dt} = Y(x,y) \tag{18}$$

where $X(x,y), Y(x,y) \in C^0$ $(G \subseteq R^2)$ guarantee that the system (18) has uniqueness of the solution. If there is a closed orbit L of (18),and

$$\oint_L div(X,Y)dt < 0 \quad (resp. > 0)$$

then L is a stable limit cycle (resp.unstable).

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3 Results

In this section, we can state our main results.

Consider the system (1) under the assumption $(A_1) - (A_4)$. The above conditions $(A_1) - (A_4)$ guarantee that the system (1) has only positive equilibrium point $E(x_0, y_0)$ inside the population quadrant, and

$$y_0 = \frac{x_0 g(x_0)}{p(x_0)}.$$

Equilibriums $E_0(0, 0)$ and $E_1(k, 0)$ are saddle(See Fig.1).

Fig.1

By local stability analysis of the equilibrium point, we can easily obtain that positive equilibrium $E(x_0, y_0)$ is unstable if $(xg(x)/p(x))'|_{x=x_0} > 0$, it is asymptotically stable if $(xg(x)/p(x))'|_{x=x_0} < 0$.

Theorem 1 Consider system (1) and let $(A_1) - (A_4)$ hold. Further assume $\frac{d}{dx}[xg(x)/p(x)] < 0$ $(0 \leq x \leq k)$ and boundness of every solution of the system (1), then the positive equilibrium point $E(x_0, y_0)$ is globally stable.

Proof. Assuming on the contrary, the positive equilibrium point $E(x_0, y_0)$ is not globally stable. Because of local stability of $E(x_0, y_0)$ under condition $\frac{d}{dx}[xg(x)/p(x)]|_{x=x_0} < 0$ and boundness of every solution of the system(1),then the unstable limit cycle $L(x(t), y(t))$ with period T exists. We obtain

$$
\begin{aligned}
\Delta &= \oint_{0 \leq t \leq T} div(X, Y) dt \\
&= \int_0^T \left\{ \frac{\partial}{\partial x}[x(t)g(x(t)) - y(t)p(x(t))] + \frac{\partial}{\partial y}[y(t)(-q(x(t))) + h(x(t))] \right\} dt \\
&= \int_0^T p(x(t)) \frac{d}{dx} \left\lceil \frac{x(t)g(x(t))}{p(x(t))} \right\rceil dt < 0
\end{aligned}
\tag{19}
$$

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The assertion (19) implies immediately that the limit cycle L is stable from lemma 4. Obviously, such an assertion is absurd from local asymptotically stable $E(x_0, y_0)$. This contradiction leads to the globally stable positive equilibrium point $E(x_0, y_0)$.

Theorem 2 If $(A_1) - (A_4)$ hold, further assume

(A_5) $\frac{d}{dx}(xg(x)/p(x))|_{x=x_0} > 0$ $\qquad\qquad\qquad\qquad$ (20)

(A_6) $\int_{x_0}^{x} xg(x)(-q(x) + h(x))$

$\qquad \exp\left(2\int_{x_0}^{x} f_2(x)dx\right) dx \to +\infty$, as $|x| \to +\infty$ $\qquad$ (21)

(A_7) there exist α, β with $\alpha < x_0 < \beta$ such that $f_1(x) \geq 0$ for $x \notin [\alpha, \beta]$,

$\qquad$ and $\int_{\alpha}^{\beta}\left[-p(x)\left(\frac{xg(x)}{p(x)}\right)' + q(x) - h(x)\right]\exp\left(\int_{x_0}^{x} f_2(x)dx\right) dx > 0 (22)$,

then system (1) has at least one stable limit cycle.

Proof Obviously, under the assumptions of theorem 2, the system (1) may be transformed into Lie'nard (15), i.e

(1) $(x - x_0)\bar{g}(x) = (x - x_0)(-g(x) + h(x))xg(x)\frac{p^2(x_0)}{p^2(x)} > 0$ $\quad (0 < x < k)$

$\qquad G(x) = \int_{x_0}^{x} xg(x)(-q(x) + h(x))\frac{p^2(x_0)}{p^2(x)}dx \to +\infty$ as $|x| \to +\infty (24)$

(2) Since $f(x) = f_1(x)\exp\left(\int_{x_0}^{x} f_2(x)dx\right)$ is continuous and

$\qquad f(x_0) = f_1(x_0)\exp\left(\int_{x_0}^{x_0} f_2(x)dx\right) = f_1(x_0)$

$\qquad\qquad = -p(x_0)\left[\frac{xg(x)}{p(x)}\right]'|_{x=x_0} < 0.$ $\qquad\qquad\qquad$ (25)

then $f(x) < 0$, as $|x - x_0| \ll 1$,

$\qquad$ hence $|x - x_0|F(x) = (x - x_0)\int_{x_0}^{x} f(x)dx < 0$, as $|x - x_0| \ll 1$.

(3) Let $\quad k_1 = F(\beta)$, $\quad k_2 = F(\alpha)$, we get

$\qquad\qquad k_1 - k_2 = \int_{\alpha}^{\beta}\left[-p(x)\left(\frac{xg(x)}{p(x)}\right)' + q(x) - h(x)\right]\frac{p(x_0)}{p(x)}dx > 0.$ $\qquad$ (26)

From (26), it follows immediately that $k_1 > k_2$. Since $f(x) \geq 0$ for $x \notin [\alpha, \beta]$, then $F(x) = \int_{x_0}^{x} f(x)dx$ is monotone nondecreasing for $x \geq \beta$, and $F(x) = \int_{x_0}^{x} f(x)dx$ is monotone nonincreasing for $x \leq \alpha$.

Let $M = \max\{|\alpha|, \beta\}$, then

$$F(x) \geq k_1 \text{ for } x > M,$$
$$F(x) \leq k_2 \text{ for } x < -M.$$

From lemma 2, we can conclude that system(15) has at least one stable limit cycle. So, we have proved the system(1) has at least one stable limit cycle. This completes the proof of theorem 2.

Theorem 3 If $(A_1) - (A_4)$ hold, further assume that

$$(A_5) \ \frac{d}{dx}\left(xg(x)/p(x)\right)|_{x=x_0} > 0 \qquad (27)$$
$$(A_6) \ \int_{x_0}^{x} xg(x)\left(-q(x)+h(x)\right)$$
$$\exp\left(2\int_{x_0}^{x} f_2(x)dx\right)dx \to +\infty \text{ as } |x| \to +\infty \quad (28)$$
$$(A_7) \ H'(x) > 0, \text{ where}$$
$$H(x) = \frac{f(x)}{g(x)} = \frac{p(x)\left(\frac{xg(x)}{p(x)}\right)' - q(x) + h(x)}{xg(x)(q(x)-h(x))} \times \frac{p(x)}{p(x_0)}, (29)$$

Then system (1) has at most one stable limit cycle around the unstable equilibrium point $E(x_0, y_0)$.

Proof. In according to assumptions of theorem 3 and the preceding analysis, the proof completes immediately from an application of lemma 3 .

References

1. G.F.Gause, *The stuggle for existence.* (Baltimore Williams and Wilkins 1934.)
2. H.I.Freedman, *Deterministic mathematical models in population ecology.* (New York: Dekker 1980.)
3. F.Rothe, *J. Reine Angew. Math.* **355**, 129-138(1985).
4. L. S. Chen *et al, Kexue Tongbao.* **29**, 521-523 (1984) (in Chinese).
5. J. P. Chen *et al, Appl. Math. & Mech.* **7**, 77-86 (1986) (English Edition).
6. X. Cao, & L. Chen, *Differ.Equations,* **2**, 415-417(1986).
7. Z. F. Zhang, T. R. Ding et al,*American Mathematical Society, Providence,* RI(Transl.Math.Monogr, Vol.101).
8. N. M. Helmar, *J.Math.Biol.* **28**, 341-354 (1990).

SYMBOLIC CALCULI FOR SEQUENCES

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In 1202, Leonardo of Pisa proposed a mathematical model for the population growth of rabbit pairs. Suppose in time period 1, a pair of young rabbit pair of opposite sex is introduced. Assume that it takes one period of time for the young rabbits to grow up, and then give birth to a pair of young rabbit pairs (of opposite sex). Assume further that rabbits never die. Let f_k count the number of rabbit pairs present in the time period k. Then it is not difficult to see that $f_0 = 0$, $f_1 = 1$, and there is a recurrence relation for f_n, $n \geq 2$, namely,

$$f_{n+2} = f_{n+1} + f_n, n = 0, 1, \cdots \tag{1}$$

By means of this recurrence relation, it is clear that $f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, \cdots$. The numbers $0, 1, 1, 2, 3, 5, \cdots$ are known as the Fibonacci numbers. These numbers have remarkable properties and arise in a great variety of places. What we will be interested in, however, is the analytic form of these numbers.

A well known method for finding the analytic form is the method of generating functions. The essence of this method is as follows. We first multiply both sides of (1) by t^n, and then sum from $n = 0$ to ∞. The resulting equation is

$$\sum_{n=0}^{\infty} f_{n+2}t^n = \sum_{n=0}^{\infty} f_{n+1}t^n + \sum_{n=0}^{\infty} f_n t^n. \tag{2}$$

We will say that the power series $f_0 + f_1 t + f_2 t^2 + \cdots$ is the *generating function* of the sequence $f_0, f_1, f_2, \cdots$ and denotes it by $\Psi(t)$. That is

$$t \sum_{n=0}^{\infty} f_{n+1}t^n = t (f_1 + f_2 t + \cdots) = (f_0 + f_1 t + f_2 t^2 + \cdots) - f_0 = \Psi(t) - f_0,$$

and

$$t^2 \sum_{n=0}^{\infty} f_{n+2}t^n = t^2 (f_2 + f_3 t + \cdots) = \Psi(t) - f_0 - f_1 t,$$

so from (2), we have

$$t^2 \sum_{n=0}^{\infty} f_{n+2}t^n = t^2 \sum_{n=0}^{\infty} f_{n+1}t^n + t^2 \sum_{n=0}^{\infty} f_n t^n,$$

or

$$\Psi(t) - f_0 - f_1 t = t\left(\Psi(t) - f_0\right) + t^2 \Psi(t).$$ (3)

From (3), we may solve for $\Psi(t)$, and obtain

$$\Psi(t) = \frac{-t}{t^2 + t - 1}.$$

Since the coefficients of the power series $\Psi(t)$ are the Fibonacci numbers, there are at least two ways to uncover them. First, recall from Calculus that

$$\Psi(0) = f_0, \quad \Psi'(0) = f_1, \quad \Psi''(0) = 2! f_2, \cdots, \quad \Psi^{(n)}(0) = n! f_n, \cdots,$$

thus

$$f_n = \frac{1}{n!} \Psi^{(n)}(0), \quad n = 0, 1, \cdots$$

In our case, this method seems to be tedious. So we turn to another method. Note that by the method of partial fractions,

$$\frac{-t}{t^2 + t - 1} = -\frac{1}{2}\left\{ \frac{1}{t + \gamma_+} + \frac{1}{t - \gamma_-} \right\},$$

where

$$\gamma_\pm = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}.$$

By expanding the rational functions $1/(t \pm \gamma_\pm)$, we see further that

$$\frac{-t}{t^2 + t - 1} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\gamma_+^n - \gamma_-^n}{\sqrt{5}} t^n.$$

By comparing coefficients of the power series $\Psi(t)$ and $-t/(t^2 + t - 1)$, we finally end up with

$$f_n = (-1)^{n+1} \frac{\gamma_+^n - \gamma_-^n}{\sqrt{5}}, \quad n = 0, 1, \cdots$$ (4)

The above method seems to be right except for the convergence questions of the power series. To adviod such questions, many theories have been introduced. In this note, we will present the synopsis of a theory, the essence of which is known to a number of authors (see for example [1,2]). For a systematic presentation, the reader is referred to one of the Chapters in the book [3] by the author.

Let N be the set of nonnegative integers, and let l^N be the set of complex sequences of the form $f = \{f_0, f_1, f_2, \cdots\}$, where each term f_n is a complex number. For the sake of convenience, the sequence $\{f_0, f_1, \cdots\}$ will be denoted by $\{f_n\}_{n \in N}$ or by $\{f_n\}$. There are a number of special sequences that deserve special notations. First of all, the sequence $\{\alpha, 0, 0, \cdots\}$ will be denoted by $\overline{\alpha}$. The sequence $\{1, \cdots, 1\}$ will be denoted by σ. The sequences $\{1, 0, 0, \cdots\}$, $\{0, 1, 0, \cdots\}$, $\{0, 0, 1, 0, \cdots\}, \cdots$, will be denoted by $\hbar^0, \hbar^1, \hbar^2, \cdots$ respectively. The sequence $\{f_1, f_2, f_3, \cdots\}$ obtained by shifting the sequence $f = \{f_0, f_1, f_2, \cdots\}$ to the left by "one unit" will be denoted by Ef. Thus $E^2 f$, which is defined by $E(Ef)$, will be the sequence $\{f_2, f_3, \cdots\}$. In general, $E^m f = \{f_{n+m}\}_{n \in N}$.

As is customary, the sum of two sequences $f = \{f_n\}$ and $g = \{g_n\}$ is defined to be the sequence $f+g = \{f_n + g_n\}$. We will also define the convolution product of f and g by

$$f * g = \left\{ f_0 g_0, f_1 g_0 + f_0 g_1, \cdots, \sum_{k=0}^{n} f_k g_{n-k}, \cdots \right\}.$$

As examples, we have

$$\sigma * f = \left\{ \sum_{k=0}^{n} f_k \right\} = \{f_0, f_0 + f_1, f_0 + f_1 + f_2, \cdots\},$$

$$\hbar * f = \{0, f_0, f_1, \cdots\},$$

$$\hbar^2 * f = \{0, 0, f_0, f_1, f_2, \},$$

$$\hbar * (Ef) = \{0, f_1, f_2, \cdots\} = f - \overline{f_0}, \tag{5}$$

$$\begin{aligned}
\hbar^2 * (E^2 f) &= \hbar * (\hbar * (E(Ef))) \\
&= \hbar * \left(Ef - \overline{(Ef)_0} \right) \\
&= \hbar * \left(Ef - \overline{f_1} \right) \tag{6} \\
&= \hbar * Ef - \hbar * \overline{f_1} \\
&= f - \overline{f_0} - \hbar * \overline{f_1}.
\end{aligned}$$

It is not difficult to establish that the set l^N endowed with the sum and convolution operations have all the essential properties of the set of integers with the usual addition and multiplication operations. In algebraic terms, l^N is a commutative integral domain. Therefore, by standard procedures, we can

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construct an extension field of *operators* by use of pairs of elements of l^N. Briefly, on the set

$$\{(f,g)\mid f,g\in l^N, g\neq \overline{0}\},$$

we define a relation $\sim$ by

$$(f,g)\sim (p,q)\Leftrightarrow f*q = p*g.$$

Then it is easily verified that $\sim$ is an equivalence relation. Each equivalence class can be represented by an ordered pair (f,g), which is written in the form of a quotient f/g in analogy with the rationals. The set of such quotients is called the field of operators and denoted by l^N/l^N. Hence, two quotients f/g and p/q are equal if, and only if, $fq = pg$. We define product and addition of quotients by

$$\frac{f}{g}\frac{p}{q} = \frac{f*p}{g*q},$$

and

$$\frac{f}{g}+\frac{p}{q} = \frac{f*q+p*g}{g*q}$$

respectively. Under these definitions, l^N/l^N is a field with the unique additive identity $\overline{0}/\overline{1}$ and the unique multiplicative identity $\overline{1}/\overline{1}$. Every $f\in l^N$ can be regarded as the operator $f/\overline{1}$, and in such a case, the corresponding operator is said to be ordinary. Not all operators are ordinary, for example, the operator $\overline{1}/(\sigma-\overline{1})$ cannot be ordinary, for otherwise, there would exist a sequence $f\in l^N$ such that $\sigma*f - f = \overline{1}$. But then $0 = f_0 - f_0 = 1$, which is a contradiction. It may happen that ordinary operators have forms other than the standard $f/\overline{1}$. As an important example, consider the sequence $\{a^n\}_{n\in N}$. Since

$$a^{n+1} - a^n = a^n(a-1), n = 0, 1, \cdots$$

hence

$$\left\{a^{n+1}\right\} - \left\{a^n\right\} = \left\{(a-1)a^n\right\},$$

or,

$$E\left\{a^n\right\} - \left\{a^n\right\} = \left\{(a-1)a^n\right\}.$$

By taking convolution with $\hbar$ on both sides, we see that

$$\hbar * E\left\{a^n\right\} - \hbar * \left\{a^n\right\} = \hbar * \left\{(a-1)a^n\right\},$$

so that by (10),

$$\left\{a^n\right\} - \overline{1} - \hbar * \left\{a^n\right\} = \overline{(a-1)}\hbar * \left\{a^n\right\}.$$

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Solving for $\{a^n\}$, we obtain

$$\frac{\{a^n\}}{\overline{1}} = \frac{\overline{1}}{\overline{1} - \overline{a} * \hbar}. \tag{7}$$

We now change the meaning of the "generating function" of a sequence f in l^N by defining it as

$$\sum_{n=0}^{\infty} \frac{\overline{f_n} * \hbar^n}{\overline{1}}. \tag{8}$$

Since we are now dealing with an infinite sum, we need to define the limits of a sequence $\{g^{[k]}\}$ of sequences in l^N and the limit of a sequence of operators. The obvious one for the former is the pointwise limit, that is,

$$\lim_{k \to \infty} g^{[k]} = g \quad \text{if} \quad \lim_{k \to \infty} g_n^{[k]} = g_n \quad \text{for} \quad n \in N.$$

We can now define the limit of a sequence of operators $\{f^{[k]}/g^{[k]}\}$. This is done by

$$\lim_{k \to \infty} \frac{f^{[k]}}{g^{[k]}} = \frac{f}{g}, \quad \text{if} \quad \lim_{k \to \infty} f^{[k]} = f \quad \text{and} \quad \lim_{k \to \infty} g^{[k]} = g \neq \overline{0}.$$

The limit

$$\sum_{n=0}^{\infty} \overline{f_n} * \hbar^n = \lim_{k \to \infty} \sum_{n=0}^{k} \overline{f_n} * \hbar^n$$

is easily found. Indeed, since $\overline{f_0} * \hbar^0 = \{f_0, 0, \cdots\}, \overline{f_0} * \hbar^0 + \overline{f_1} * \hbar^1 = \{f_0, f_1, 0, \cdots\}$,

$$\overline{f_0} * \hbar^0 + \overline{f_1} * \hbar^1 + \overline{f_2} * \hbar^2 = \{f_0, f_1, f_2, 0, \cdots\},$$

etc., this limit is just f itself. In other words, the infinite series (100) is just the ordinary operator $f/\overline{1}$. In particular, in view of (100),

$$\frac{\overline{1}}{\overline{1} - \overline{a} * \hbar} = \sum_{n=0}^{\infty} \frac{\overline{a^n} * \hbar^n}{\overline{1}}. \tag{9}$$

Once we identify the generating function of a sequence as the ordinary operator $f/\overline{1}$, we need to find a "derivative" that will help us to generate the terms of our sequence (as in the case of $\Psi(t)$ mentioned above). This can be done as follows. For any $f = \{f_n\}_{n \in N}$ in l^N, we define the algebraic derivative

$$Df = \{(n+1)f_{n+1}\}_{n \in N} = \{f_1, 2f_2, 3f_3, \cdots\}.$$

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For an operator f/g, we will define

$$D(f/g) = \frac{g * Df - f * Dg}{g * g}.$$

For examples,

$$D\sigma = \{1, \cdots, 1\},$$

$$D\hbar^n = \overline{n} * \hbar^{n-1}, \quad n = 1, \cdots$$

and

$$D\left(\sum_{n=0}^{\infty} \overline{f_n} * \hbar^n\right) = Df = \{(n+1)f_{n+1}\} = \sum_{n=1}^{\infty} \overline{nf_n} * \hbar^{n-1}.$$

From the last example, we see that

$$\frac{\overline{m!f_m}}{\overline{1}} = \left\lceil D^m \left(\sum_{n=0}^{\infty} \frac{\overline{f_n} * \hbar^n}{\overline{1}}\right)\right\rceil_{\hbar=\overline{0}}, \quad m = 0, 1, 2, ...,$$

"formally" holds, where $D^0 f = f$, $D^{m+1}f = D(D^m f)$ for $m = 1, \cdots$ For instance,

$$\frac{\overline{0!f_0}}{\overline{1}} = \left\lceil D^0 \frac{\overline{1}}{\overline{1} - \overline{a} * \hbar}\right\rceil_{\hbar=\overline{0}} = \left\lceil \frac{\overline{1}}{\overline{1} - \overline{a} * \hbar}\right\rceil_{\hbar=\overline{0}} = \frac{\overline{1}}{\overline{1}},$$

and

$$\frac{\overline{1!f_1}}{\overline{1}} = \left\lceil D\frac{\overline{1}}{\overline{1} - \overline{a} * \hbar}\right\rceil_{\hbar=\overline{0}} = \left\lceil \frac{\overline{a}}{(\overline{1} - \overline{a} * \hbar) * (\overline{1} - \overline{a} * \hbar)}\right\rceil_{\hbar=\overline{0}} = \frac{\overline{a}}{\overline{1}},$$

which show that $f_0 = 1$, and $f_1 = a$. This is expected in view of (100).

We now return to our original recurrence relation for the Fibonacci numbers. Let $f = \{f_0, f_1, f_2, ...\}$. Following the same method of generating functions described before, we treat $f_0 = 0, f_1 = 1$ as $\overline{f_0} = \overline{0}, \overline{f_1} = \overline{1}$ respectively and treat equation (1) as

$$\overline{f_{n+1}} = \overline{f_{n+1}} + \overline{f_n}, \quad n = 0, 1, \cdots \tag{10}$$

Convoluting both sides of (200) with $\hbar^n$, then sum from $n = 0$ to ∞, and then taking the convolution product of the resulting equation with $\hbar^2$, we obtain

$$\hbar^2 * \sum_{n=0}^{\infty} \overline{f_{n+2}} * \hbar^n = \hbar^2 * \sum_{n=0}^{\infty} \overline{f_{n+1}} * \hbar^n + \hbar^2 * \sum_{n=0}^{\infty} \overline{f_n} * \hbar^n,$$

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or

$$\hbar^2 * E^2 f = \hbar^2 * Ef + \hbar^2 * f.$$

In view of (10) and (11), we see further that

$$f - \overline{f}_0 - \overline{f_1} * \hbar = \hbar * (f - \overline{f}_0) + \hbar^2 * f,$$

which shows that

$$\frac{f}{\overline{1}} = \frac{-\hbar}{\hbar * \hbar + \hbar - \overline{1}}.$$

By the method of partial fractions again,

$$\frac{f}{\overline{1}} = \frac{\overline{-1}}{\overline{2}} \left(\frac{\overline{1}}{\hbar + \overline{\gamma_+}} + \frac{\overline{1}}{\hbar - \overline{\gamma_-}} \right).$$

In view of (102), we may "expand" the last term to obtain

$$\frac{f}{\overline{1}} = \sum_{n=0}^{\infty} \frac{\overline{(-1)^{n+1}(\gamma_+^n - \gamma_-^n)/\sqrt{5}} * \hbar^n}{\overline{1}},$$

so that the same formula (4) holds.

Once the method of generating functions is justified for ordinary recurrence equations such as (1), the same ideas can be carried to the set of infinite complex matrices of the form

$$f = \{f_{ij}\}_{i,j=0}^{\infty} = \begin{bmatrix} f_{00} & f_{01} & \cdots \\ f_{10} & f_{11} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

Denoting the set of all such matrices as $l^{N \times N}$, we may also define a summation $+$ and a convolution product $*$ for two of its elements $f = \{f_{ij}\}$ and $g = \{g_{ij}\}$:

$$\{f_{ij}\} + \{g_{ij}\} = \{f_{ij} + g_{ij}\},$$

and

$$\{f_{ij}\} * \{g_{ij}\} = \left(\sum_{u=0}^{i} \sum_{v=0}^{j} f_{uv} g_{i-u} g_{j-v} \right).$$

Again, the triplet $(l^{N \times N}, +, *)$ is a commutative integral domain and an extension field of operators can be defined. A systematic theory [3] can then be developed to yield theoretical justification for heuristic methods of finding explicit solutions to partial difference equations such as the discrete heat equation

$$u_{m,n+1} = au_{m-1,n} + bu_{m,n} + cu_{m+1,n} + du_{m,n-1} + p_{m,n},$$

where $m = 0, \pm 1, \pm 2, \cdots$ and $n = 0, 1, \cdots$

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References

1. I. Niven, *Amer. Math. Monthly*, **76**, 871 (1969).
2. S. Okamoto, *Proc. Japan Acad.*, **54A**, 303 (1978).
3. S. S. Cheng, *Partial Difference Equations*, (Lecture Notes, Tsing Hua University (R.O.C.), 1999).

UNIFORMLY ASYMPTOTIC STABILITY FOR NONAUTONOMOUS DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAY

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Consider the following delay differential equation

$$x'(t) + \lambda x(t) = F(t, x_i)$$

where $\lambda > 0, F : [0, \infty) \times C([-r, 0], R) \to R$ continuous, we obtain sufficient conditions for the uniform stability and uniformly asymptotic stability of the zero solution of the equation.

1 Introduction

Consider the delay differential equation

$$x'(t) + \lambda x(t) = F(t, x_i) \tag{1}$$

Here $\lambda > 0, F : [0, \infty) \times C([-r, 0], R) \to R$ continuous and satisfies the following condition

$$-\sum_{i=0}^{n} \alpha_i \max_{s \in [-r_i, 0]} (\varphi(s)) \leq F(t, \varphi)$$

$$\leq \sum_{i=0}^{n} \alpha_i \max_{s \in [-r_i, 0]} (-\varphi(s)) \tag{2}$$

or

$$-\sum_{i=0}^{n} \alpha_i \max\{0, \max_{s \in [-r_i, 0]} (\varphi(s))\} \leq F(t, \varphi)$$

$$\leq \sum_{i=0}^{n} \alpha_i \ \max\{0, \max_{s \in [-r_i, 0]} (-\varphi(0))\} \tag{3}$$

where

$$\varphi \in C(H) = \{\varphi \in C([-r, 0], R); \| \varphi \| \leq H\}$$

$\alpha_i, \ r_i, \ i = 0, ..., n$ are nonnegative constants and $r = \max_{l \leq i \leq n} r_i$.

When $\lambda = 0$, the stability of zero solution of $Eq(1)$ has been extensively investigated in paper [1-2]. Recently, these results has been developed to the case $\lambda > 0$ by paper [3]. The purpose of this paper is to investigate the stability conditions when F has several delays. When $\lambda = 0$, the stability conditions have been obtained in paper [4-5]. In this paper, we will consider the case $\lambda > 0$.

For the sake of convenience, let

$$\mu_0 = \sum_{i=0}^{n} \alpha_i, \qquad 0 = r_0 < r_1 < r_2 < \cdots < r_n = r,$$

$$\mu_1 = \sum_{i=0}^{n} \alpha_i e^{\lambda r_i}, \qquad g(x) = x + (1-x)\ln(1-x)$$

We now have the following main results:

Theorem 1: Assume (2) holds,

(i) if $\lambda \geq \mu_0$ or

$$\frac{1}{\lambda}(\mu_1 - \mu_0) \leq 1 + \frac{\mu_1}{2\mu_0} \tag{4},$$

then the zero solution of $Eq.(1)$ is uniformly stable.

(ii) if $\lambda > \mu_0$ or

$$\frac{1}{\lambda}(\mu_1 - \mu_0) < 1 + \frac{\mu_1}{2\mu_0} \tag{5}$$

then the zero solution of $Eq.(1)$ is uniformly asymptotic stable.

Theorem 2: Assume (3) holds,

(i) if $\lambda \geq \mu_0$ or

$$\frac{1}{\lambda}(\mu_1 - \mu_0) \leq 1 + \frac{\mu_1}{2\mu_0} - \frac{\mu_0^2}{\lambda^2} g(\frac{\lambda}{\mu_0}) \tag{6}$$

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then the zero solution of $Eq.(1)$ is uniformly stable.

(ii) if $\lambda < \mu_0$ or

$$\frac{1}{\lambda}(\mu_1 - \mu_0) < 1 + \frac{\mu_1}{2\mu_0} - \frac{\mu_0^2}{\lambda^2}g(\frac{\lambda}{\mu_0}) \qquad (7)$$

then the zero solution of $Eq.(1)$ is uniformly asymptotic stable.

Theorem 3: Assume (3) holds and

$$\frac{\lambda}{\mu_0} \leq 1 - e^{-\lambda\eta} \qquad (8)$$

(i) if

$$\frac{1}{\lambda}(\mu_1 - \mu_0) \leq \frac{1}{2} + \frac{\mu_1}{\mu_0} \qquad (9)$$

then the zero solution of $Eq.(1)$ is uniformly stable.

(ii) if

$$\frac{1}{\lambda}(\mu_1 - \mu_0) < \frac{1}{2} + \frac{\mu_1}{\mu_0} \qquad (10)$$

then the zero solution of $Eq.(1)$ is uniformly asymptotic stable.

2 Proofs of the main results

When $\lambda \geq \mu_0$ or $\lambda > \mu_0$, the proofs of the Theorem 1 and Theorem 2 can be completed by using Razumikhin method [6]. For the case $\lambda < \mu_0$, the proofs of Theorem 1 and Theorem 3 are similar to [25] and are a consequence of the following lemma 1-4.

Lemma 1. If (6) (7) holds, then there exists $\gamma^* \in [0,1]$ ($\gamma^* \in [0,1)$) such that $h(\gamma^*) = 0$ and $h(\gamma) \leq 0$ for $\gamma \in [\gamma^*, 1]$, where

$$h(\gamma) = \frac{\gamma}{\lambda}(\mu_1 - \mu_0) - \frac{\mu_1}{2\mu_0}\gamma^2 + \frac{\mu^2}{\lambda^2}g(\frac{\lambda}{\mu_0}\gamma) - \gamma,$$

and let

$$\gamma_0 = \frac{2\mu_0}{\mu_1}(\frac{\mu_1 - \mu_0}{\lambda} - 1),$$

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it is easy to see that under condition $(4) - (7)$, $\gamma_0 \leq 1$.

Lemma 2: Suppose $t_1 > t_0$, $x(t_1) = 0$, $|x(s)| > 0$, $s \in (t_1, t]$, $t \in (t_1, t_1 + r]$. If (2), (4) or (3), (6) hold, then

$$|x(t)| \leq \gamma \max_{s \in [t_1 - 2r, t_1]} |x(s)|$$

where $\gamma = \gamma_0$, if (2), (4) hold; $\gamma = \max\{\gamma_0, \gamma^*\}$, if (3), (6) hold.

Lemma 3. Suppose that (3), (8) and (9) hold, then

$$|x(t)| \leq \gamma \max_{s \in [t_1 - 2r, t_1]} |x(s)|$$

where

$$\gamma = \frac{\mu_0}{\lambda} \cdot \frac{\mu_1 - \mu_0}{\mu_1} - \frac{\mu_0}{2\mu_1}$$

Lemma 4. Let

$$\delta_0 H_0 \leq H, \qquad H_0 = \exp(\lambda r + 2r\mu_0 e^{\lambda r}), \qquad \varepsilon_0 \in (0, \delta_0 H_0),$$

(2), (4) or (3), (6) or (3), (8) and (9) hold, then there exist $T > 0$ for any solutions $x(t)$ of $Eq.(1)$ and for any $t \geq t_0$, from

$$\| x_{t_0} \| \leq \delta_0, \qquad |x(t+T)| \geq \varepsilon_0,$$

it follows that there exist $\tau \in [t, t+T]$ such that

$$x(\tau) = 0 \quad and \quad |x(s)| > 0, \quad s \in [\tau, t+T].$$

Finally, by the Lemma 2,3 and 4, the proofs of uniform stable and uniformly asymptotic stability can be completed by similar method that in paper [5]. However, due to the length limitation, we only show the lemmas in above which are key for proving Theorem 1 to Theorem 3.

1. J. A. York, *Asymptotic Stability for One Dimensional Differential-delay Equations*, J.Diff.Eqs **7**, 89-202 (1970)

2. T. Yoneyama, *On the 3/2 Stability Theorem for One- dimensional Delay-differential- Equations*, J.Math.Anal.Appl. **125**, 161-173 (1987)

3. J.S.Yu, *Asymptotic Stability for Nonautonomous Delay Differential Equations*, Kexue Tongbao **42**, 1248-1252 (1997)

4. T.Yoneyama, J. Sugie, *On the Stability Region of differential Equations with Two Delays*, Funk Ekvac. **31**, 233-240 (1988)

5. T. Krisztin, *On Stability Properties for One Dimension Functional Differential Equations*, Funk Ekvac. **34**, 241-256 (1991)

6. J.K.Hale, *Theory of Functional Differential Equation*, Springer-Verlag, New York(1977)

LIMITING PROFILES OF PERIODIC SOLUTIONS OF NEURAL NETWORKS WITH SYNAPTIC DELAYS

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Let $f(\cdot, \lambda) : R \to R$ be given so that $f(0, \lambda) = 0$ and $f(x, \lambda) = (1 + \lambda)x + ax^2 + bx^3 + o(x^3)$ as $x \to 0$. We characterize those small values of $\varepsilon > 0$ and $\lambda \in R$ for which there are periodic solutions of periods approximately $\frac{2}{k}$ with $k \in N$ of the following system arising from a network of neurons

$$\begin{cases} \varepsilon \dot{x}(t) = -x(t) + f(y(t-1), \lambda), \\ \varepsilon \dot{y}(t) = -y(t) + f(x(t-1), \lambda). \end{cases}$$

The periodic solutions are synchronized if k is even and phase-locked if k is odd. As $\varepsilon \to 0$, these periodic solutions approach square waves if $a = 0$ and $b < 0$, and pulses if $a = 0$ and $b > 0$ or if $a \neq 0$. Moreover, same results for the scalar case (a single neuron)

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t-1), \lambda)$$

can be deduced from the fact of synchronization.

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1 Introduction

For $\varepsilon > 0$ and $f \in C^m(R \times R)$, $m \geq 3$, consider the following system of delay differential equations

$$\begin{cases} \varepsilon \dot{x}(t) = -x(t) + f(y(t-1), \lambda), \\ \varepsilon \dot{y}(t) = -y(t) + f(x(t-1), \lambda), \end{cases} \qquad (1)$$

which describes the dynamics of a network of two identical amplifiers (or neurons) with delayed outputs. See, for example, Hopfield[8], Marcus and Westervelt[9] and Wu[2].

We have recently obtained some results about the global dynamics of system (1) under some minor technical hypotheses [1,2,4]. It is shown that system (1) has at least two periodic orbits when ε is less than a certain value, one is synchronized and has the minimal period between 1 and 2 and the other one is phase-locked and has the minimal period larger than 2. Here a solution (x, y) of (1) is *synchronized* if $x \equiv y$ in their domains of definition, and a *phase-locked* T-periodic solution of (1) is one satisfying $x(t) = y\left(t - \frac{T}{2}\right)$ for all $t \in R$. The purpose of this paper is to study the limiting profiles of these periodic solutions of (1) as $\varepsilon \to 0$.

More specifically, we assume that

$$f(x, \lambda) = (1 + \lambda)x + ax^2 + bx^3 + o(x^3) \qquad \text{as } x \to 0. \qquad (2)$$

Our work is inspired by that of Chow, Hale and Huang[5] and Hale and Huang[7], where they studied

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t-1), \lambda) \qquad (3)$$

with $f \in C^m(R \times R)$, $m \geq 3$, satisfying

$$f(x, \lambda) = -(1 + \lambda)x + ax^2 + bx^3 + o(x^3) \qquad \text{as } x \to 0. \qquad (4)$$

2 Main Results

It follows from (2) that, when $a \neq 0$, $f(\cdot, \lambda)$ has only one nontrivial fixed point $c_{0\lambda}$ in a small neighborhood of 0; when $a = 0$ and $\lambda b < 0$, $f(\cdot, \lambda)$ has two distinct nonzero fixed points $c_{1\lambda}$ and $c_{2\lambda}$ in a small neighborhood of 0; when $a = 0$ and $\lambda b > 0$, 0 is the only fixed point of f in a small neighborhood of 0. Furthermore, $c_{0\lambda}$, $c_{1\lambda}$, $c_{2\lambda} \to 0$ as $\lambda \to 0$. One of our objectives here is to understand how these fixed points of the map $f(\cdot, \lambda)$ is reflexed into the bifurcation from the origin of periodic solutions whose periods are approximately $\frac{2}{k}$ with $k \in N$.

Our main results are summarized in the following theorem.

Theorem 1. *Suppose that $f(x, \lambda)$ satisfies (2) with $a^2 - b \neq 0$. Then, for any $k \in \mathbb{N}$, there is a neighborhood U_k of $(0,0)$ in the (λ, ε) plane and a sectorial region S_k in U_k such that, if $(\lambda, \varepsilon) \in U_k$, then there is a periodic solution $(\tilde{x}^{(k)}_{\lambda, \varepsilon}, \tilde{y}^{(k)}_{\lambda, \varepsilon})$ of (1) with period $\frac{2}{k}(1+\varepsilon) + O(|\varepsilon|(|\lambda| + |\varepsilon|))$ as $(\lambda, \varepsilon) \to (0,0)$ if and only if $(\lambda, \varepsilon) \in S_k$. Furthermore, this orbit is unique and the solution is synchronized if k is even and phase-locked if k is odd.*

When $a = 0$ and $b < 0$, for small and fixed $\lambda = \lambda_0 > 0$, the set $\{\varepsilon; (\lambda_0, \varepsilon) \in S_k\}$ is an interval $(0, \varepsilon_k(\lambda_0))$. At the point $(\lambda_0, \varepsilon_k(\lambda_0))$, there is a Hopf bifurcation and the periodic solution approaches a square wave as $\varepsilon \to 0$; that is, the periodic solution $(\tilde{x}^{(k)}_{\lambda_0, \varepsilon}, \tilde{y}^{(k)}_{\lambda_0, \varepsilon})$ has the property that $\tilde{x}^{(k)}_{\lambda_0, \varepsilon}(t) \to c_{1\lambda_0}$ (respectively, $c_{2\lambda_0}$) as $\varepsilon \to 0$ uniformly on compact subsets of $\left(0, \frac{1}{k}\right)$ (respectively, $\left(\frac{1}{k}, \frac{2}{k}\right)$) (possibly after a translation, same for the other cases). When $a = 0$ and $b > 0$, for small and fixed $\lambda = \lambda_0 > 0$, the set $\{\varepsilon; (\lambda_0, \varepsilon) \in S_k\}$ is an interval $(\varepsilon_k(\lambda_0), \beta_k(\lambda_0))$. At the point $(\lambda_0, \varepsilon_k(\lambda_0))$, there is a Hopf bifurcation. For small and fixed $\lambda = \lambda_0 < 0$, the set $\{\varepsilon; (\lambda_0, \varepsilon) \in S_k\}$ is an interval $(0, \alpha_k(\lambda_0))$. As $\varepsilon \to 0$, the unique periodic solution becomes pulse-like in the following sense: the periodic solution $(\tilde{x}^{(k)}_{\lambda_0, \varepsilon}, \tilde{y}^{(k)}_{\lambda_0, \varepsilon})$ has the property that $\tilde{x}^{(k)}_{\lambda_0, \varepsilon}(t) \to 0$ as $\varepsilon \to 0$ uniformly on compact subsets of $\left(0, \frac{1}{k}\right) \bigcup \left(\frac{1}{k}, \frac{2}{k}\right)$. The magnitude of the pulse exceeds $\max\{|c_{1\lambda_0}|, |c_{2\lambda_0}|\}$. When $a \neq 0$, for small and fixed $\lambda = \lambda_0 > 0$, the set $\{\varepsilon; (\lambda_0, \varepsilon) \in S_k\}$ is an interval $(\varepsilon_k(\lambda_0), \beta_k(\lambda_0))$. At the point $(\lambda_0, \varepsilon_k(\lambda_0))$, there is a Hopf bifurcation. For small and fixed $\lambda = \lambda_0 < 0$, the set $\{\varepsilon; (\lambda_0, \varepsilon) \in S_k\}$ is an interval $(0, \alpha_k(\lambda_0))$. As $\varepsilon \to 0$, the unique periodic solution becomes pulse-like with the magnitude of the pulse exceeds $|c_{0\lambda_0}|$.

Remark 2. It follows from the fact of synchronization that same results hold for equation (3) with f satisfying (2). A comparison of our results here with those of Chow, Hale and Huang [5] and Hale and Huang [7] indicates a difference between excitatory and inhibitory networks of neurons.

In the following, we only outline the proof of Theorem 1. For the details, we refer to Chen and Wu [3].

The key to prove Theorem 1 is to relate the existence of periodic solutions to that of a perturbed Hamiltonian system. The main steps are described as follows.

Step 1. *Local analysis.* Study the characteristic equation to obtain Hopf bifurcation. At $\varepsilon_k = \frac{\sqrt{\lambda^2 + 2\lambda}}{k\pi - \arccos \frac{1}{1+\lambda}}$ for $k \in \mathbb{N}$, periodic solutions with periods around $\frac{2\pi}{k\pi - \arccos \frac{1}{1+\lambda}} \in \left(\frac{2}{k}, \frac{4}{2k-1}\right)$ branched out.

Step 2. *Rescaling.* The purpose is to transform the problem to a perturbation problem with two parameters described by a system of delay differential equations.

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Step 3. *Analysis of the perturbation of a linear equation.* The objective here is to decompose the phase space into the direct sum of two invariant subspaces. In this step, we can show that the periodic solutions are synchronized when k is even and phase-locked when k is odd.

Step 4. *Computing the normal form equation (a perturbed Hamiltonian system) on the center manifold.* Using the normal form theory developed by Faria and Magalhães [3], we obtain the normal form on the center manifold, which is given by

$$\begin{cases} \dot{x}_1 = \left(\frac{4}{3}\lambda + 2h\right) x_1 + 2\lambda x_2 - \frac{8}{3}ax_1x_2 - 2ax_2^2 \\ \qquad + \left(2b + \frac{46}{9}a^2\right)x_2^3 - \frac{5}{3}a\lambda x_2^2 - \frac{4}{3}ahx_2^2 \\ \qquad + \left(4b + \frac{428}{45}a^2\right)x_1x_2^2 - \frac{28}{27}a\lambda x_1x_2 - \frac{28}{9}ahx_1x_2 \\ \dot{x}_2 = -x_1 \end{cases} \tag{5}$$

up to terms of $O((\lambda + h)^2|x| + (\lambda + h)|x|^3 + |x|^4)$.

Step 5. *Using results about the phase-portraits of perturbed Hamiltonian systems to get the main results.*

References

1. Y. Chen and J. Wu, Existence and attraction of a phase-locked oscillation in a delayed network of two neurons, to appear in *Advances in Differential Equations*.
2. Y. Chen and J. Wu, Minimal instability and unstable set of a phase-locked periodic orbit in a delayed neural network, to appear in *Physica D*.
3. Y. Chen and J. Wu, The asymptotic shapes of periodic solutions to a singular delay differential system, submitted to *Journal of Differential Equations*.
4. Y. Chen, J. Wu and T. Krisztin, Connecting orbits from synchronous periodic solutions to phase-locked periodic solutions in a delay differential system, to appear in *Journal of Differential Equations*.
5. S.-N. Chow, J. K. Hale and W. Huang, *Proc. Royal Soc. Edinburgh* **120A**, 223 (1992).
6. T. Faria and L. T. Magalhães, *J. Diff. Equ.* **122**, 181 (1995).
7. J. K. Hale and W. Huang, *J. Diff. Equ.* **114**, 1 (1994).
8. J. J. Hopfield, *Proc. Natl. Acad. Sci.* **81**, 3088 (1984).
9. C. M. Marcus and R. M. Westervelt, *Phys. Rev. A* **39**, 347 (1989).
10. J. Wu, *Trans. Amer. Math. Soc.* **350**, 4799 (1998).

GLOBAL UNIFORM ASYMPTOTIC STABILITY OF NEUTRAL VOLTERRA EQUATIONS

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We study a class of neutral Volterra equations and give a criterion for global asymptotic stability and global uniform asymptotic stability.

By using the technique in [1], we reduce the neutral Volterra equations to delay integral inequalities by the variation of parameter formula, further we get a criterion for stability by the asymptotic properties of solutions of delay integral inequalities.

Let $C = C([\alpha, 0], R^n)$, in which $\alpha \leq 0$ could be $-\infty$. For $\varphi \in C$ we define $\| \varphi \|_\alpha = \sup_{\alpha \leq u \leq t} |\varphi(u)|$. $f(t, s) \in UC_t$ means that $f \in C[R^+ \times R, R^+]$ and that for any given α and $\epsilon > 0$, there exist positive numbers B, T, and A satisfying $\int_\alpha^t f(t, s)ds \leq B$, $\int_\alpha^{t-T} f(t, s)ds < \epsilon, \forall t \geq A$.

Consider the neutral Volterra equation

$$\frac{d}{dt}[x_i(t) - \int_\alpha^t G_i^{(1)}(t, s, x(r_1(s)))ds] = A_i(t)x_i(t) + \int_\alpha^t G_i^{(2)}(t, s, x(r_2(s)))ds,$$
$$i = 1, \cdots, m, \qquad (1)$$

where $x_i(t) \in R^i$, $\sum_{i=1}^m n_i = n$, $A_i(t) \in C[R^+, R^{n_i \times n_i}]$, $G_i^{(k)}(t, s, 0) \equiv 0$, $G_i^{(k)} \in C[R^+ \times R \times C, R^i]$, $(k = 1, 2)$, $r(t) \leq r_1(t), r_2(t) \leq t$ and $r(t) \to \infty$ as $t \to \infty$.

Let $D_i(t, x_t) = x_i(t) - \int_\alpha^t G_i^{(1)}(t, s, x(r_1(s)))ds$, we get

$$\frac{d}{dt}D_i(t, x_t) = A_i(t)D_i(t, x_t) + \int_\alpha^t A_i(t)G_i^{(1)}(t, s, x(r_1(s)))ds$$

$$+ \int_\alpha^t G_i^{(2)}(t, s, x(r_2(s)))ds \qquad (2)$$

Using the variation of parameter formula, we get

$$D_i(t, x_t) = \Phi_i(t, t_0)D(t_0, \varphi_i) + \int_{t_0}^t \Phi_i(t, u) \int_\alpha^u [A_i(u)G_i^{(1)}(u, v, x(r_1(v)))$$

$$+ G_i^{(2)}(u, v, x(r_2(v)))]dvdu$$

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Then

$$|x_i(t)| \le |\Phi_i(t,t_0)||D_i(t_0,\varphi_i)| + \int_\alpha^t |G_i^{(1)}(t,u,x(r_1(u)))|du + \int_{t_0}^t |\Phi_i(t,u)| \cdot$$

$$\cdot \int_\alpha^u |A_i(u)G_i^{(1)}(u,v,r_1(v)) + G_i^{(2)}(u,v,x(r_2(v)))|dvdu,$$

where $\Phi_i(t,t_0)$ be a fundamental matrix of $\dot{z}_i(t) = A_i(t)z_i(t)$. We suppose that

(A) $\begin{cases} |G_i^{(1)}| \le \sum_{j=1}^m b_{ij}^{(1)}(t)\xi_{ij}^{(1)}(t,u) \parallel x_{ju} \parallel_s \\ |A_iG_i^{(1)} + G_i^{(2)}| \le \sum_{j=1}^m b_{ij}^{(2)}(t)\xi_{ij}^{(2)}(t,u) \parallel x_{ju} \parallel_s, \end{cases}$

where $b_{ij}^{(k)} \in C[R^+, R^+]$, $\xi_{ij}^{(1)}, \xi_{ij}^{(2)} \in C[R^+ \times R, R^+] \cap UC_t, (k = 1,2)$, and $s = r(t)$.

(B) $\begin{cases} |\varphi_i(t,t_0)| \le M_i \ \exp\{\int_u^t \alpha_i(v)dv\}, \\ b_{ij}^{(k)}(t) \le \bar{b}_{ij}^{(k)}\alpha_i(t) \end{cases}$

where $\alpha_i(u) \ge 0$ satisfies

$$\int_{t-T}^t \alpha_i(u)du \to \infty \text{ as } T \to \infty \text{ uniformly in } t \ge \tau.$$

By means of the delay integral inequality in Xu[1], we can get the following result.

Theorem 1. If (A),(B) holds and suppose there are nonnegative numbers π_{ij} such that for any $t \ge \tau \in R^+$

$$\int_\alpha^t b_{ij}^{(1)}(t)\xi_{ij}^{(1)}(t,u)du + \int_{t_0}^t |\Phi_i(t,u)| \int_\alpha^u b_{ij}^{(2)}(u)\xi_{ij}^{(2)}(u,v)dvdu \le \pi_{ij}$$

If the spectral radius $\rho(\Pi)$ of the matrix $\Pi = (\pi_{ij})$ is less than one, then the following hold

(i) The zero solution is globally asymptotic stability.

(ii) The zero solution is globally uniform asymptotic stability if $|\Phi_i(t,t_0)| \to 0$ as $t \to \infty$ uniformly in $t_0 \ge \tau \in R^+$, and there are $r > 0, b > 0$ such that $t - r(t) \ge r$ for any $t \ge \tau$ and $|\Phi_i(t,u)| \le b$ for $\tau \le u \le t < \infty$.

References

1. Xu Dao Yi, Integro-Differential Equations and Delay Integral Inequalities, Tohoku Math. J. **44**, 365-378 (1992)

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BIFURCATION AND CHAOS OF THE DYNAMICAL MODELS OF MANTLE CONVECTION IN AN IMPOSED VERTICAL MAGNETIC FIELD

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In this paper, a fifth-order set of ODEs is derived using Galerkin truncation methods from the PDEs of model about the mantle convection in an imposed vertical magnetic field, then a third-order set of ODEs, which provides an accurate description of the model in some certain parameter regimes, is obtained while the convection motion is small amplitude. Finally, the phenomena such as Hopf bifurcation and chaos in the third-order model are discussed using theoretical methods and numerical computations.

1 Introduction

The mantle convection theory is a basis of the research on plate tectonic and tectonic dynamics of the earth, according to which the driving forces of plate motion should be promoted and controlled by the motion of the mantle mass. It provides the plate with not only the forces but the mass so that some mineral deposits could be formed in the earth's surface. So it is very important to investigate the principles of the earth mantle for understanding the whole earth including the construction, the motion of the interior and the surface of the earth, the change of the natural environment, the exploration of natural and energy resources, the prevention of natural calamities [3,4].

The PDEs of the model which describe the behavior of mantle convection are very complicated, containing many terms of nonlinearity, large scales of time and space. All these make it more difficult to study the behavior of mantle convection. In this paper, lower order sets of ODEs can be derived from a kind of PDEs of the model describing mantle convection in an imposed vertical magnetic field. Under the circumstances are the phenomena such as Hopf bifurcation and chaos in the lower order ODEs discussed using theoretical and numerical methods. It is of necessity for us to further explore the principles of mantle convention.

2 The PDEs and their reduction

It is known that the mantle mass is a special fluid, which is the melting mass with high temperature, high pressure, multi-composition and existence in the form of liquid, vapour and solid. It's constitutive relations are so complicated that the considered PDEs can only be studied under some simplified conditions. For examples, some assumptions below are made to get a kind of simplified PDEs when the action of vertical magnetic field is considered.

Suppose that the model is two dimensional convection and the convection motion is confined between two plates, which are keeping at a distance of h. The lower is near to the core part of the earth, which belongs to higher temperature boundary, and the upper is near to the shell part of that, which belongs to lower temperature boundary. The mantle convection can be formed because of the inhomogeneous density and unstable weight caused by the difference of temperature. Further supposing that the Boussiesq approximation is satisfied for the fluid. The boundary conditions are chosen in order that the eigenfunctions of linearized problem are harmonic. The heigh scale in the y-direction is h, and the width scale in x-direction is L. Under the consideration above are the governing non-dimensional PDEs of the mantle convection in an imposed vertical magnetic field obtained [4].

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial y}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial \psi}{\partial x}$$
$$= P_r\left[\left(\frac{\partial^4 \psi}{\partial x^4} + \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right) - \frac{\partial \theta}{\partial x}\right] + f_1(E), \tag{1}$$

$$\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) + R\frac{\partial \psi}{\partial x}, \tag{2}$$

$$\frac{\partial E}{\partial x} - \frac{\partial \psi}{\partial y}\frac{\partial E}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial E}{\partial y} = \beta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)E + f_2(E). \tag{3}$$

Here $f_1(E) = -P_r Q\beta(\frac{\partial \nabla^2 E}{\partial y} + \frac{\partial E}{\partial x}\frac{\partial \nabla^2 E}{\partial y} - \frac{\partial E}{\partial y}\frac{\partial \nabla^2 E}{\partial x})$, $f_2(E) = \frac{\partial \Psi}{\partial y}$. E is the magnetic flux function. θ is non-dimensional temperature. β is the ratio of the magnetic diffusivity(e) to the heat diffusivity($\overline{k}$). Q is the Chandrasekhar number, which is a parameter related with magnetic strength, permeability in fluid and the density of fluid etc. $R = \frac{\rho \alpha g(T_1 - T_2)h^3}{\overline{k}\mu}$ is the Rayleigh number, which describes the ratio of the strength of buoancy forces to damping forces, and $P_r = \frac{\mu}{k\rho}$ is the Prandtl number, which describes the ratio of the strength of momentum diffusion to heat diffusion. It is of great significance to study the dynamical behavior when R and P_r are larger, as both R and P_r are usually large enough in most cases while the mantle convection is considered [4].

Here are the boundary conditions. Stress-free boundary means $\psi = 0$; and $\psi = 0, E = 0, \frac{\partial E}{\partial y} = 0$ when $y = 0, h$. No heat and magnetic across the sidewalls mean $\frac{\partial \theta}{\partial x} = 0, E = 0, \frac{\partial E}{\partial x} = 0$.

Galerkin trnucation methods are used to reduce the PDEs above, and some double Fourier sums are used here.

$$\psi = C_1 A(t) sin\frac{\pi x}{L} sin\pi y,$$
$$\theta = C_2 B(t) sin2\pi y + C_3 C(t) cos\frac{\pi x}{L} sin\pi y, \qquad (4)$$
$$E = C_4 D(t) sin\frac{\pi x}{L} cos\pi y + C_5 E(t) sin\frac{2\pi x}{L},$$

where t is a scaled time. All constants above can be chosen for the final equations being simple[4], and the reduced model is

$$\dot{A} = -P_r A - P_r C + P_r \beta q D + P_r \beta q \gamma DE,$$
$$\dot{B} = -\beta_2 AC - \overline{\omega} B,$$
$$\dot{C} = AB - C + rA, \qquad (5)$$
$$\dot{D} = -\beta_3 AE - \beta D + A,$$
$$\dot{E} = \overline{\omega} AD - \beta\beta_1 E,$$

in which there are many non-linearities from the original PDEs.

Consider the eigenvalue equation of (5) and let $r_c := -\frac{(\beta+P_r)}{P_r(1-\beta)}$, $q_c = -\frac{\beta(1+P_r)}{P_r(1-\beta)}$, a critical point C can be obtained. If $\beta < 1$, oscillatory convection can exist. There is the codimension two bifurcation near point C, whose eigenvalues are $0, 0, -\overline{\omega}, -\beta\beta_1, -(1 + \beta + P_r)$. It is the total number of eigenvalues with zero real part that determine the final dimension of the ODEs. When small parameter $\overline{\omega} \to 0$ is introduced, namely tall thin rolls existing in the convection, the dynamical system (5) can be reduced to a centre manifold and discussed according to the centre manifold reduction theory.

Consider the situations of solution near the point $r = r_c(1 + \mu)$, $q = q_c(1 + \nu)$, where $\mu \ll 1$ and $\nu \ll 1$, and the following

$$A = \overline{\omega} m, B = \overline{\omega}^2 n, C = \overline{\omega} p, D = \overline{\omega} q, E = \overline{\omega}^2 s, t^* = \overline{\omega} t \qquad (6)$$

are introduced. (6) is substituted into (5), and at the same time, some of (5) are substituted back into themselves several times. Finally, a third-order set of ODEs, a convenient form with two parameters λ and k, can be derived.

$$\dot{x} = y, \dot{y} = ky - \lambda x - xz, \dot{z} = -z + xz. \qquad (7)$$

3 Discussion of the third-order dynamical system

The equations of (7) are symmetric under the substitution $(x, y, z) \to (-x, -y, z)$, which give the invariance of the PDEs under inversion of the sense

of the flow. The divergence of (7) is $k - 1$ when $k < 1$, which manifests that the system is dissipative. The volume element of it's phase space contracts in forms of exponential and the dissipative structure expresses the whole stability. If there is unstability in part, chaos may be produced. That is to say, the dissipative structure is necessary for the existence of an attractor.

It is easy to obtain the solution to three equilibria, which are $(0, 0, 0)$ and $(\pm\sqrt{-\lambda}, 0, -\lambda)$. For the first equilibrium, it is easy to prove that there is subcritical Hopf bifurcation when $k = 0(\lambda \geq 0)$, corresponding to heat transfer without any flow; for the other equilibria, it is easy to prove that there is supercritical Hopf bifurcation when $\lambda = 0.5k(1 - k)$ and $k \leq 0$, and a pitchfork bifurcation when $\lambda = 0$. The system belongs to three-dimensional saddle-focus, and there would be homoclinic trajectory if the parameter λ goes on changing. Further a Poincare back map might can be constructed while some conditions are added according to Shilnikov methods. The map has the behaviour of Smale horseshoes transfer, which manifests there would be chaos in the sense of Smale horseshoes[6].

All these theoretical results such as periodic solution, stability and chaos have been given in references [1,4,5,6]. For the model, some numerical computations have been done using the Runge-Kutta methods to make program[2]. Here have some numerical results been done, consistent with theoretical analysis, seeing figure 1,2,3,4.

Fig.1 Phase portrait of (7)
for $(\lambda, k) = (5.0, 0.0)$

Fig.2 Phase portrait of (7)
for $(\lambda, k) = (-0.375, -0.50)$

Fig.3 Phase portrait of (7)
for $(\lambda, k) = (-4.775, -0.5)$

Fig.4 Chaotic trajectory of (7)
for $(\lambda, k) = (-12.8, -2.5)$

4 Conclusion

There are many non-linear characteristics in mantle convection according to the discussion and numerical results. If some related parameters are made to be translated into the original PDEs, the behavior of mantle convection can be observed through changing some geological parameters. When some parameters are taken as certain values, the phenomena such as upsurge of flow abruptly can exist, namely Hopf bifurcation or chaos. All these might explain how some minerals are formed. It is important to provide it with theoretical and numerical methods to study on the principles of mantle convection,although the considered model is very simple, only an approximation for the complicated mantle convection.

Acknowledgments

References

1. J.,Guckenheimer and P.J.,Homes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York,1983.
2. M.Kubicek and M.Marek,Computational Methods in Bifurcation Theory and Dissipative Structures, Spinger-Verlag, New York,1983.
3. The NNSFC,The report on the development strategy of mechanics, Mechanics and Practice, 20(1998).
4. Jun Dong, Studies on bifurcation and chaos of the dynamical models of mantle convection in an imposed vertical magnetic field and atmosphere on the earth, doctorial dissertation of Kunming University of Science and Technology, 1999.3.
5. In-Ding Hsu and Nicholas D.Kazarinoff,Existence and stability of periodic of a third-order nonlinear autonomous system simulating immune response in animals,Proceeding of the Royal Society of Edinburgh,77A,163-175, 1977.
6. Zengrong Liu,Perturbation Criteria for Chaos, Shanghai scientific and technological eduction publishing house,Shanghai,P.R.China,1994.

PERTURBATIONS FROM AN ELLIPTIC HAMILTONIAN OF DEGREE FOUR

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The paper deals with a complete study of the number of zeros of Abelian integrals, related to perturbations of the Hamilonian vector fields with an elliptic Hamiltonian of degree four: $\dot{x} = y$, $\dot{y} = P(x) + \delta Q(x)y$, where P and Q are polynomials of degree respectively 3 and 2, and δ small. We prove that if the unperturbed Hamiltonian vector field has a saddle loop, a cuspidal loop or a global center, then for the perturbed system the corresponding maximum number of zeros of Abelian integral is respectively 2, 4 and 4. In the last case, the perturbed system may have a quadruple limit cycle, as was conjectured in [11].

1 Introduction

It is well known that for polynomial planar vector fields there is the famous 16th problem of Hilbert asking for an upper bound on the number of limit cycles depending on the degree of the vector field. It is even not known whether a finite upper bound exists. Also for the limited class of (generalized) Liénard equations $\dot{x} = y$, $\dot{y} = P(x) + Q(x)y$, with P and Q polynomial, Hilbert's 16th problem is still unsolved. These Liénard equations can be met in many constructions and applications. They are e.g. unavoidable in the study of local bifurcations by means of rescaling techniques. We say to have a Liénard equations of type (m,n) if $\deg P = m$ and $\deg Q = n$. A complete study has been made for the cases $m + n \leq 4$, except for $(m,n) = (1,3)$, we refer to [8], [3], [4] and [14]. In all these cases it has been proved that there is at most one limit cycle and for type (1,3) the same has been conjectured (see [14]). For $m + n \geq 5$ no general results have been obtained, except for local ones, near non-degenerate singularities. We refer to [3] for a recent account of the known results.

In case $(m,n) = (3,2)$ the maximum number of such local limit cycles is

two. This local analysis is for sure a starting point in a global approach but it can clearly not be expected that the local results will trivially extend, even not in case there is globally only one singularity. For this we refer to [5] where strong numerical evidence has been given for the existence of systems with four limit cycles; we also refer to [11] where the occurrence of a quadruple limit cycle, together with a full unfolding, has been conjectured.

In this paper we consider Liénard equation of type (3,2) that are small perturbations of Hamiltonian vector fields with an elliptic Hamiltonian of degree four. The Hamiltonians are given by the function $H(x, y) = \frac{y^2}{2} + \Phi(x)$, where $\Phi(x) = \frac{a}{4}x^4 + \frac{b}{3}x^3 + \frac{c}{2}x^2, a \neq 0$. The perturbations are given by adding $\delta y(\alpha + \beta x + x^2)\frac{\partial}{\partial y}$ for small $\delta > 0$. It is well known that a first step in studying the limit cycles consists in calculating the zeros of the Abelian integrals, or more precisely the elliptic integrals, obtained by integrating the related 1-form $y(\alpha + \beta x + x^2)dx$ over the compact level curves of the Hamiltonian H.

The study of the zeros of Abelian integrals obtained by integrating polynomial 1-form over level curves of polynomial Hamiltonian is called the weak 16th problem of Hilbert or the Arnold-Hilbert problem. With our study we aim at proving a complete and sharp estimate in the cases under study.

2 Main Results

In this section, we state our main result and give an outline of the proof.

As mentioned above we consider the Hamiltonian function $H(x, y) = \frac{y^2}{2} + \Phi(x)$, where $\Phi(x) = \frac{a}{4}x^4 + \frac{b}{3}x^3 + \frac{c}{2}x^2, a \neq 0$. After linear rescaling, we may change $\Phi(x)$ to one-parameter function $\Phi_\lambda^-(x)$ or $\Phi_\lambda^+(x)$, depending on $a < 0$ or $a > 0$ respectively, where $\Phi_\lambda^-(x) = -\frac{1}{4}x^4 + \frac{1-\lambda}{3}x^3 + \frac{\lambda}{2}x^2, \lambda \in [1, +\infty)$, and $\Phi_\lambda^+(x) = \frac{1}{4}x^4 - \frac{2\lambda}{3}x^3 + \frac{1}{2}x^2, \lambda \in [0, \frac{3}{2\sqrt{2}}]$. The perturbation system is given by

$$\dot{x} = y, \qquad \dot{y} = -\Phi'(x) + \delta(\alpha + \beta x + x^2)y, \qquad (1)$$

and the related elliptic integral is given by

$$I(h) = \int_{\Gamma_h} (\alpha + \beta x + x^2)ydx = \alpha I_0(h) + \beta I_1(x) + I_2(x), \qquad (2)$$

where $\Gamma_h : \{(x, y)|H(x, y) = h, h_0 < h < h_1\}$ compact.

The unperturbed system (1) with $\delta = 0$ has 5 topologically different phase portraits:

(1) with a Two Saddle Cycle, if $\Phi(x) = \Phi_\lambda^-(x)$, and $\lambda = 1$;

(2) with a Saddle Loop, if $\Phi(x) = \Phi_\lambda^-(x)$, and $\lambda > 1$;

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(3) with a Global Center, if $\Phi(x) = \Phi_\lambda^+(x)$, and $0 < \lambda < 1$; .

(4) with a Cuspidal Loop, if $\Phi(x) = \Phi_\lambda^+(x)$, and $\lambda = 1$;

(5) with "Figure 8" Saddle Loops, if $\Phi(x) = \Phi_\lambda^+(x)$, and $1 < \lambda < \frac{3}{2\sqrt{2}}$.

The problem is to find the least (sharp) upper bound of the number of zeros of the elliptic integral (2). For case (1) it was solved by Horozov in 1979 $(^2)$ and the answer is one. For other cases there were only some partial results in ^{9} and ^{17}. Let us state our main result, which gives an answer to cases (2)-(4) $(^5, ^6$ and $^7)$. The answer to case (5) will appear in a forthcoming paper.

Theorem 3 *For all constants α and β the least upper bound of the number of zeros of the Abelian integral (2) is two if system (1) with $\delta = 0$ has a saddle loop; is four if system (1) with $\delta = 0$ has a cuspidal loop or a global center. In any case, the multiplicity of the zeros is included.*

Remark 1. For the perturbation from cuspidal loop case the sharp upper bound of the number is four. If restricting to "inside" or "outside" the cuspidal loop, then the sharp upper bound is respectively 2 and 3. Hence there is no possibility to exhibit a quadruple limit cycle. But for the perturbation from global center case, the above result permits to give a formal proof that in Liénard equations of type (3,2) one encounter quadruple limit cycles, as was conjectured in ^{11}.

Remark 2. Extending the results to statements on limit cycles for system (1), with $\delta > 0$ but small, can for sure be realized.

Main steps of the proof.

1. By using the standard technique we get the Picard-Fuchs equation

$$G(h)\frac{d}{dh}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix},$$

where $G(h)$ and a_{ij} are polynomials of h with coefficients depending on λ.

2. Since $I_0(h) > 0$ for $h > h_0$, we may change the Abelian integral (2) to the form

$$I(h) = I_0(h)(\alpha + \beta P(h) + Q(h)),$$

where $P(h) = I_1(h)/I_0(h)$ and $Q(h) = I_2(h)/I_0(h)$, which have certain limits as h tends to h_0 and h_1 . From the Picard-Fuchs equation we obtain a system of differential equations

$$\dot{h} = G(h), \ \dot{P} = f(h, P, Q), \ \dot{Q} = g(h, P, Q),$$

where f and g are polynomials of h, P and Q. The trajectory $(h, P(h), Q(h))$, which we are interested, connects two singularities of the system. Hence for $h > h_0$ the number of zeros of the Abelian integral (2) is equal to the number of intersection points of the curve $\Sigma_\lambda = \{(P, Q)(h)|h_0 < h < h_1\}$ with the straight line $\Pi_{\alpha,\beta} : \alpha + \beta P + Q = 0$ in (P, Q)-plane.

3. By using the criterion in[12] and some techniques in[7] we prove that both $P(h)$ and $Q(h)$ are monotonically increasing.

4. Since $I_0''(h) \neq 0$ (see[2] and[5]), we can write

$$I''(h) = I_0''(h)(\alpha + \beta\omega(h) + \nu(h)),$$

where $\omega(h) = I_1''(h)/I_0''(h)$ and $\nu(h) = I_2''(h)/I_0''(h)$. On the other hand, $I_2''(h)$ can be expressed as a linear combination of $I_0''(h)$ and $I_1''(h)$ with linear functions of h as their coefficients. This gives a Ricatti equation of $\omega(h)$; and we may rewrite

$$I''(h) = I_0''(h)L(h)(\omega(h) - U(h)),$$

where $L(h)$ is a linear function, and $U(h) = \frac{a_1 h + b_1}{a_2 h + b_2}$ in cases (2) and (4), and $U(h) = \frac{a_1 h^2 + b_1 h + c_1}{a_2 h^2 + b_2 h + c_2}$ in case (3), a_i and b_i depend on α, β and λ.

By using these results, it is possible to study the number of zeros of $I''(h)$ precisely.

5. At last, we find out the geometric shape of the curve Σ_λ as follows:

(1) in the Saddle Loop case, Σ_λ is globally convex;

(2) in the Cuspidal Loop case, Σ_λ has two inflection points, and one of them corresponds to the value of the cuspidal loop;

(3) in the Global Center case, Σ_λ has at most two inflection points. When λ is decreasing from 1 to 0, the two inflection points coalesce for a certain value of λ and become a quadruple point, then Σ_λ is globally convex.

For a full proof see[5],[6] and[7].

1. C. Christopher and N. Lloyd, Small-amplitude limit cycles in Liénard systems, *Nonlinear Differential Equations and Applications* , to appear.
2. S.-N. Chow and J. A. Sanders, *J. Diff. Equs.* **64** (1986) 51–66.
3. F. Dumortier and C. Li, *Nonlinearity* **9** (1996) 1489–1500.
4. F. Dumortier and C. Li, *J. Diff. Equs.* **139** (1997) 41–59.

5. F. Dumortier and C. Li, Perturbations from an elliptic Hamltonian of degree four : (I) Saddle Loop and Two Saddle Cycle, *preprint*.
6. F. Dumortier and C. Li, Perturbations from an elliptic Hamltonian of degree four : (II) Cuspidal Loop, *preprint*.
7. F. Dumortier and C. Li, Perturbations from an elliptic Hamltonian of degree four : (III) Global Center, *preprint*.
8. F. Dumortier and C. Rousseau, *Nonlinearity* **3** (1990) 1015–1039.
9. L. Gavrilov, *J. Diff. Equs.* **101** (1993) 58–65.
10. E. Horozov, *Trudy Seminar Petrovskii* **5** (1979) 163–192.
11. A. Khibnik, B. Krauskopf and C. Rousseau, Nonlinearity **11** (1998) 1505-1519.
12. C. Li and Z. Zhang, *J. Diff. Equs.* **127** (1996) 407–424.
13. J. Li, Two results on Liénard equations, *Ph.D thesis Peking University* (1998).
14. A. Lins, W. de Melo and C. Pugh, *Lecture Notes in Math.* **597** (1977) 335–357.
15. S. Melo, Rigorous computer verification of planar vector field structure, *Ph.D thesis Cornell University* (1994).
16. G. S. Petrov, *Funct. Anal. Appl.* **24** (1990) 205–210.
17. Y. Zhao, Abelian integrals for cubic Hamiltonian vector fields, *Ph.D thesis Peking University* (1998).

GLOBAL STABILITY OF SEIR MODELS
WITH SATURATION INCIDENCE

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Two SEIR epidemiological models with a constant population size are considered. The incidences of infection are given by saturation forms which work for the situations that the host population is saturated with infectious individuals or susceptible individuals, respectively. It is proved that the disease-free equilibrium is globally stable and the disease always dies out if a threshold number $\sigma \leq 1$; and that, if $\sigma > 1$, a unique endemic equilibrium is globally stable in the interior of the feasible region and the disease persists at the constant endemic equilibrium level.

1 Introduction

Li and Muldowney [13] studied an SEIR epidemic model for the dynamics of an infectious disease that spreads in a population. A case of the model is described by the following system of differential equations

$$
\begin{aligned}
S' &= \mu - \mu S - \lambda IS \\
E' &= \lambda IS - (\epsilon + \mu)E \\
I' &= \epsilon E - (\gamma + \mu)I \\
R' &= \gamma I - \mu R,
\end{aligned}
\tag{1}
$$

where $S(t), E(t), I(t)$ and $R(t)$ denote the fractions

of the population that are susceptible, exposed (not yet infectious), infectious, and recovered individuals at time t, respectively. The transmission is assumed to occur through direct contact of hosts and the incidence is described by the standard mass action form λIS, where $\lambda > 0$ is the transmission coefficient. The birth rate and the death rate are assumed to be equal and denoted by μ, and the disease is assumed not to be fatal. Consequently the total population is constant; $S + E + I + R = 1$. The parameter $\epsilon > 0$ is the rate at which exposed individuals become infectious, and $\gamma > 0$ is the rate that infectious individuals become recovered. The immunity is assumed to be permanent, so once recovered, an individual will not become susceptible again.

Li and Muldowney [13] proved that the dynamics of (1) is completely determined by the basic reproduction number

$$
\sigma = \frac{\lambda \epsilon}{(\epsilon + \mu)(\gamma + \mu)}
\tag{2}
$$

If $\sigma \leq 1$ the disease always dies out, whereas when $\sigma > 1$ the disease persists at a unique endemic equilibrium if it is initially present.

Nonlinear incidence forms other than the standard bilinear incidence form λIS have been used in the literature. Busenberg and Cooke [2] use a saturation incidence form $\frac{\lambda IS}{1+aI}$, where $a \geq 0$, which describes the situation that the population is saturated with infectious individuals; May and Anderson [1,1] use another saturation incidence term $\frac{\lambda IS}{1+aS}$, when a large number of susceptible

individuals are present in population. We refer the reader to [1,6,8,9,14,16] for background on epidemic models and surveys of results.

In this paper, we study model (1) with the two different saturation incidence forms mentioned above. We find the basic reproduction numbers σ for each model, and then establish that the disease-free equilibrium is globally stable in the feasible region Γ if $\sigma \leq 1$, and the unique endemic equilibrium is globally stable in $\overset{\circ}{\Gamma}$ when $\sigma > 1$. The global stability of the unique endemic equilibrium is proved using the method of Li and Muldowney[13].

2   Model One

Consider the SEIR model of the form

$$
\begin{aligned}
S' &= -\frac{\lambda IS}{a+I} + \mu - \mu S \\
E' &= \frac{\lambda IS}{a+I} - (\epsilon + \mu)E \\
I' &= \epsilon E - (\gamma + \mu)I \\
R' &= \gamma I - \mu R,
\end{aligned}
\tag{3}
$$

where $\lambda, \mu, \epsilon, \gamma$ are nonnegative parameters as described in Section 1. The constant a is nonnegative.

Adding all the equations in (3), we have

$$
(S + E + I + R)' = -\mu(S + E + I + R - 1),
$$

which implies that the 3-dimensional simplex $S + E + I + R = 1$ is invariant with respect to (3). Using the relation $R(t) = 1 - S(t) - E(t) - I(t)$, we can reduce (3) to the following equivalent 3-dimensional system

$$
\begin{aligned}
S' &= -\frac{\lambda IS}{a+I^p} + \mu - \mu S \\
E' &= \frac{\lambda IS}{a+I^p} - (\epsilon + \mu)E \\
I' &= \epsilon E - (\gamma + \mu)I.
\end{aligned}
\tag{4}
$$

We study system (4) in the feasible region

$$
\Omega = \{(S, E, I) \in R_+^3 \ : \ 0 \leq S, E, I \leq 1, S + E + I \leq 1\},
$$

which is positively invariant with respect to (4).

There are two possible equilibria of (4): the disease-free equilibrium $P_0 = (1, 0, 0)$ and the endemic equilibrium $P^* = (S^*, E^*, I^*)$, where

$$
S^* = \frac{a + I^*}{a\sigma}, \qquad E^* = \frac{\gamma + \mu}{\epsilon}I^*, \qquad I^* = \frac{2a\mu(\sigma - 1)}{\lambda + \mu}.
$$

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and

$$\sigma = \frac{\lambda \epsilon}{a(\epsilon + \mu)(\gamma + \mu)},$$

which can be regarded as the basic reproduction number since $\sigma S^* = 1$ (see [1])]. If $0 \le \sigma \le 1$, then P_0 is the only equilibrium of (4) in Ω; if $\sigma > 1$, then both P_0 and P^* exist.

Theorem 4 *If $\sigma \le 1$, then P_0 is the only equilibrium in Ω and it is globally stable in Ω. If $\sigma > 1$, then P_0 becomes unstable and there exists a unique endemic equilibrium P^* in $\overset{\circ}{\Omega}$. Furthermore, all solutions starting in $\overset{\circ}{\Omega}$ and sufficiently close to P_0 move away from P_0.*

Proof. The global stability of P_0 can be verified using a Lyapunov function

$$L = \epsilon E + (\epsilon + \mu)I.$$

The derivative of L along a solution of (4) is

$$L' = I \left\lceil \frac{\lambda \epsilon S}{1 + aI} - (\epsilon + \mu)(\gamma + \mu) \right\rceil = \frac{I}{(\epsilon + \mu)(\gamma + \mu)} \left(\frac{\sigma S}{1 + aI} - 1 \right)$$
$$\le 0, \quad \text{if} \quad \sigma \le 1.$$

Furthermore, $L' = 0$ if and only if $I = 0$. Since $\{P_0\}$ is the largest compact invariant set in $\{(S, E, I) \mid L' = 0\}$, P_0 is globally stable by LaSalle's Invariance Principle ([11]). If $\sigma > 1$ and $I > 0$, then $L' > 0$ for $S < 1$ but sufficiently close to 1. This implies the instability of P_0 and also leads to the last claim of the theorem. □

Theorem 2.1 completely determines the global dynamics of (4) in Ω for the case $\sigma \le 1$. Its epidemiological implication is that the infected fraction (the sum of the latent and the infectious fractions) of the population vanishes in time so the disease dies out. In the rest of this section, we show that the disease persists when $\sigma > 1$. We say the disease is *endemic* if the infected fraction of the population persists above a certain positive level for sufficiently large time. The endemicity of disease can be well captured and analyzed through the notion of uniform persistence. System (4) is said to be *uniformly persistent* (see [3,20]) if there exists a constant $0 < c < 1$ such that any solution $(S(t), E(t), I(t))$ with $(S(0), E(0), I(0)) \in \overset{\circ}{\Omega}$ satisfies

$$\min \left\{ \liminf_{t \to \infty} S(t), \quad \liminf_{t \to \infty} E(t), \quad \liminf_{t \to \infty} I(t) \right\} \ge c.$$

The disease is endemic if (4) is uniformly persistent. In this case, both the infective and the latent fractions persist above a certain positive level. The local behavior near the only boundary equilibrium P_0 as stated in Theorem 2.1

allows us to use a similar argument as in the proof of Proposition 3.3 in [12] and a uniform persistence criteria in [5], Theorem 4.3, to establish the following persistence result.

Proposition 1 *System (4) is uniformly persistent in $\overset{\circ}{\Omega}$ if and only if $\sigma > 1$.*

Next, we establish that any periodic solution to (4), if one exists, is orbitally asymptotically stable (see [7] for definition) using the following stability criterion of Muldowney [17].

Theorem 5 *A sufficient condition for a periodic orbit $\gamma = \{p(t) \ : \ 0 \leq t \leq \omega\}$ of $x' = f(x)$ to be orbitally asymptotically stable is that the linear system*

$$z'(t) = \left(\frac{\partial f^{[2]}}{\partial x}(p(t))\right) z(t) \tag{5}$$

be asymptotically stable.

Here, $A^{[2]}$ denote the second additive compound matrix of an $n \times n$ matrix A. We refer the reader to [4,17] for the definition and properties of compound matrices. Pertinent to our study is a spectral property of $A^{[2]}$. Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of A. Then the spectrum $\sigma(A^{[2]}) = \{\lambda_i + \lambda_j \ : \ 1 \leq i < j \leq n\}$. The second additive compound matrix of a 3×3 matrix is provided in the Appendix.

Proposition 2 *Any periodic solution to (4), if one exists, is orbitally asymptotically stable if $\sigma > 1$.*

Proof. The Jacobian matrix $J(S, E, I)$ of (4) is given by

$$J(S, E, I) = \begin{bmatrix} -\frac{\lambda I}{a+I} - \mu & 0 & -\frac{\lambda a S}{(a+I)^2} \\ \frac{\lambda I}{a+I} & -(\epsilon + \mu) & \frac{\lambda a S}{(a+I)^2} \\ 0 & \epsilon & -(\gamma + \mu) \end{bmatrix}.$$

Using the second additive compound matrix in the Appendix, we can write the linear system (5) with respect to a periodic solution $(S(t), E(t), I(t))$ to (4) as

$$\begin{aligned} X' &= -\left(\frac{\lambda I}{a+I} + \epsilon + 2\mu\right) X + \frac{\lambda a S}{(a+I)^2}(Y + Z) \\ Y' &= \epsilon X - \left(\frac{\lambda I}{a+I} + \gamma + 2\mu\right) Y \\ Z' &= \frac{\lambda I}{a+I} Y - (\epsilon + \gamma + 2\mu) Z. \end{aligned} \tag{6}$$

To prove that (6) is asymptotically stable, we consider the following function

$$V(X, Y, Z; S, E, I) = |P(S, E, I) \cdot (X, Y, Z)^*|,$$

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where the matrix $P = diag(1, E/I, E/I)$ and $|\cdot|$ is the norm in R^3 defined by

$$|(X, Y, Z)| = \sup\{|X|, \ |Y| + |Z|\}. \tag{7}$$

The uniform persistence of (4) when $\sigma > 1$ implies that the orbit Γ of the periodic solution $(S(t), E(t), I(t))$ remains at a positive distance from the boundary of Ω. The matrix P and its inverse are thus well defined and smooth along Γ. There exists a constant $c_1 > 0$ such that

$$V(X, Y, Z; S, E, I) \geq c_1 |(X, Y, Z)| \tag{8}$$

for any $(X, Y, Z) \in R^3$ and $(S, E, I) \in \Gamma$. Let $(X(t), Y(t), Z(t))$ be a solution to (6) and

$$\begin{aligned} V(t) &= V(X(t), Y(t), Z(t); S(t), E(t), I(t)) \\ &= \sup\{|X(t)|, \tfrac{E}{I}(|Y(t)| + |Z(t)|)\}. \end{aligned}$$

The right-hand derivative of $V(t)$ exists. In fact, direct calculation yields

$$\begin{aligned} D^+|X(t)| &\leq -\left(\tfrac{\lambda I}{a+I} + \epsilon + 2\mu\right)|X(t)| + \tfrac{\lambda a S}{(a+I)^2}(|Y(t)| + |Z(t)|) \\ &\leq -\left(\tfrac{\lambda I}{a+I} + \epsilon + 2\mu\right)|X(t)| + \tfrac{I}{E}\tfrac{\lambda a S}{(a+I)^2}\left(\tfrac{E}{I}(|Y(t)| + |Z(t)|)\right) \end{aligned} \tag{9}$$

and

$$\begin{aligned} D^+|Y(t)| &\leq \epsilon|X(t)| - \left(\tfrac{\lambda I}{a+I} + \gamma + 2\mu\right)|Y(t)| \\ D^+|Z(t)| &\leq \tfrac{\lambda I}{a+I}|Y(t)| - (\epsilon + \gamma + 2\mu)|Z(t)|, \end{aligned}$$

. Then

$$\begin{aligned} D^+\left\{\tfrac{E}{I}(|Y(t) + |Z(t)|)\right\} &\leq \left(\tfrac{E'}{E} - \tfrac{I'}{I}\right)(|Y(t)| + |Z(t)|) \\ &\quad + \tfrac{E}{I}\left[\epsilon|X(t)| - (\gamma + 2\mu)(|Y(t)| + |Z(t)|)\right] \\ &= \left(\tfrac{E'}{E} - \tfrac{I'}{I}\right)\tfrac{E}{I}(|Y(t)| + |Z(t)|) \\ &\quad + \tfrac{E}{I}\left[\epsilon|X(t)| - (\gamma + 2\mu)(|Y(t)| + |Z(t)|)\right] \\ &= \tfrac{\epsilon E}{I}|X(t)| + \left(\tfrac{E'}{E} - \tfrac{I'}{I} - \gamma - 2\mu\right)\tfrac{E}{I}(|Y(t)| + |Z(t)|). \end{aligned} \tag{10}$$

We claim that (9) and (10) imply that

$$D^+V(t) \leq \left(\tfrac{E'}{E} - \mu\right)V(t). \tag{11}$$

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In fact, if $|X(t)| > \frac{E}{I}(|Y(t)| + |Z(t)|)$, then $V(t) = |X(t)|$. From (4) and (9) it follows that

$$
\begin{aligned}
D^+V(t) &= D^+|X(t)| \\
&\leq -\left(\frac{\lambda I}{a+I} + \epsilon + 2\mu\right)|X(t)| + \frac{I}{E}\frac{\lambda a S}{(a+I)^2}|X(t)| \\
&\leq \left(-\frac{\lambda I}{a+I} - \epsilon - 2\mu + \frac{\lambda I S}{E(a+I)}\right)|X(t)| \\
&= \left(-\frac{\lambda I}{a+I} - \epsilon - 2\mu + \frac{E'}{E} + \epsilon + \mu\right)|X(t)| \\
&\leq \left(\frac{E'}{E} - \mu\right)|X(t)| = \left(\frac{E'}{E} - \mu\right)V(t).
\end{aligned}
$$

If $|X(t)| < \frac{E}{I}(|Y(t)| + |Z(t)|)$, then $V(t) = \frac{E}{I}(|Y(t)| + |Z(t)|)$. By (4) and (10), we have

$$
\begin{aligned}
D^+V(t) &= D^+\left\{\frac{E}{I}(|Y(t)| + |Z(t)|)\right\} \\
&\leq \left(\frac{\epsilon E}{I} + \frac{E'}{E} - \frac{I'}{I} - \gamma - 2\mu\right)\frac{E}{I}(|Y(t)| + |Z(t)|) \\
&= \left(\frac{I'}{I} + \gamma + \mu + \frac{E'}{E} - \frac{I'}{I} - \gamma - 2\mu\right)\frac{E}{I}(|Y(t)| + |Z(t)|) \\
&= \left(\frac{E'}{E} - \mu\right)V(t).
\end{aligned}
$$

If $|X(t)| = \frac{E}{I}(|Y(t)| + |Z(t)|)$, we have $D^+|X(t)| = D^+E/I(|Y(t)| + |Z(t)|)$, and thus (11) follows from either (9) or (10).

Note that

$$
\int_0^\omega \left(\frac{E'}{E} - \mu\right)dt = -\mu\omega < 0,
$$

which, together with (11) implies that $V(t) \to 0$ as $t \to \infty$, and in turn that $(X(t), Y(t), Z(t)) \to 0$ as $t \to \infty$ by (8). Thus the linear system (6) is asymptotically stable. By Theorem 2.3, the periodic solution $(S(t), E(t), I(t))$ of (4) is orbitally asymptotically stable. The proof is complete. □

Corollary 1 *If $\sigma > 1$, then the unique endemic equilibrium P^* is locally asymptotically stable.*

Proof. Regard the equilibrium P^* as a periodic solution, then system (6) at $(S(t), E(t), I(t)) = (S^*, E^*, I^*)$ is autonomous. The same analysis as in the proof of Proposition 2.4 shows that this system is asymptotically stable, and thus its coefficient matrix $J(S^*, E^*, I^*)^{[2]}$, the second additive compound matrix of the Jacobian matrix $J(S^*, E^*, I^*)$ of (4) at P^*, is stable, namely, all the eigenvalues of $J(S^*, E^*, I^*)^{[2]}$ have negative real parts. By the spectral property of additive compound matrices stated preceding Proposition 2.4, this implies that $\operatorname{Re}(\lambda_i + \lambda_j) < 0$, $i \neq j$, where $\lambda_1, \lambda_2, \lambda_3$

are eigenvalues of $J(S^*, E^*, I^*)$. Thus, at most one of λ_i can have nonnegative real part. All λ_i will have negative real parts if we can show that $\lambda_1 \lambda_2 \lambda_3 = \det(J(S^*, E^*, I^*)) < 0$. In fact,

$$
\det(J(S^*, E^*, I^*)) = \begin{vmatrix} -\frac{\lambda I^*}{a+I^*} - \mu & 0 & -\frac{\lambda a S^*}{(a+I^*)^2} \\ \frac{\lambda I^*}{a+I^*} & -\epsilon - \mu & \frac{\lambda a S^*}{(a+I^*)^2} \\ 0 & \epsilon & -\gamma - \alpha - \mu \end{vmatrix}
$$

$$
= -\left(\frac{\lambda I^*}{a+I^*} + \mu \right)(\epsilon + \mu)(\gamma + \mu) + \frac{\lambda \epsilon S^*}{a+I^*} \frac{a\mu}{a+I^*}.
$$

Using the equilibrial equations for (S^*, E^*, I^*) we can derive that $\lambda \epsilon S^*/(a + I^*) \leq (\epsilon + \mu)(\gamma + \mu)$. Therefore, $\det(J(P^*)) \leq -\lambda I^*(\epsilon + \mu)(\gamma + \mu)/(a+I^*) < 0$. This shows that P^* is locally asymptotically stable. $\qquad \square$

To show that the unique endemic equilibrium P^* is globally asymptotically stable, we take the advantage of the fact that (4) is a competitive system when $\sigma > 1$. We refer the reader to [18] for definition of competitive systems. In fact, from Jacobian matrix of (4), one may verify that $HJ(S, E, I)H$ has nonpositive off-diagonal entries, where $H = \text{diag}(-1, 1, -1)$. This implies, under a linear change of variables $y = Hx$, the flow of (4) preserves, for $t < 0$, the partial ordering in R^3_+ defined by the orthant $\{(S, E, I) \in R^3 \; : \; S \leq 0, E \geq 0, I \leq 0\}$ (see [18]). An important characteristic of a three-dimensional competitive system is the following Poincaré-Bendixson property, which follows from a result of Hirsch [10] and Smith [19] for general three-dimensional competitive systems. A proof may be found in [18], Chapter 3, Theorem 4.1.

Theorem 6 *Let L be a nonempty compact omega limit set of (4). If L contains no equilibria, then L is a closed orbit.*

Theorem 7 *Suppose that $\sigma > 1$. Then the unique endemic equilibrium P^* is globally asymptotically stable in $\overset{\circ}{\Omega}$. Moreover, P^* attracts all trajectories in Ω except those on the invariant S-axis which converge to P_0 along this axis.*

Proof. By inspecting the vector field given by (4), we see that all trajectories originating from the boundary $\partial \Omega$ enter $\overset{\circ}{\Omega}$ except those on the S-axis which converge to P_0 along this invariant axis. It remains to show that P^* attracts all points in $\overset{\circ}{\Omega}$. Let $U \subset \overset{\circ}{\Omega}$ be the set of points that are attracted by P^*. Then U is an open subset of $\overset{\circ}{\Omega}$ by the asymptotic stability of P^*. The theorem is proved if we establish that $\overset{\circ}{\Omega} \subset U$. Assume the contrary; then the boundary ∂U of U has a nonempty intersection $\mathcal{I}$ with $\overset{\circ}{\Omega}$. Since both U and its closure $\overline{U}$ are invariant and U is open, $\partial U = \overline{U} - U$ is also invariant. As the intersection of ∂U with the positively invariant $\overset{\circ}{\Omega}$, $\mathcal{I}$ is positively invariant,

and thus $\mathcal{I}$ contains a nonempty compact omega limit set Γ. By the uniform persistence, we must have $\Gamma \cap \partial\Omega = \emptyset$. Since it contains no equilibria, by Theorem 2.6 and Proposition 2.4, Γ is a closed orbit and is asymptotically orbitally stable. We thus obtain a contradiction since Γ belongs to the alpha limit set of a trajectory in U. This completes the proof. $\qquad\square$

3 Model Two

Consider the SEIR model of the form

$$
\begin{aligned}
S' &= -\frac{\lambda IS}{a+S} + \mu - \mu S \\
E' &= \frac{\lambda IS}{a+S} - (\epsilon + \mu)E \\
I' &= \epsilon E - (\gamma + \mu)I \\
R' &= \gamma I - \mu R.
\end{aligned}
\tag{12}
$$

The model (12) is identical to model (3) except the incidence form.

Using the relation $S + E + I + R = 1$, system (12) can be reduced to the following system

$$
\begin{aligned}
S' &= -\frac{\lambda IS}{a+S} + \mu - \mu S \\
E' &= \frac{\lambda IS}{a+S} - (\epsilon + \mu)E \\
I' &= \epsilon E - (\gamma + \mu)I,
\end{aligned}
\tag{13}
$$

which will be studied in the region

$$
\Omega = \{(S, E, I) \in R_+^3 \ : \ 0 \le S, E, I \le 1, S + E + I \le 1\}
$$

There are two possible equilibria to (13): the disease-free equilibrium $P_0 = (1, 0, 0)$ and the endemic equilibrium $P^* = (S^*, E^*, I^*)$, where

$$
S^* = \frac{a(\epsilon+\mu)(\gamma+\mu)}{\lambda\epsilon - (\epsilon+\mu)(\gamma+\mu)}, \qquad E^* = \frac{\gamma+\mu}{\epsilon}I^*,
$$

$$
I^* = \frac{\mu\epsilon}{(\epsilon+\mu)(\gamma+\mu)} - \frac{a\epsilon\mu}{\lambda\epsilon - (\epsilon+\mu)(\gamma+\mu)}
$$

. Let

$$
\sigma = \frac{\lambda\epsilon}{(\epsilon+\mu)(\gamma+\mu)(a+1)}.
$$

Then P_0 is the only equilibrium of (13) in Ω if $0 \le \sigma \le 1$; and both P_0 and P^* exist if $\sigma > 1$. The analysis of (13) follows very closely to that of (4). We state our main result in the following and only remark on its proof.

Theorem 8 *If $\sigma \le 1$, then the disease-free equilibrium P_0 is globally asymptotically stable in Ω. If $\sigma > 1$, then the unique endemic equilibrium P^* is globally asymptotically stable in $\overset{\circ}{\Omega}$.*

Remark. The proof of Theorem 3.1 can be done in the same fashion as in the previous section. The global stability of P_0 when $\sigma \leq 1$ can be proof using the same Lyapunov function L used in the proof of Theorem 2.1. The same Lyapunov function can be used to show that (13) is uniformly persistent when $\sigma > 1$. It can be checked that (13) is competitive and hence satisfies the Poincaré-Bendixson property. The same function V as in the proof of Proposition 2.4 can then be used to show the orbital asymptotic stability of any periodic orbit of (13), if one exists, and hence to prove the global stability of P^* when $\sigma > 1$ as in the proof of Theorem 2.7.

Acknowledgments

Appendix

The second additive compound matrix $A^{[2]}$ of a 3×3 matrix $A = (a_{ij})$ is given by

$$\begin{bmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{bmatrix}.$$

References

1. R. M. Anderson and R. M. May, *Infectious Diseases of Humans: Dynamics and Control*, (Oxford University Press, Oxford, 1992).
2. S. Busenberg and K. Cooke, *Vertically Transmitted Diseases*, (Biomathematics 23, Springer-Verlag, Berlin, 1993).
3. G. J. Butler and P. Waltman, *Proc. Amer. Math. Soc.*964251996.
4. M. Fiedler, *Czech Math. J.*993921974.
5. H. I. Freedman, M. X. Tang, and S. G. Ruan, *J. Dynam. Diff. Equat.* 65831994.
6. D. Greenhalgh, *Math. Comput. Modeling*25851997.
7. J. K. Hale, *Ordinary Differential Equations*, (McGraw-Hill, New York, 1969).
8. H. W. Hethcote and S. A. Levin, in *Applied Mathematical Ecology*, eds. L.Gross and S. A. Levin, (Springer, New York, 1989).
9. H. W. Hethcote, H. W. Stech and P. Van den Driessche, in *Differential Equations and Applications in Ecology, Epidemics, and Population Problems*, ed. K. L. Cooke, (Academic Press, New York, 1981).
10. M. W. Hirsch, *SIAM J. Math. Anal.*2112251990.
11. J. P. LaSalle, The Stability of Dynamical Systems, (SIAM, Philadelphia, 1976).
12. M. Y. Li, J. R. Graef, L. Wang and J. Karsai, *J. Math. Biosci.*1601911999.
13. M. Y. Li and J. S. Muldowney, *Math. Biosci.*1251551995.
14. W-M. Liu, H. W. Hethcote and S. A. Levin, *J. Math. Biol.*1 253591987.
15. R. M. May and R. M. Anderson, *J. Anim. Ecol.*472191978.
16. J. Mena-Lorca and H. W. Hethcote, *J. Math. Biol.*306931992.
17. J. S. Muldowney, *Rocky Mountain J. Math.*208571990.
18. H. L. Smith, *Monotone Dynamical Systems, An Introduction to the Theory of Competitive and Cooperative Systems*, (Amer. Math. Soc., Providence, 1995).
19. H. L. Smith, *J. Differential Equations*653611986.
20. P. Waltman, in *Delay Differential Equations and Dynamical Systems*, eds. S. Busenberg and M. Martelli, (Springer-Verlag, New York, 1991).

GLOBAL EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR N-SPECIES LOTKA-VOLTERRA-TYPE COMPETITION SYSTEM DEVIATING ARGUMENTS

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In this paper, we employ some new techniques to study the global existence of positive periodic solutions for a class of n-species Lotka-Volterra-type competition system with deviating arguments.

1 Introduction

We consider the following n-species Lotka-Volterra-type competition system with deviating arguments

$$N_i'(t) = N_i(t)[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)N_j(t - \tau_{ij}(t))], i = 1, 2, \cdots, n, \quad (1)$$

where $a_i(t), b_{ij}(t), \tau_{ij}(t)(i, j = 1, \cdots, n)$ are nonnegative continuous T-periodic functions. Since we are interested in the positive solutions of the system, we assume the system (1) is supplemented with initial conditions of the form

$$N_i(t) = \phi_i(t) \in C([-r, 0], R^+), \phi_i(0) > 0, i = 1, 2, \cdots, n, \quad (2)$$

where $r = \max_{1 \le i, j \le n} \{ \max_{t \in [0,T]} \tau_{ij}(t) \}$.

It is easy to see that all solutions $N(t) = (N_1(t), N_2(t), \cdots, N_n(t))$ of (1) and (2) exist and satisfy $N(t) > 0$, i.e. $N_i(t) > 0$ $(i = 1, 2, \cdots, n)$ for $t \in (0, +\infty)$. Therefore, under the transformations $N_i(t) = \exp x_i(t)$ $(i = 1, 2, \cdots, n)$, (1) can be reduced to

$$x_i'(t) = a_i(t) - \sum_{j=1}^{n} b_{ij}(t)e^{x_j(t - \tau_{ij}(t))}, i = 1, 2, \cdots, n. \quad (3)$$

The purpose of this paper is to establish the global existence of positive periodic solutions of the system (1). When $\tau_{ij}(t) \equiv const.$, such a problem was considered by K.Gopalsamy and X.Z.He [1]. In the paper, by using a continuation theorem based on Mawhin's coincidence degree, we establish a new existence theorem.

2   Main Results

In this section, we denote $\bar{g} = \frac{1}{T} \int_0^T g(t)dt, g_m = \min\limits_{t \in [0,T]} g(t), |g|_0 = \max\limits_{t \in [0,T]} |g(t)|$
for $g \in \{g \in C(R,R) : g(t+T) = g(t) \text{ for } t \in R\}$.

Theorem 2.1 *Suppose that the following conditions are satisfied:*

(a) $a_i(t) \in C(R, (0, +\infty)), b_{ij}(t) \in C(R, R^+), \tau_{ij}(t) \in C^1(R, R^+), \tau'_{ij}(t) < 1$,
$i, j = 1, 2, \cdots, n;$

(b)

$$m = \sum_{p \in S_1} m_{p_1 1} m_{p_2 2} \cdots m_{p_n n} - \sum_{p \in S_2} M_{p_1 1} M_{p_2 2} \cdots M_{p_n n} > 0,$$

$$m_j = \sum_{p \in S_1} m_{p_1 1} \cdots m_{p_{j-1},j-1} \bar{a}_j m_{p_{j+1},j+1} \cdots m_{p_n n}$$

$$- \sum_{p \in S_2} M_{p_1 1} \cdots M_{p_{j-1},j-1} \bar{a}_j M_{p_{j+1},j+1} \cdots M_{p_n n} > 0;$$

or

$$Q = \sum_{p \in S_1} M_{p_1 1} M_{p_2 2} \cdots M_{p_n n} - \sum_{p \in S_2} m_{p_1 1} m_{p_2 2} \cdots m_{p_n n} < 0,$$

$$Q_j = \sum_{p \in S_1} M_{p_1 1} \cdots M_{p_{j-1},j-1} \bar{a}_j M_{p_{j+1},j+1} \cdots M_{p_n n}$$

$$- \sum_{p \in S_2} m_{p_1 1} \cdots m_{p_{j-1},j-1} \bar{a}_j m_{p_{j+1},j+1} \cdots m_{p_n n} < 0;$$

where $M_{ij} = |\frac{b_{ij}}{1-\tau'_{ij}}|_0, m_{ij} = (\frac{b_{ij}}{1-\tau'_{ij}})_m,$

$S_1 = \{p = p_1 p_2 \cdots p_n : p \text{ is an even permutation of } \{1, 2, \cdots, n\}\},$

$S_2 = \{p = p_1 p_2 \cdots p_n : p \text{ is an odd permutation of } \{1, 2, \cdots, n\}\}.$

Then (1) has at least one positive T-periodic solution.

Proof: Let C_T^0 denote the linear space of real valued continuous T-periodic functions on R. The linear space C_T^0 is a Banach space with the usual norm for $x(t) = (x_1(t), \cdots, x_n(t)) \in C_T^0$ given by $\|x\|_0 = \max\limits_{t \in R} |x(t)| = \max\limits_{t \in R} \sum\limits_{i=1}^n |x_i(t)|.$

Let $X = Y = C_T^0$. We define the following maps:

$$Lx = \frac{dx(t)}{dt}, Px = Qx = \frac{1}{T} \int_0^T x(t)dt, x \in X;$$

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$$Nx = (a_1(t) - \sum_{j=1}^{n} b_{1j}(t)e^{x_j(t-\tau_{1j}(t))}, \cdots, a_n(t) - \sum_{j=1}^{n} b_{nj}(t)e^{x_j(t-\tau_{nj}(t))}),$$

$$x \in X.$$

It is easy to verify that L is a Fredholm operator of index zero and for any open bounded subset Ω of X, N is L-compact on $\bar{\Omega}$.

Let $Lx = \lambda Nx, \lambda \in (0,1)$, for $x(t) \in X$ i.e.

$$x_i'(t) = \lambda[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)e^{x_j(t-\tau_{ij}(t))}], i = 1, 2, \cdots, n, \lambda \in (0,1). \quad (4)$$

Integrating these identities, we have

$$\int_0^T \sum_{j=1}^{n} b_{ij}(t)e^{x_j(t-\tau_{ij}(t))}dt = \int_0^T a_i(t)dt, i = 1, 2, \cdots, n. \quad (5)$$

From (4),(5), we have

$$\int_0^T |x_i'(t)|dt \leq \lambda[\int_0^T a_i(t)dt + \int_0^T \sum_{j=1}^{n} b_{ij}(t)e^{x_j(t-\tau_{ij}(t))}dt] < 2T\bar{a}_i. \quad (6)$$

Again, noticing that $\int_0^T b_{ij}(t)e^{x_j(t-\tau_{ij}(t))}dt = \frac{b_{ij}(\eta_{ij})}{1-\tau_{ij}'(\eta_{ij})}\int_0^T e^{x_j(s)}ds$, for some $\eta_{ij} \in [0,T]$, where $t = \sigma_{ij}(s)$ is the inverse function of $s = t - \tau_{ij}(t)(t \in [0,T])$, hence, from (5), we have

$$\sum_{j=1}^{n} \frac{b_{ij}(\eta_{ij})}{1-\tau_{ij}'(\eta_{ij})} \int_0^T e^{x_j(t)}dt = \int_0^T a_i(t)dt, i = 1, 2, \cdots, n.$$

Since $\int_0^T e^{x_j(t)}dt = Te^{x_j(\delta_j)}$, for some $\delta_j \in [0,T](j = 1, 2, \cdots, n)$, we have

$$\sum_{j=1}^{n} \frac{b_{ij}(\eta_{ij})}{1-\tau_{ij}'(\eta_{ij})}e^{x_j(\delta_j)} = \bar{a}_i, i = 1, 2, \cdots, n. \quad (7)$$

Clearly, by (b) of Theorem 2.1, the system (7) has one unique positive solution

$$e^{x_j\delta_j} = \frac{\sum_{p \in S_1} g_{p_11}\cdots g_{p_{j-1},j-1}\bar{a}_j g_{p_{j+1},j+1}\cdots g_{p_n n} - \sum_{p \in S_2} g_{p_11}\cdots g_{p_{j-1},j-1}\bar{a}_j g_{p_{j+1},j+1}\cdots g_{p_n n}}{\sum_{p \in S_1} g_{p_11}g_{p_22}\cdots g_{p_n n} - \sum_{p \in S_2} g_{p_11}g_{p_22}\cdots g_{p_n n}},$$

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where $g_{ij} = \frac{b_{ij}(\eta_{ij})}{1-\tau'_{ij}(\eta_{ij})}(i,j = 1,2,\cdots,n)$.

It is clear that $\ln\frac{m_j}{Q} \le x_j(\delta_j) \le \ln\frac{Q_j}{m}$, or $\ln\frac{Q_j}{m} \le x_j(\delta_j) \le \ln\frac{m_j}{Q}, j = 1,2,\cdots,n$. Therefore, we have $|x_j(\delta_j)| \le \max\{|\ln\frac{m_j}{Q}|,|\ln\frac{Q_j}{m}|\} =: K_j, j = 1,2,\cdots,n$.

Combining (6) with the above inequalities, we have

$$|x_i| \le |x_i(\delta_i)| + \int_0^T |x'_i|dt \le K_i + 2T\bar{a}_i, i = 1,2,\cdots,n.$$

Hence, we have $\|x\|_0 \le \sum_{i=1}^n K_i + 2T\sum_{i=1}^n \bar{a}_i = M_0$.

Noticing that $\bar{b}_{ij} = \frac{1}{T}\int_0^T b_{ij}(t)dt = \frac{1}{T}\int_{-\tau_{ij}(0)}^{T-\tau_{ij}(0)} \frac{b_{ij}(\sigma_{ij}(s))}{1-\tau'_{ij}(\sigma_{ij}(s))}ds = \frac{b_{ij}(\mu_{ij})}{1-\tau'_{ij}(\mu_{ij})}$ for some $\mu_{ij} \in [0,T]$, where $t = \sigma_{ij}(s)$ is the inverse function of $s = t - \tau_{ij}(t)(t \in [0,T])$, hence, under the assumption (b), the system $\sum_{j=1}^n \bar{b}_{ij}u_j = \bar{a}_i(i = 1,2,\cdots,n)$ must have one unique positive solution $u^* = (u_1^*,\cdots,u_n^*)$.

Now we take $\Omega = \{x \in X : \|x\|_0 < M\}$, where $M = \max\{M_0, \sum_{j=1}^n |\ln u_j^*|\}$.

One can easily verify that all of the conditions required in Mawhin's Continuation Theorem (see [2] p40) hold. It follows by Mawhin's Continuation Theorem that (3) has at least one T-periodic solution. By the transformations $N_i(t) = e^{x_i(t)}(i = 1,2,\cdots,n)$, we obtain that (1) has at least one positive T-periodic solution. The proof is complete.

References

1. K. Gopalsamy and X. Z. He, Global stability in N-species competition modelled by "pure-delay-type" systems II: nonautonomous case, *Research Report 96-15, School of Math. And Stat., University of Sydney* (1996).
2. R. E. Gaines and J. L. Mawhin, em Lecture Notes in Mathematics, **568** (Springer-Verlag, Berlin, 1977).

BOUNBEDNESS AND PERIODIC SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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In this paper, we consider the Volterra integro-differential equations

$$x'(t) = A(t)x(t) + \int_0^t C(t,s)x(s)ds + f(t), \ t \in R^+$$

and

$$x'(t) = A(t)x(t) + \int_{-\infty}^t C(t,s)x(s)ds + f(t), \ t \in R,$$

where A, C and f are at least continuous. In Section 1, first we state two theorems without proofs concerning the boundedness of solutions of the former equation by employing a Liapunov functional and the Liapunov-Razumikhin method. Then we give two examples to our theorems, and show that our two theorems are independent. In Section 2, we discuss the existence of periodic solutions of the latter equation by employing the theory of minimal solutions, and give an example to our result.

1 Boundedness

Many results have been obtained for boundedness in functional differential equations (for instance, [1-6, 8] and references cited therein). In particular, concerning boundedness in Volterra integro-differential equations, we can find many interesting results in the books [2,3] by Burton and many papers in their references. In this section, first we state two theorems without proofs concerning the boundedness of solutions of linear Volterra integro-differential equations by employing a Liapunov functional and the Liapunov-Razumikhin method. Then we give two examples to our theorems, and show that our two theorems are independent.

Let $R := (-\infty, \infty)$ and $R^+ := [0, \infty)$. Consider a system of linear Volterra integro-differential equations with variable coefficients

$$x'(t) = A(t)x(t) + \int_0^t C(t,s)x(s)ds + f(t), \tag{1}$$

where $x \in R^n$, $A(t) = A + P(t)$, A and P are $n \times n$ matrices. We suppose

that there is a symmetric matrix B with

$$A^T B + BA = -I, \tag{2}$$

where A^T denotes the transpose of A, and I denotes the $n \times n$ identity matrix. Moreover, we suppose that there are positive constants K, k and r with

$$|Bx| \leq K(x^T Bx)^{\frac{1}{2}}, \tag{3}$$

where $|\cdot|$ denotes the Euclidean norm of R^n, and

$$r|x| \leq (x^T Bx)^{\frac{1}{2}} \leq \frac{|x|}{2k}. \tag{4}$$

We ask that $C(t,s)$ is an $n \times n$ continuous matrix function defined for $0 \leq s \leq t < \infty$, and that $f : R^+ \to R^n$ is bounded and continuous, $P(t)$ is a continuous matrix with

$$|P(t)| \leq \rho, \ t \in R^+, \tag{5}$$

where $|P| := \sup\{|Px| : |x| = 1\}$ and ρ is a constant with $0 < \rho \leq 1$. Then we obtain the following result by using a Liapunov functional.

Theorem 9 *Let (2)-(5) hold and suppose that*
(a) there is $\overline{K} > K$ with

$$|x|(k - K\rho - \overline{K} \int_t^\infty |C(u,t)|du) \geq (\overline{K} - K)(|Ax| + |x|)$$

and
(b) for some D with $0 < D < \overline{K} - K + r$,

$$\overline{K} \int_0^t \int_t^\infty |C(u,s)|duds \leq D.$$

Then all solutions of Eq.(1) are bounded.

This theorem can be proved easily by employing a Liapunov functional

$$V(t, x(\cdot)) = (x(t)^T Bx(t))^{\frac{1}{2}} + \overline{K} \int_0^t \int_t^\infty |C(u,s)|du|x(s)|ds,$$

where x is a solution of Eq.(1). So we omit the proof. For the details, see [5].

Now we give an example to Theorem 1.1.

Example 1.1 *Consider the equation*

$$x'(t) = A(t)x(t) + \int_0^t C(t,s)x(s)ds + f(t), \qquad (6)$$

where $A(t) = A + P(t)$ with

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \ and \ P(t) = \frac{1}{3} \begin{pmatrix} \cos t & 0 \\ 0 & \sin t \end{pmatrix}$$

and where $C(t,s) = e^{2(s-t)}I$ and

$$f(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Then it is easy to see that

$$B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

satisfies (2) and we can take $K = k = 1/\sqrt{2}$, $r = 1/2$, and $\rho = 1/3$. For these numbers, Assumption (a) is satisfied with $\overline{K} = 11\sqrt{2}/21 > K$, and Assumption (b) is satisfied with $D = 11\sqrt{2}/84$, where $0 < D < \overline{K} - K + r$. Thus, all solutions of Eq.(6) are bounded.

Now we discuss boundedness of solutions of Eq.(1) by using a Liapunov function instead of a Liapunov functional. Again we consider the linear system (1)

$$x'(t) = A(t)x(t) + \int_0^t C(t,s)x(s)ds + f(t),$$

where $A(t) = A + P(t)$, A is constant, and all characteristic roots of A have negative real parts. Let B be a symmetric $n \times n$ matrix which satisfies Eq.(2), and let α^2 ($\alpha > 0$) and β^2 ($\beta > 0$) be the smallest and largest characteristic roots of B, respectively. By using the Liapunov-Razumikhin method, we obtain the following result.

Theorem 10 *Let the above stated conditions hold, and suppose that there is $M > 0$ with*

$$\int_0^t |BC(t,s)|ds \le M, \ t \ge 0,$$

where $2\beta M/\alpha + 2\rho|B| < 1$. If, in addition, f is bounded, then all solutions of Eq.(1) are bounded.

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This theorem can be easily proved by employing the Liapunov function

$$V(t,x) = x^T B x$$

and the Liapunov-Razumikhin method. So we omit the proof. For the details, see [5].

Now we show an example to Theorem 1.2.

Example 1.2 *Consider the scalar equation*

$$x'(t) = (-2 + \cos t)x(t) + \int_0^t C(t,s)x(s)ds + \sin t, \qquad (7)$$

where $C(t,s)$ is defined by

$$C(t,s) = \begin{cases} st, & 0 \le t < 1 \\ \frac{s}{t^2}, & t \ge 1. \end{cases}$$

Then we can take $B = 1/4$, $\alpha = \beta = 1/2$, and $\rho = 1$, and we obtain

$$\int_0^t |BC(t,s)|ds = \frac{1}{4} \int_0^t st\, ds = \frac{t^3}{8} < \frac{1}{8}, \ 0 \le t < 1,$$

and

$$\int_0^t |BC(t,s)|ds = \frac{1}{4} \int_0^t \frac{s}{t^2}ds = \frac{1}{8}, \ t \ge 1.$$

Thus we can take $M = 1/8$. Since we have

$$2M + 2\rho|B| = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} < 1$$

and $\sin t$ is bounded, all assumptions of Theorem 2 are satisfied, and hence, all solutions of Eq.(7) are bounded.

Remark 1.1. *Eq.(7) is a special case of Eq.(1). Assumption (3) is satisfied with $K = 1/2$, and Assumption (4) is satisfied with $r = 1/2$ and $k = 1$. Since we have*

$$\int_t^\infty |C(u,t)|du = \int_t^\infty \frac{t}{u^2}du = 1, \ t \ge 1,$$

there is no $\overline{K}$ which satisfies Assumption (a) of Theorem 1.1. Thus, Theorem 1.1 is not applicable to Eq.(7).

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On the other hand, again we consider Eq.(6) in Example 1.1. Then we obtain

$$\int_0^t |BC(t,s)|ds = \frac{1}{2} \int_0^t e^{2(s-t)}ds = \frac{1}{4}(1-e^{-2t}) \leq \frac{1}{4} =: M$$

which implies $2\beta M/\alpha + 2\rho|B| = \sqrt{2}/2 + 1/3 = (3\sqrt{2}+2)/6 > 1$. *Therefore an assumption in Theorem 1.2 is not satisfied, and hence, Theorem 1.2 is not applicable to Example 1.1. Thus, Theorems 1.1 and 1.2 are independent.*

2 Periodic Solutions

In this section, we discuss the existence of periodic solutions by employing the theory of minimal solutions. Corresponding to Eq.(1), consider a system of linear Volterra equations

$$x'(t) = A(t)x(t) + \int_{-\infty}^t C(t,s)x(s)ds + f(t), \tag{8}$$

where $A(t+T) = A(t)$, $C(t+T,s+T) = C(t,s)$, and $f(t+T) = f(t)$ for some $T > 0$. In addition to the assumptions in Section 1, we assume the condition

$$\int_{-\infty}^{t-\tau} |C(t,s)|ds \to 0 \ uniformly \ for \ t \in R \ as \ \tau \to \infty. \tag{9}$$

In [7], it is shown that Assumption (9) is equivalent to the continuity of $\int_{-\infty}^t |C(t,s)|ds$ on R. First we prepare the following lemma.

Lemma 2.1 *Let the above stated conditions hold, and suppose that Eq.(1) has an R^+-bounded solution. Then Eq.(8) has an R-bounded solution.*

Proof. Let $\xi(t)$ be an R^+-bounded solution of Eq.(1), and let $H := \sup\{|\xi(t)| : t \in R^+\}$. For each positive integer k, define a function $x_k : [-kT, \infty) \to R^n$ by

$$x_k(t) = \xi(t+kT), \ t \geq -kT.$$

It is easy to see that the set of functions $\{x_k(t)\}$ is uniformly bounded and equicontinuous on each compact interval in R. Taking a subsequence if necessary, we may assume that $\{x_k(t)\}$ converges to a bounded continuous function $x(t)$ uniformly on any compact interval in R.

Now we show that x satisfies Eq.(8) on R. From Assumption (9), for any $\epsilon > 0$ there is $\tau > 0$ with

$$\int_{-\infty}^{t-\tau} |C(t,s)|ds \leq \epsilon, \ t \in R.$$

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Let $t_0 \in R$ be any number. For $t \geq t_0$ and j with $-jT \leq t_0$ we have

$$x_j(t) = x_j(t_0) + \int_{t_0}^{t} (A(s)x_j(s) + \int_{-jT}^{s} C(s,u)x_j(u)du + f(s))ds. \qquad (10)$$

Here we note that

$$\limsup_{j \to \infty} | \int_{-jT}^{s} C(s,u)x_j(u)du - \int_{-\infty}^{s} C(s,u)x(u)du |$$

$$\leq \limsup_{j \to \infty} | \int_{s-\tau}^{s} C(s,u)(x_j(u) - x(u))du | + 2M \int_{-\infty}^{s-\tau} |C(s,u)|du \leq 2M\epsilon.$$

Since ϵ is arbitrary, this inequality yields that

$$\lim_{j \to \infty} \int_{-jT}^{s} C(s,u)x_j(u)ds = \int_{-\infty}^{s} C(s,u)x(u)du, \ s \in [t_0, t].$$

Moreover, for some constant $\Gamma > 0$ we have

$$| \int_{-jT}^{s} C(s,u)x_j(u)du | \leq H \int_{-\infty}^{s} |C(s,u)|du \leq \Gamma, \ s \in [t_0, t].$$

Thus, letting $j \to \infty$ in (10), we obtain

$$x(t) = x(t_0) + \int_{t_0}^{t} (A(s)x(s) + \int_{-\infty}^{s} C(s,u)x(u)du + f(s))ds, \ t \geq t_0,$$

which together with the continuity of $\int_{-\infty}^{s} C(s,u)x(u)du$ in s implies that $x(t)$ satisfies Eq.(8) on $[t_0, \infty)$. Since t_0 is arbitrary, $x(t)$ satisfies Eq.(8) on R.

Now let $h(t)$ be a continuous, positive, integrable function on R. For each R-bounded function $x : R \to R^n$, we define a functional $\lambda(x)$ by

$$\lambda(x) = \sup\{ \int_{-\infty}^{\infty} |x(t+s)|^2 h(s)ds : t \in R \},$$

and a number Λ by

$$\Lambda = \inf\{\lambda(x) : x(t) \ is \ a \ solution \ of \ Eq.(8) \ with \ \sup_{t \in R} |x(t)| \leq H \},$$

where H is a positive constant. Then we have the following two lemmas.

Lemma 2.2. *Under the assumptions in Lemma 2.1, Eq.(8) has a minimal solution, that is, an R-bounded solution of Eq.(8) which attains the value Λ.*

Proof. Lemma 2.1 assures the existence of an R-bounded solution $x(t)$ of Eq.(8) with $\sup_{t \in R} \leq H$ for some $H > 0$. From the definition of Λ, there is a sequence $\{x_j(t)\}$ of R-bounded solutions of Eq.(8) which satisfies

$$\lambda(x_j) \leq \Lambda + \frac{1}{j} \text{ and } \sup_{t \in R} |x_j(t)| \leq H.$$

Clearly the set of functions $\{x_j(t)\}$ is uniformly bounded and equicontinuous on R. Thus the sequence $\{x_j(t)\}$ has a subsequence which converges to an R-bounded solution $y(t)$ of Eq.(8) with $\sup_{t \in R} |y(t)| \leq H$. Moreover, since we have

$$\int_{-\infty}^{\infty} |x_j(s+t)|^2 h(s) ds \leq \lambda(x_j) \leq \Lambda + \frac{1}{j},$$

we obtain

$$\int_{-\infty}^{\infty} |y(s+t)|^2 h(s) ds \leq \Lambda,$$

and hence $\lambda(y) \leq \Lambda$. Thus we have $\lambda(y) = \Lambda$, because $\lambda(y) \geq \Lambda$ from the definition of Λ.

Lemma 2.3. *In addition to the assumptions of Lemma 2.1, if $y_j(t)$ $(j = 1, 2)$ are minimal solutions of Eq.(8), then there is a sequence $\{t_j\}$ with*

$$y_1(t + t_j) - y_2(t + t_j) \to 0$$

uniformly on any compact interval in R as $j \to \infty$.

Proof. Define functions $p, q : R \to R^n$ by

$$p(t) = \frac{y_1(t) + y_2(t)}{2} \text{ and } q(t) = \frac{y_1(t) - y_2(t)}{2}, \ t \in R.$$

Then we have

$$\int_{-\infty}^{\infty} |p(s+t)|^2 h(s) ds + \int_{-\infty}^{\infty} |q(s+t)|^2 h(s) ds$$

$$= \frac{1}{2} \Big(\int_{-\infty}^{\infty} |y_1(s+t)|^2 h(s) ds + \int_{-\infty}^{\infty} |y_2(s+t)|^2 h(s) ds \leq \Lambda,$$

which implies

$$\int_{-\infty}^{\infty} |p(s+t)|^2 h(s) ds \leq \Lambda - \inf \Big\{ \int_{-\infty}^{\infty} |q(s+t)|^2 h(s) ds : t \in R \Big\}.$$

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This together with the definition of Λ yields $\inf\{\int_{-\infty}^{\infty} |q(s+t)|^2 h(s)ds : t \in R\} = 0$, and hence,

$$\inf\{\int_{-\infty}^{\infty} |y_1(s+t) - y_2(s+t)|^2 h(s)ds : t \in R\} = 0.$$

Thus, there is a sequence $\{t_j\}$ with

$$\lim_{j \to \infty} \int_{-\infty}^{\infty} |y_1(s+t_j) - y_2(s+t_j)|^2 h(s)ds = 0. \qquad (11)$$

Since the set of functions $\{y_1(t+t_j) - y_2(t+t_j)\}$ is uniformly bounded and equicontinuous on R, taking a subsequence if necessary, we may assume that the sequence $\{y_1(t+t_j) - y_2(t+t_j)\}$ converges to a continuous function $\eta(t)$ uniformly on any compact interval in R as $j \to \infty$. From (11), for any τ_1 and τ_2 we have

$$\int_{\tau_1}^{\tau_2} |\eta(s)|^2 h(s)ds = 0,$$

which together with the positivity of $h(t)$ on R implies $\eta(t) \equiv 0$ on R. This completes the proof.

Remark 2.1. *In this proof, we need that the norm $|\cdot|$ of R^n is the Euclidean norm.*

From these lemmas, we have the following theorem.

Theorem 11 *Under the assumptions of Lemma 2.1, Eq.(2.1) has a T-periodic solution.*

Proof. Let $y(t)$ be a minimal solution of Eq.(8) ensured in Lemma 2.2. Clearly, $y(t+T)$ is also a minimal solution of Eq.(8). Thus, from Lemma 2.3 there is a sequence $\{t_j\}$ in R with

$$y(t+t_j) - y(t+T+t_j) \to 0 \text{ as } j \to \infty$$

uniformly on any compact interval in R. For each positive integer j, let ν_j be an integer with

$$\nu_j T \le t_j < (\nu_j + 1)T,$$

and let $\sigma_j := t_j - \nu_j T$. Taking a subsequence if necessary, we may assume that

$$\sigma_j \to \sigma \text{ as } j \to \infty$$

for some σ with $0 \le \sigma \le T$, and that for some bounded continuous function $\eta(t)$ on R,

$$y(t+t_j) \to \eta(t) \text{ as } j \to \infty$$

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uniformly on any compact interval in R, since the set of functions $\{y(t + t_j)\}$ is uniformly bounded and equicontinuous on R. Clearly, $\eta(t)$ is T-periodic.

Now we show that $\eta(t - \sigma)$ is a T-periodic solution of Eq.(8). Let $t_0 \in R$ be any number, and let $t \geq t_0$. Since $y(t)$ satisfies Eq.(8), by integrating the both sides of Eq.(8) on $[t_0 - \sigma + t_j, t - \sigma + t_j]$, we have

$$y(t - \sigma + t_j) = y(t_0 - \sigma + t_j) + \int_{t_0 - \sigma + t_j}^{t - \sigma + t_j} (A(s)y(s) + \int_{-\infty}^{s} C(s, u)y(u)du + f(s))ds$$

$$= y(t_0 - \sigma + t_j) + \int_{t_0}^{t} (A(s - \sigma + \sigma_j)y(s - \sigma + t_j)ds$$

$$+ \int_{t_0}^{t} (\int_{-\infty}^{s} C(s - \sigma + \sigma_j, u - \sigma + \sigma_j)y(u - \sigma + t_j)du + f(s - \sigma + \sigma_j))ds.$$

By letting $j \to \infty$ and using the same arguments as in the proof of Lemma 2.1, for $t \geq t_0$ we have

$$\eta(t - \sigma) = \eta(t_0 - \sigma) + \int_{t_0}^{t} (A(s)\eta(s - \sigma) + \int_{-\infty}^{s} C(s, u)\eta(u - \sigma)du + f(s))ds.$$

Since t_0 is arbitrary, $\eta(t - \sigma)$ satisfies Eq.(8) on R. Thus, $\eta(t - \sigma)$ is a T-periodic solution of Eq.(8).

Now, by combining Theorems in Sections 1 and 2, we obtain the following corollary, which we state without proof.

Corollary 2.1. *Under the assumptions of Theorem 1.1 (or Theorem 1.2) and Lemma 2.1, Eq.(8) has a T-periodic solution.*

Finally we show an example to Corollary 2.1.

Example 2.1. *Consider the scalar equation*

$$x'(t) = (-2 + \frac{1}{3}\cos t)x(t) + \int_{-\infty}^{t} e^{2(s-t)}x(s)ds + \sin t. \qquad (12)$$

Then we can take $B = 1/4$, $K = r = 1/2$, $k = 1$, $\rho = 1/3$, and $T = 2\pi$. It is easy to see that all assumptions of Corollary 2.1 are satisfied. Thus, Eq.(12) has a 2π-periodic solution.

1. T. A. Burton, Boundedness in functional differential equations, *Funkcialaj Ekvacioj* **25** (1982), 51-77.

2. T. A. Burton, Volterra Integral and Differential Equations, *Academic Press* 1983.

3. T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, *Academic Press* 1985.

4. T. Furumochi, Stability and boundedness in functional differential equations, *J. Math. Anal. Appl.* **113** (1986), 473-489.

5. T. Furumochi and S. Matsuoka, Stability and boundedness in Volterra integro-differential equations, *Mem. Fac. Sci. Eng. Shimane Univ. Series B:Mathematical Science* **32**(1999), 25-40.

6. R. Grimmer and G. Seifert, Stability properties of Volterra integro-differential equations, *J. Differential Equations* **19**(1975), 142-166.

7. Y. Hino and S. Murakami, Periodic solutions of a linear Volterra system, *Proc. Equadiff Conference, ed. by C. M. Dafermos, G. Ladas, and G. Papanicolaou, Dekker* (1989), 319-326.

8. R. K. Miller, Nonlinear Volterra Integral Equations, *Benjamin* 1971.

NEW CRITERIA OF STABILITY AND BOUNDEDNESS FOR FUNCTIONAL DIFFERENTIAL SYSTEMS[d]

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In this paper, we present some new Razumikhin type criteria that guarantee the solutions of retarded functional differential systems with finite or infinite delay to be stable or bounded. Here, the restrictions on the time derivatives of Liapunov-Razumikhin function are weakened in some sense.

1 Introduction

It is well-known that most of stability and boundedness criteria are given by the Liapunov second method[3,4], and the Razumikhin technique is one of the useful tools. However, in some cases, the restrictions on derivatives of Liapunov-Razumikhin functions are too strict to be used.The purpose of our work is to weaken these restrictions and to give some new criteria. We consider

$$x^{'}(t) = f(t, x_t) \tag{1}$$

where $x \in R^n$, $x_t = x(t + \theta)$, $\theta \in [-r, 0]$, $r = \text{const.} > 0$, $f : R^+ \times C([-r, 0], R^n) \to R^n$, $R^+ = [0, +\infty)$. If the stability problem is considered, we assume $f(t, 0) \equiv 0$. Also, we consider retarded functional differential systems with infinite delay:

$$x'(t) = f(t, x(s)) \qquad \alpha \le s \le t. \tag{2}$$

where $\alpha \ge -\infty$, $f : R^+ \times C([\alpha, 0], R^n) \mapsto R^n$.

Let $|\cdot|$ be the Euclid norm of R^n,

$$\|\varphi\| = \begin{cases} \sup_{\theta \in [-r, 0]} |\varphi(\theta)| & \text{for} & \varphi(\theta) \in C([-r, 0], R^n) \\ \sup_{\theta \in [\alpha, 0]} |\varphi(\theta)| & \text{for} & \varphi(\theta) \in C([\alpha, 0], R^n) \end{cases} . \tag{3}$$

We always assume that system (1) or (2) satisfies the conditions[1] which guarantee existence, uniqueness and extendedness of the solution through (t_0, φ). The uniform boundedness and uniform stability definitions are considered as the traditional definitions. A continuous function $u : R^+ \mapsto R^+$ is called to be a wedge function, if $u(0) = 0$ and $u(t)$ increases strictly. A Liapunov-Razumikhin function is a continuous function $V : R^+ \mapsto R^+$. The time derivative of V along the solution $x(t)$ of system (1) or (2) is defined as

$$D^+ V(t, x(t))|_{(*)} := \lim_{h \to 0^+} \sup \frac{1}{h} [V(t + h, x(t + h)) - V(t, x(t))] \tag{4}$$

where (*) means (1) or (2).

2 Main Results

In this section, we present some new criteria of boundedness and stability for functional differential systems with both finite and infinite delay.

Theorem 12 Suppose that there exist two wedge functions u, w, a Liapunov-Razumikhin function $V(t, x)$ and two real number sequences $\{\underline{H}_i\}$, $\{\overline{H}_i\}$, $i \in I^+ = 0, 1, 2, \cdots$, satisfying

    i) $u(|x|) \le V(t, x(t)) \le w(|x|)$, $t \ge t_0 - r$, $u(s) \to +\infty$ as $s \to +\infty$;

    ii) $\underline{H}_i < \overline{H}_i$, $[\underline{H}_i, \overline{H}_i] \bigcap [\underline{H}_j, \overline{H}_j] = \phi, i \ne j$, $i, j \in I^+$, $\lim_{i \to +\infty} \underline{H}_i = +\infty$;

    iii) $D^+ V(t, x(t))|_{(1)} \le 0$, whenever $V(t, x(t)) \in [\underline{H}_i, \overline{H}_i]$, and $V(t - s, x(t - s)) \le V(t, x(t))$, $s \in [-r, 0]$; then the solutions of system (1) are uniformly bounded.

In the above theorem, if condition $\lim_{i \to +\infty} \underline{H}_i = +\infty$ is replaced by $\lim_{i \to +\infty} \overline{H}_i = 0$, the zero solution of system (1) is uniformly stable.

For systems with infinite delay, similarly, we can gave the criteria of boundedness and stability. For instance, we obtain

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Theorem 13 Suppose that there exist two wedge functions u, w, a Liapunov-Razumikhin function $V(t, x)$ and two real number sequences $\{\underline{H}_i\}$, $\{\overline{H}_i\}$, $i \in I^+ = 1, 2, ...$, satisfies

i) $u(|x|) \leq V(t, x(t)) \leq w(|x|)$, $t \geq \alpha, u(s) \rightarrow +\infty$ as $s \rightarrow +\infty$;

ii) $\underline{H}_i < \overline{H}_i$, $[\underline{H}_i, \overline{H}_i] \cap [\underline{H}_j, \overline{H}_j] = \phi, i \neq j, i, j \in I^+, \lim_{i \to +\infty} \underline{H}_i = +\infty$;

iii) $D^+ V(t, x)|_{(2)} \leq 0$, whenever $V(t, x) \in [\underline{H}_i, \overline{H}_i]$, and $V(s, x(s)) \leq V(t, x(t))$, $s \in [\mu, t]$, where $\mu = \max\{\alpha, t - r(L)\}$, $L > 0$ is arbitrary, $r(L) > 0$ is assumed to be existed such that $\sup_{s \in [\mu, t]} |x(s)| \leq L$; then the solutions of (2) are uniformly bounded.

Analogously, we can give the stability theorem for (2). It is omitted here. Finally, by two examples we show the effect of our theorems.

Example 1 Consider delay system

$$\begin{cases} \frac{dx(t)}{dt} = \frac{\sqrt{2}}{2}x(t - r) + x(t) \sin(x^2(t) + y^2(t)) \\ \frac{dy(t)}{dt} = \frac{\sqrt{2}}{2}y(t - r) + y(t) \sin(x^2(t) + y^2(t)) \end{cases} \tag{5}$$

Taking $u(s) = v(s) = \frac{1}{2}s^2, V(x, y) = \frac{1}{2}(x^2 + y^2)$, we have

$$D^+ V(x, y)|_{(5)} \leq \sqrt{2}(V(x(t), y(t)) + V(x(t - r), y(t - r)))$$
$$+ 2V(x(t), y(t)) \sin(x^2(t) + y^2(t)).$$

Hence, $D^+ V(x, y)|_{(5)} \leq 0$, whenever $x^2 + y^2 \in [2k\pi + \frac{5\pi}{4}, 2k\pi + \frac{7\pi}{4}]$ and $V(x(s), y(s)) \leq V(x(t), y(t))$, for $s \in [t - r, t]$. By Theorem 1, we conclude that all the solutions of system (5) are uniformly bounded.

Example 2 Consider system

$$\begin{cases} \frac{dx(t)}{dt} = \begin{cases} \frac{\sqrt{2}}{2}x(t - r) + x(t) \sin(\frac{1}{\sqrt[3]{x^2(t) + y^2(t)}}), & x^2 + y^2 \neq 0 \\ \frac{\sqrt{2}}{2}x(t - r), & x^2 + y^2 = 0 \end{cases} \\ \frac{dy(t)}{dt} = \begin{cases} \frac{\sqrt{2}}{2}y(t - r) + y(t) \sin(\frac{1}{\sqrt[3]{x^2(t) + y^2(t)}}), & x^2 + y^2 \neq 0 \\ \frac{\sqrt{2}}{2}y(t - r), & x^2 + y^2 = 0 \end{cases} \end{cases} \tag{6}$$

Also, we take the wedges and Razumikhin-Liapunov function as in Example 1. We have $D^+ V(x(t), y(t))|_{(6)} \leq V(x(t), y(t))(\sqrt{2} + 2\sin(\frac{1}{\sqrt[3]{x^2 + y^2}}) \leq 0$, whenever, $2V(x, y) \in [(\frac{1}{2k\pi + \frac{7\pi}{4}})^2, (\frac{1}{2k\pi + \frac{5\pi}{4}})^2], k \in I^+$, and $V(x(s), y(s)) \leq V(x(t), y(t))$, for $s \in [t - r, t]$. By Theorem 2, we conclude that the zero solution of system (??) is uniformly stable.

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References

1. T. A. Burton & Shunian Zhang, Ann. Mat. Pura. Appl. **145**, 129-158 (1986).
2. J.K. Hale, Theory of functional differential equations, *springer-verlag, New York*,1977.
3. Dingwen Lai, Chinese Science Bull. **19**, 1161-1163 (1984).
4. Xiaoxin Liao, Absolute stability of nonlinear control systems, *Kluwer Academic Publishers, Netherlands*, 1993.
5. X. Mao, Stochastic Processes and their Applications **65**, 233-250 (1996).
6. Shunian Zhang, Chinese Ann. of Math. **19**, 73-82 (1998).
7. Zhang Shunian, Razumikhin type theorems on boundedness, Northeastern Math. J. **2**, 235-246 (1996).

H_∞ OBSERVER DESIGN FOR DISCRETE DELAY SYSTEMS

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This paper deals with the observer-based H_∞ controller design for discrete-time systems with time-delay in state. In terms of modified Riccati inequality for discrete-time linear systems, we design two kind of observer based feedback control laws which guarantee the quadratic stability of the closed-loop control system and reduce the effect of the disturbance input on the controlled output to a prescribed level.

1 Introduction

Time delay is commonly encountered in various engineering systems, such as chemical processes, long transmission lines in pneumatic, hydraulic and rolling mill systems. And time delay usually results in unsatisfactory performances and is frequently a source of instability[1]. Recently, by extending the state space H_∞ controller design methods, several authors have proposed H_∞ control methods for linear systems with time delay[2−8]. The problem of robust H_∞ state feedback control design for linear continuous-time system with delay was addressed in paper [2] − [6], and the problem of static out put-feedback control for this system was considered by Shaked et.al.[7]. Besides, Song[8] dealt with the H_∞ control problem for discrete-time linear system with delayed state.

In this paper, we extend the result of Song in order to design a state observer and observer state feedback control laws for discrete-time linear system with time-delay. We present two methods for designing observer-based H_∞ controller for such system. In terms of modified Riccati inequality for discrete-time linear systems, we give a sufficient condition for quadratic stability with H_∞ norm-bound. According to the given sufficient condition one can obtain control parameters of an observer-based H_∞ control law by solving two discrete Riccati-like inequality.

2 Model description and background results

Let the system to be controlled be represented by the following equation:

$$\begin{cases} x(k+1) = A_1 x(k) + A_2 x(k-h) + B_1 w(k) + B_2 u(k), \\ z(k) = C_1 x(k) + D_1 u(k), \\ y(k) = C_2 x(k), \end{cases} \tag{1}$$

where $x(k) \in \Re^n$ is the state, $w(k) \in \Re^{m_1}$ is exogenous disturbance which belongs to $l_2[0, \infty)$, $u(k) \in \Re^{m_2}$ is the control input, $z(k) \in \Re^{p_1}$ is the controlled output, $y(k) \in \Re^{p_2}$ is the measured output and h is a positive integer for time delay. $A_1, A_2, B_1, B_2, C_1, C_2$ and D_1 are known real constant matrices of appropriate dimensions. We also assume that A_2 has been factored as $A_2 = A_m \cdot A_n$, where $A_m \in \Re^{n \times l}, A_n \in \Re^{l \times n}$ and $l = Rank(A_2)$. For the sake of technical simplification, without loss of generality, we shall make the following assumption:

Assumption 1: $D_1^T \lceil C_1 \; D_1 \rceil = \lceil 0 \; I_{m_2} \rceil$.

Before proceeding further, we will give some preliminary results. Let us consider the following discrete-time delay system:

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$$\begin{cases} x(k+1) = A_1 x(k) + A_2 x(k-h) + B_1 w(k), \\ z(k) = Cx(k), \end{cases} \tag{2}$$

where A_1, A_2, B_1 and C are constant matrices with appropriate dimensions. And A_2 can be factored as $A_2 = A_m \cdot A_n$.

Definition 1: For some given positive constant γ, the discrete-time delay system (2) is said to be quadratically stable with an H_∞-norm bound γ if the following properties (P_1) and (P_2) hold:

$(P_1.)$ The system (2) is asymptotically stable when $w = 0$;

$(P_2.)$ $\|G_{zw}\|_\infty < \gamma$.

In Definition 1, G_{zw} is the transfer function matrix of the system (2), and

$$G_{zw}(z) = C(zI_n - A_1 - A_2 z^{-h})^{-1} B_1, \qquad z = e^{j\theta}, \qquad \theta \in [0, 2\pi].$$

Lemma 1: For some given positive constant γ, if there exist a positive definite matrix P and a positive scalar ϵ satisfying the following inequalities:

$$A_1^T (P^{-1} - \gamma^{-2} B_1 B_1^T - \epsilon A_m A_m^T)^{-1} A_1 - P + C^T C + \tfrac{1}{\epsilon} A_n^T A_n < 0,$$
$$P^{-1} - \gamma^{-2} B_1 B_1^T - \epsilon A_m A_m^T > 0.$$

Then, the system (2) is quadratically stable with an H_∞-norm bound γ.

From the results given in Song(1998), one can get the above conditions.

3 Main results

Now, consider the following observer-based output feedback control law:

$$\begin{cases} \hat{x}(k+1) = (A_1 - LC_2) \hat{x}(k) + A_2 \hat{x}(k-h) + Ly(k) + B_2 u(k), \\ u(k) = -K \hat{x}(k), \end{cases} \tag{3}$$

where $\hat{x}(k) \in \Re^n$ is estimated state of $x(k)$, and L, K are the gain matrices with appropriate dimensions. By introducing the observer error $e(k) = x(k) - \hat{x}(k)$, we get the following augmented system:

$$\begin{cases} \overline{x}(k+1) = \overline{A}_1 \overline{x}(k) + \overline{A}_2 \overline{x}(k-h) + \overline{B}_1 w(k), \\ z(k) = \overline{C}_1 \overline{x}(k), \end{cases} \tag{4}$$

where $\overline{x}^T(k) = [\hat{x}^T(k), e^T(k)]$, $\overline{C}_1 = [C_1 - D_1 K \quad C_1]$ and

$$\overline{A}_1 = \begin{bmatrix} A_1 - B_2 K & LC_2 \\ 0 & A_1 - LC_2 \end{bmatrix}, \quad \overline{A}_2 = \begin{bmatrix} A_2 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \overline{B}_1 = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}.$$

Theorem 1: Consider the system (1) and suppose that the control parameters are given by

$$K = (I_{m_2} + B_2^T N B_2)^{-1} B_2^T N A_1, \tag{5}$$

$$L = A_1 M C_2^T (C_2 M C_2^T + \varphi I_{p_2})^{-1}, \tag{6}$$

where

$$M = (P_2^{-1} - \frac{1}{\epsilon_1} A_n^T A_n - (1 + \frac{1}{\epsilon_2}) C_1^T C_1 - \frac{1}{\epsilon_3} C_2^T C_2)^{-1} > 0,$$

$$N = (P_1^{-1} - \epsilon_1 A_m A_m^T - \epsilon_3 L L^T)^{-1} > 0, \quad \varphi, \ \epsilon_i (i = 1, 2, 3)$$

are some positive constants, P_1 and P_2 are, respectively, positive-definite solution matrices to the following Riccati-like inequalities:

$$A_1^T (P_1^{-1} - \epsilon_1 A_m A_m^T - \epsilon_3 L L^T + B_2 B_2^T)^{-1} A_1 - P_1 + \frac{1}{\epsilon_1} A_n^T A_n + (1 + \epsilon_2) C_1^T C_1 < 0, \tag{7}$$

$$A_1 [M^{-1} + C_2^T C_2 (\frac{2}{\varphi} + \frac{1}{\varphi^2} M C_2^T C_2)]^{-1} A_1^T - P_2 + \epsilon_1 A_m A_m^T + \gamma^{-2} B_1 B_1^T < 0. \tag{8}$$

Then, the closed-loop system is quadratically stable with H_∞-norm bound γ.

Proof: By Lemma 1, if we define P as $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2^{-1} \end{bmatrix}$, where P_1 and P_2 are, respectively, positive-definite solution matrices to the Riccati-like inequalities(7) and (8), then through some routine algebraic manipulations one can show that P_1 and P_2 satisfy

$$\begin{bmatrix} P_1 - \frac{1}{\epsilon_1} A_n^T A_n - (1 + \epsilon_2) C_1^T C_1 - K^T K & (A_1 - B_2 K)^T \\ (A_1 - B_2 K) & P_1^{-1} - \epsilon_1 A_m A_m^T - \epsilon_3 L L^T \end{bmatrix} > 0,$$

$$\begin{bmatrix} P_2 - \gamma^{-2} B_1 B_1^T - \epsilon_1 A_m A_m^T & A_1 - L C_2 \\ (A_1 - L C_2)^T & P_2^{-1} - \frac{1}{\epsilon_1} A_n^T A_n - (1 + \frac{1}{\epsilon_2}) C_1^T C_1 - \frac{1}{\epsilon_3} C_2^T C_2 \end{bmatrix} > 0.$$

Choose $K = (I_{m_2} + B_2^T N B_2)^{-1} B_2^T N A_1$, $L = A_1 M C_2^T (C_2 M C_2^T + \varphi I_{p_2})^{-1}$, where $M = (P_2^{-1} - \frac{1}{\epsilon_1} A_n^T A_n - (1 + \frac{1}{\epsilon_2}) C_1^T C_1 - \frac{1}{\epsilon_3} C_2^T C_2)^{-1} > 0$, $N = (P_1^{-1} - \epsilon_1 A_m A_m^T - \epsilon_3 L L^T)^{-1} > 0$, and $\varphi, \ \epsilon_i (i = 1, 2, 3)$ are some positive constants.

From the well-know Schur complement, one can conclude that the above two matrix inequalities are equivalent to (7) and (8).   Q. E. D.

It should be noted that the above control law requires instantaneous values of the estimated state and past values of the estimated state. Now, we construct another observer-based output feedback control law which requires only instantaneous values as:

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$$\begin{cases} \hat{x}(k+1) = (A_1 - LC_2)\,\hat{x}(k) + B_2 u(k) + Ly(k), \\ u(k) = -K\,\hat{x}(k), \end{cases} \tag{9}$$

where $\hat{x}(k) \in \Re^n$ is estimated state of $x(k)$. Corresponding, we have the following augmented system:

$$\begin{cases} \bar{x}(k+1) = \widetilde{A}_1 \bar{x}(k) + \widetilde{A}_2 \bar{x}(k-h) + \widetilde{B}_1 w(k) \\ z(k) = \widetilde{C}_1 \bar{x}(k) \end{cases} \tag{10}$$

where $\bar{x}^T(k) = [\hat{x}^T(k), e^T(k)]$, and

$$\widetilde{A}_1 = \begin{bmatrix} A_1 - B_2 K & LC_2 \\ 0 & A_1 - LC_2 \end{bmatrix}, \qquad \widetilde{A}_2 = \begin{bmatrix} 0 & 0 \\ A_2 & A_2 \end{bmatrix},$$

$$\widetilde{B}_1 = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}, \qquad\qquad \widetilde{C}_1 = \begin{bmatrix} C_1 - D_1 K & C_1 \end{bmatrix}.$$

$\widetilde{A}_2$ can be factored as $\widetilde{A}_2 = \begin{bmatrix} 0 \\ A_m \end{bmatrix} \begin{bmatrix} A_n & A_n \end{bmatrix}$, so we have the following theorem.

Theorem 2: Consider the system (1) and the observer-based feedback control law (9), and suppose that the control parameters are given by

$$K = (I_{m_2} + B_2^T N' B_2)^{-1} B_2^T N' A_1, \tag{11}$$

$$L = A_1 M' C_2^T (C_2 M' C_2^T + \varphi' I_{p_2})^{-1}, \tag{12}$$

where $M' = (P_2^{-1} - \frac{2}{\epsilon_1} A_n^T A_n - (1 + \frac{1}{\epsilon_2}) C_1^T C_1 - \frac{1}{\epsilon_3} C_2^T C_2)^{-1} > 0$, $N' = (P_1^{-1} - \epsilon_3 LL^T)^{-1} > 0$, and φ', $\epsilon_i (i = 1, 2, 3)$ are some positive constants. P_1 and P_2 are, respectively, positive-definite solution matrices to the following Riccati-like inequalities:

$$A_1^T (P_1^{-1} - \epsilon_3 LL^T + B_2 B_2^T)^{-1} A_1 - P_1 + \frac{2}{\epsilon_1} A_n^T A_n + (1 + \epsilon_2) C_1^T C_1 < 0, \tag{13}$$

$$A_1 [M'^{-1} + C_2^T C_2 (\frac{2}{\varphi'} + \frac{1}{\varphi'^2} M' C_2^T C_2)]^{-1} A_1^T - P_2 + \epsilon_1 A_m A_m^T + \gamma^{-2} B_1 B_1^T < 0. \tag{14}$$

Then, the closed-loop system is quadratically stable with an H_∞-Norm bound γ.

Proof: As proved in the Theorem 1, through the same manipulation, one can conclude that the closed-loop system is quadratically stable with an H_∞-Norm bound γ.

Remark 1: In this paper, we construct two kind of observer-based output feedback control laws. The first type need instantaneous values and past values of the estimated state, while the second type requires only instantaneous values. When we don't know the time delay clearly, the first one is more difficult to implement, but the second one can. However, the first one is less conservative than the second one in designing the control parameters.

Remark 2: The conditions in Theorem 1 or Theorem 2 are developed from the well-known Riccati inequality which plays an important role in H_∞ control of discrete-time linear systems. So one can solve the Riccati-like inequalities (7), (8) or (13), (14) to get the gain matrices which solve the problem of H_∞ control .

Acknowledgments

References

1. Choi H. H. and Chung M. J., " Robust observer-based H_∞ controller design for linear uncertain time-delay systems", *Automatica,* 1997, 33(9); 1749-1752
2. Pila A., Shaked U. and Desouza C. E., " Robust H_∞ control of continuous time-varying linear systems with time delay " , *Proceedings of the 35th conference on Decision and Control,* Kobe, Japan, 1996; 1368-1369
3. Kokame H., Konishi K. and Mori T., " Robust H_∞ control for linear delay-differential systems with time-varying uncertainties ", *Proceedings of the 35th conference on Decision and Control,* Kobe, Japan, 1996; 2097-2102
4. Ge J.H., Frank P.M. and Lin C.F., "Robust H_∞ state feedback control for linear systems with state delay and parameter uncertainy",*Automatica,* 1996, 32(8);1183-1185
5. Niculescu S. I., "H$_\infty$ memoryless control with α-stability constraint for time-delay systems: An LMI approach" , *IEEE Trans. Automat. Contr.,* 1998, 43(5), 739-743
6. Gu Y., Li Q., and Cheng Z. et. al., " Robust reliable H_∞ control for linear uncertain systems with delayed state " , *Control theory and Applications,* 1999, 16(1); 149-151(in Chinese)
7. Shaked U. and Yaesh I., "H$_\infty$ static output feedback control of linear continuous-time systems with delay" , *IEEE Trans. Automat. Contr.,* 1998, 43(10); 1431-1436

8. Song S. H. and Kim J. K., "H_∞ control of discrete-time linear systems with norm-bounded uncertainties and time-delay in state", *Automatica*, 1998, 34(1); 137-139

9. Choi H. H., and Chung M. J., " Observer-based H_∞ control design for state delayed linear systems ", *Automatica*, 1996,32; 1073-1075

10. Niculesw S. I. et. al , " Robust H_∞ memoryless control for uncertain linear systems with time-varying delay ", *In 3rd European Control Conf.*, Roma, Italy, 1995; 1814-1818

HOPF BIFURCATION FOR A ECOLOGICAL MATHEMATICAL MODEL ON MICROBE POPULATIONS[e]

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The ecological Model of a class of the two microbe populations with second-order growth rate is studied.The methods of qualitative theory of ordinary differential equations are used in the four-dimension phase space. The qualitative property and stability of equilibrium points are analysed. The center manifold and criterion quantity of Hopf bifurcation are computed by using symbolic software based on computer algebra.

1 Model and background

In microbe biochemical reactions there are complex metabolism processes and usually many populations[1]. Among these populations there may be a relation that the metabolate in the first process construct the nutriment of the second process.Now we consider only two important populations and denote them by x_1 and x_2. The model is discrebed as follows:

$$\ddot{x}_1 = a_{01}\dot{x}_1 + a_1 x_1 + a_{11} x_1^2$$
$$\ddot{x}_2 = a_{02}\dot{x}_2 + a_2 x_2 + a_{21} x_1 x_2 + a_{22} x_2^2 \tag{1.1}$$

In order to use the method of qualitative theory of ordinary differential equations, we set $y_i = \dot{x}_i$ (i=1,2) It follows immediately that

$$\begin{aligned}
\dot{x}_1 &= y_1 \\
\dot{y}_1 &= a_{01} y_1 + a_1 x_1 + a_{11} x_1{}^2 \\
\dot{x}_2 &= y_2 \\
\dot{y}_2 &= a_{02} y_2 + a_2 x_2 + a_{21} x_1 x_2 + a_{22} x_2^2
\end{aligned} \qquad (1.2)$$

In the next sections we will analyse this set of 4-dimension equations in detail.

2 Equilibrium points and their stability

Using matric notations we can write (1.2) as
$$\dot{X} = F(X) \qquad (2.1)$$
where $X = (x_1, x_2, y_1, y_2)^T$

Let F(X)=0, we have found that equilibrium points of (2.1):

$O_1(0,0,0,0)$ $O_2(-\frac{a_1}{a_{11}}, 0, 0, 0)$ $O_3(0, -\frac{a_2}{a_{22}}, 0, 0)$ $O_4(m_1, m_2, 0, 0)$

where $m_1 = x_1^* = -a_1/a_{11}$ $m_2 = x_2^* = \frac{a_1 a_{21} - a_{11} a_2}{a_{11} a_{22}}$

Condition I Supose $a_1 > 0, a_2 > 0, a_{11} < 0, a_{22} < 0$ and let a_{21} be approporitely small.

Under condition I the points $O_1, O_2, O_3, O_4 \in \mathbf{R}_4{}^+ = \{(x_1, x_2, y_1, y_2) | x_i \geq 0, y_i \geq 0\}$ when $x_1 > 0, x_2 > 0$, we call the point O_4 as positive equilibrium point.

The first order variational equation of (2.1) is
$$\dot{X} = AX \qquad (2.2)$$
$A = DF(X^*)$ is the Jacobian matrix of F(X), X^* denotes one of the four equilibrium points. The eigenpolynomial is

$$\Phi(\lambda) = |\lambda E - A| = (\lambda^2 - \lambda a_{01} - b_1)(\lambda^2 - a_{02}\lambda - b_2) \qquad (2.3)$$

where $b_1 = a_1 + 2a_{11} x_1^*$, $b_2 = a_2 + a_{21} x_1^* + 2a_{22} x_2^*$

Condition II Suppose $a_{01} a_{02} \neq 0, b_1 b_2 \neq 0$ in (2.1).

Lemma 2.1 under condition II. All the equilibrium points are hyperbolic equilibrium points[2][3].

Lemma 2.2 suppose that the condition II holds thus the classification of the singular points is determined by sign of $a_0 i$ and b_i. I). If there is a b_i such that $b_i(X^*) > 0$ then the singular point is a generalized saddle point . II).If $b_i(X^*) < 0$ for all i, then A) When $a_{0i} < 0$ for all i, X^* is generalized node

with asymptotic stability. B) When $a_{0i} > 0$ for all i, then X^* is a generalized node with nonstability. C) When $a_{01}\, a_{02} < 0$, then X^* is a generalized saddle point.

Theorem 2.3 Suppose condition I, II hold and $a_{0i} < 0$ for all i, then all the singular points of (2.2) are hyporbolic singular points: O_1, O_2, O_3 are generalized saddle points. They are unstable. O_4 is generalized node point with asymptotic stability. It is also called O^+ attractor.[3]

We have completed the local qualitative analysis on the linear variational equation. The topological structure of the set of solutions of (2.1)in the neighborhood of the equilibrium points are clear according to Hartman theorem [2].

3 Hopf bifurcation under small perturbation

From(1.2) we know that population x_1 can be completely determined by the subsystem

$$
\begin{aligned}
\dot{x_1} &= y_1 \\
\dot{y_1} &= a_{01}\, y_1 + a_1 x_1 + a_{11}{x_1}^2
\end{aligned}
\tag{3.1}
$$

On above equations the interaction between x_1 and x_2 has not been considered. In the practical biochemical reaction population x_2 may react to x_1 with the increase of x_2. Consequently we will take a small pertrubation on the origin system. Thus the considered system will be more complete.

$$
\begin{aligned}
\dot{x_1} &= y_1 \\
\dot{y_1} &= a_{01}\, y_1 + a_1 x_1 + a_{11}{x_1}^2 + a_{12}(x_1 - m_1)(x_2 - m_2) \\
\dot{x_2} &= y_2 \\
\dot{y_2} &= a_{02}\, y_2 + a_2 x_2 + a_{21}x_1 x_2 + a_{22}x_2^2
\end{aligned}
\tag{3.2}
$$

Where a_{12} sufficently small. The equilibrium persists and m_1 and m_2 are the same as section 2. Let $\xi = x_1 - m_1$ $\eta = x_2 - m_2$ $\overline{X} = (\xi, \eta, y_1, y_2)^T$,the (3.2) is replaced by

$$
\dot{\overline{X}} = A\overline{X} + G
\tag{3.3}
$$

$$
A\mid_{o_4} =
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
b_1 & 0 & a_{01} & 0 \\
c_1 & b_2 & 0 & a_{02}
\end{pmatrix}
\quad
G =
\begin{pmatrix}
0 \\
0 \\
a_{11}\,\xi^2 + a_{12}\,\xi\,\eta \\
a_{21}\,\xi\,\eta + a_{22}\,\eta^2
\end{pmatrix}
$$

Where $b_1 = a_{11}\, m_1, b_2 = a_{22}\, m_2, c_1 = a_{21}\, m_1$

Condition III Suppose $a_{01} = 0, a_{02} < 0$.

Theorem 3.1[4] suppose the condition I, III are satisfied. For system (3.3)

then there are a 2-dimensional local stable manifold S and a 2-dimensional local center manifold M. These manifolds are all tangent to the respective stable and central eigensubspace of the linear system.

It is very complex and difficult to calculate the centre manifolds manually. Using sign algebra software we completed this work on computer successfully. For simplifying we denote $\lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^-$ by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ respectively. Let
$D_{12} = \lambda_1^2 - a_{02}\,\lambda_1 - b_2 \quad D_{22} = \lambda_2^2 - a_{02}\,\lambda_2 - b_2$
Under conditions I, III . We also assume $a_{01} = 0, a_{02}^2 + 4b_2 > 0$. Then we have $\lambda_{1,2} = \pm i\beta (\beta > 0)$ and λ_3, λ_4 are negative real roots.

On the above supposes there are four linearly independent eigenvectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$. They construct the transform matrix.

$$
(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}) = \begin{pmatrix} D_{12} & D_{22} & 0 & 0 \\ c_1 & c_1 & 1 & 1 \\ \lambda_1\,D_{12} & \lambda_2\,D_{22} & 0 & 0 \\ \lambda_1\,c_1 & \lambda_2\,c_1 & \lambda_3 & \lambda_4 \end{pmatrix}
$$

Where $\lambda_1, \lambda_2, D_{12}, D_{22}$ are complex number. $\mathbf{e_1}, \mathbf{e_2}$ are complex conjugate vectors. By dividing real and imaginary parts we obtained the real form transform matrix denoted by

$$
P = \begin{pmatrix} s & -t & 0 & 0 \\ c_1 & 0 & 1 & 1 \\ -\beta t & -\beta s & 0 & 0 \\ 0 & -c_1\beta & \lambda_3 & \lambda_4 \end{pmatrix}
$$

Here $s = -\beta^2 - b_2$, $t = -a_{02}\,\beta$, $\beta = \sqrt{-b_1}$
Using the transformation $\overline{X} = P\,\mathbf{u}, \mathbf{u} = (u_1, u_2, u_3, u_4)^T$ We obtained

$$\dot{\mathbf{u}} = P^{-1}A\,P\,\mathbf{u} + P^{-1}\,\mathbf{G} = J\,\mathbf{u} + \overline{\mathbf{G}} \tag{3.4}$$

Where

$$
J = \begin{pmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}
$$

the inverse of the martric P is as follows

$$
P^{-1} = \frac{1}{|P|} \begin{pmatrix} \beta s(\lambda_4 - \lambda_3) & 0 & -t(\lambda_4 - \lambda_3) & 0 \\ -\beta t(\lambda_4 - \lambda_3) & 0 & -s(\lambda_4 - \lambda_3) & 0 \\ \beta c_1(\beta t - s\lambda_4) & \beta \lambda_4(s^2 + t^2) & c_1(\beta s + t\lambda_4) & -\beta(s^2 + t^2) \\ -\beta c_1(\beta t - s\lambda_3) & -\beta \lambda_3(s^2 + t^2) & -c_1(\beta s + t\lambda_3) & \beta(s^2 + t^2) \end{pmatrix}
$$

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$$|P| = detP = \beta\,(s^2 + t^2)(\lambda_4 - \lambda_3)$$

For simplifying we denote P^{-1} by $\mathbf{Q} = (q_{ij})$

$$\overline{\mathbf{G}} = \mathbf{Q}\mathbf{G} = (g_1, g_2, g_3, g_4)^T \tag{3.5}$$

Now we write (3.4) in the form

$$\dot{\mathbf{u}}_{\mathbf{c}} = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = C\,\mathbf{u}_{\mathbf{c}} + \overline{\mathbf{G}}_{\mathbf{c}} \tag{3.6}$$

$$\dot{\mathbf{u}}_{\mathbf{B}} = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} + \begin{pmatrix} g_3 \\ g_4 \end{pmatrix} = B\,\mathbf{u}_{\mathbf{B}} + \overline{\mathbf{G}}_{\mathbf{B}} \tag{3.7}$$

Where $\mathbf{u}_{\mathbf{c}} = (u_1, u_2)^T$, $\mathbf{u}_{\mathbf{B}} = (u_3, u_4)^T$ the centre maniforld is
$$\mathbf{u}_{\mathbf{B}} = \mathbf{h}(\mathbf{u}_{\mathbf{c}})$$
where $\mathbf{h}(\mathbf{u}_{\mathbf{c}})$ satisfy the following equation

$$D\mathbf{h}(\mathbf{u}_{\mathbf{c}})[C\,\mathbf{u}_{\mathbf{c}} + \overline{\mathbf{G}_{\mathbf{c}}}(\mathbf{u}_{\mathbf{c}}, \mathbf{h}(\mathbf{u}_{\mathbf{c}}))] = B\,\mathbf{h}(\mathbf{u}_{\mathbf{c}}) + \overline{\mathbf{G}_{\mathbf{B}}}(\mathbf{u}_{\mathbf{c}}, \mathbf{h}(\mathbf{u}_{\mathbf{c}})) \tag{3.8}$$

$$\mathbf{h}(0)=0 \qquad D\mathbf{h}(0)=0 \qquad\qquad \text{Let}$$

$$h_i = h_{i1}\,u_1^2 + h_{i2}u_1u_2 + h_{i3}u_2^2 + h_{i4}u_1^3 + \dots \quad (i = 3, 4) \tag{3.9}$$

Substituting (3.9) into (3.8), the h_{ij},(i=3,4,j=1,2,3,4,...) can be determined by balancing powers of coeficients for each component. When paramenters of the system are given the h_{ij} can be actually computed. We substitute these h_{ij} into (3.9) and obtained the approximate expressions of the center manifold through second-order.
In the actual problem the parameter a_{0i} relates with growth rate μ and constance b. It can be written as $a_{01} = \mu - b$, when $\mu = b$, $a_{01} = 0$, a_{01} changes with change of μ. Under some conditions the periodic solution of Hopf type may occur. The vector field restricted to the center manifold is given by

$$\dot{\mathbf{u}}_{\mathbf{c}} = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} g_1(u_1, u_2, h_3(u_1, u_2), h_4(u_1, u_2)) \\ g_2(u_1, u_2, h_3(u_1, u_2), h_4(u_1, u_2)) \end{pmatrix}$$

Lemma 3.2 If the system satisfies the following conditions: 1.Transversality condition $d = d\alpha(\mu)/d\mu \neq 0$, 2.The criterion quantity $a \neq 0$ (the expression of a is below) then Hopf bifurcation occur.
By analysing signs of d and a we can obtain the bifurcation director and stability of the periodic solutions. There are four possibilities : case 1: $d > 0, a > 0$.

case 2: $d > 0, a < 0$. case 3: $d < 0, a > 0$. case 4: $d < 0, a < 0$.

Remark 1. For $a < 0$, the bifurcating periodic orbit is always stable, the case $a < 0$ is referred to as supercritical bifurcation and for $a > 0$ the bifurcating periodic is always unstable. This case is referred to as a sub- critical bifurcation.

For our system d is easy to calculate. To calculate a we use the explicit formular. It can be found [4] the criterion quantity a is given by

$$
\begin{aligned}
a &= \frac{1}{16}[g_{1u_1u_1u_1} + g_{1u_1u_2u_2} + g_{2u_1u_1u_2} + g_{2u_2u_2u_2}] \\
&+ \frac{1}{16\beta}[g_{1u_1u_2}(g_{1u_1u_1} + g_{1u_2u_2}) - g_{2u_1u_2}(g_{2u_1u_1} + g_{2u_2u_2}) \\
&- g_{1u_1u_1}g_{2u_1u_1} + g_{1u_2u_2}g_{2u_2u_2}]
\end{aligned}
$$

Where
$$a = \frac{1}{16}aa + \frac{1}{16\beta}bb$$

$$
\begin{aligned}
aa &= (-2\,a_{12}\,t\,(h_{32} + h_{42}) + 3\,a_{12}\,s\,(2\,h_{31} + 2\,h_{41}) + a_{12}\,s\,(2\,h_{33} + 2\,h_{43}))\,q_{13} \\
&+ (2\,a_{12}\,s\,(h_{32} + h_{42}) - 3\,a_{12}\,t\,(2\,h_{33} + 2\,h_{43}) - a_{12}\,t\,(2\,h_{31} + 2\,h_{41}))\,q_{23}
\end{aligned}
$$

$$
\begin{aligned}
bb &= (-2\,a_{11}\,ts - a_{12}\,tc_1)\,(2\,a_{11}\,s^2 + 2\,a_{12}\,sc_1 + 2\,a_{11}\,t^2)\,q_{13}{}^2 \\
&+ \left(-(2\,a_{11}\,s^2 + 2\,a_{12}\,sc_1)^2 + 4\,a_{11}{}^2 t^4\right) q_{23}\,q_{13} + (2\,a_{11}\,ts + a_{12}\,tc_1) \\
&\quad (2\,a_{11}\,s^2 + 2\,a_{12}\,sc_1 + 2\,a_{11}\,t^2)\,q_{23}{}^2
\end{aligned}
$$

Theorem 3.3 At the origin of the system (3.4) the Hopf transverse condition holds: $d > 0$. If $a \neq 0$ then there is a Hopf bifurcation. Case $a < 0$ is referred to as a supercritical bifurcation:an asymptotically stable periodic orbit will occur near the origin when $\mu > b$ and $|\mu - b|$ is sufficiently small. Case $a > 0$ is referred to as a subcritical bifurcation: an unstable periodic orbit will occur near the origin when $\mu < b$ and $|\mu - b|$ is sufficiently small.

4 Example

We take:$a_{01} = 0$, $a_{02} = -10$, $a_1 = 25$, $a_{11} = -0.1$, $a_2 = 16$, $a_{21} = 0.01$, $a_{22} = -0.1$

There is a positive equilibrium point with cordilation $m_1 = 250, m_2 = 185$. The eigenvalues are as follows: $\lambda_1, \lambda_2 = \pm 5i$, $\lambda_3 = -2.45049$, $\lambda_4 = -7.54951$

We have found the criterion quantity :

when $a_{12} = 0.005$, $a = 0.000098358105$,if $a_{12} = -0.005$, $a = -0.000098307776$ Therefore,for $a_{12} = 0.005$ we abtained the subcritical bifurcation for $a_{12} = -0.005$ we abtained the supercritical bifurcation,a stable periodic orbit will occure near the equilibrium point O_4.

Refrences

1. Yuan Xiaofeng, Liu Shize e.t,J.Sichuan Univ. 34(3) 373-376 (1997).
2. Hartman.P. Ordinary Differential Equation, Birkhauser,228-250(1982).
3. Liu Shize. Mathematical Advance, 8(3), 217-242 (1965).
4. Wiggins.S,Introduction to Applied Nonlinear Dynamical System and Chaos,Springer-verlag(1990).

A NEW Z_p INDEX THEORY AND ITS APPLICATIONS

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In this paper, we introduce a new geometrical Z_p index theory and give an example to illustrate its applications.

1 Definition of Z_p index and its properties

Let $p > 1$, a positive integer, X be a Banach space and μ be a linear isometric action of Z_p on X, where Z_p is a cyclic group with order p. A subset A of X will be called μ-invariant if $\mu(A) \subset A$. A continuous map $f : A \to X$ is called μ-equivariant if $f(\mu x) = \mu f(x), \forall x \in A$. Set

$$\Sigma = \{A \subset X \text{ is closed and } \mu\text{-invariant}\},$$

For any given positive integer m, we denote the set of all divisors of m by E_m.

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For any set $A \in \Sigma$ and positive integer k, if the map $\varphi : A \to C^k$ is continuous and there exist $m_1, \cdots, m_k \in E_m$, such that $\forall x \in A, \varphi_j(\mu x) = e^{i(2\pi m_j/p)} \varphi_j(x), j = 1, 2, \cdots, k$, we call φ a $(\mu, E_m)^k$-type map, where $\varphi(x) = (\varphi_1(x), \varphi_2(x), \cdots, \varphi_k(x))$. Set

$$Y = \{z \in C | \arg z = \frac{2\pi j}{p}, j = 0, 1, \cdots, p - 1\}.$$

Now, we define our index as follows, $i_m : \Sigma \to N \cup \{+\infty\}, \forall A \in \Sigma$,

$$i_m(A) = \min\{k \in N | \text{there exists a } (\mu, E_m)^k\text{-type map } \varphi : A \to Y^k \backslash \{\theta\}\}.$$
(1.1)

If no such map exists, we define $i_m(A) = +\infty$ and set $i_m(\emptyset) = 0$.

Theorem 1.1 The index i_m, as usual, has following properties:

(i) $i_m(A) = 0 \Longleftrightarrow A = \emptyset$.

(ii) (Monotonicity) If $\psi : A \to B$ is a continuous μ-equivariant map, then $i_m(A) \leq i_m(B)$, where $A, B \in \Sigma$.

(iii) (Subadditivity) $i_m(A \cup B) \leq i_m(A) + i_m(B)$.

(iv) (Superinvariance) $i_m(A) \leq i_m(\overline{h(A)})$, where h is continuous and μ-equivariant.

(v) (Continuity) $\forall A \in \Sigma$, if A is a compact set, then there exists a closed neighborhood $\Omega(A)$ of A such that $i_m(\Omega(A)) = i_m(A)$.

Theorem 1.2 The index i_m also have normality under certain conditions, i.e., $\forall x \in X$, denote $[x] = \{\mu^j x | j = 0, 1, \cdots, p - 1\}, k = \min\{j : \mu^j x = x\}$, if $k \neq 1$ and $p/k \in E_m$, then $i_m([x]) = 1$.

Denote by X_{2a} the μ-invariant linear subspace of X with dimension $2a$. We identify X_{2a} with C^a by $x = (x_1, x_2, \cdots, x_{2a}) \in X_{2a}$ with

$$(x_1 + ix_{a+1}, \cdots, x_a + ix_{2a}) \in C^a.$$

A Z_p action μ on X_{2a} is given by

$$\mu z = (e^{i(2\pi k_1/p)} z_1, \cdots, e^{i(2\pi k_a/p)} z_a),$$

for $z = (z_1, \cdots, z_a) \in C^a$. where $k_j \neq 0$ integers, $j = 1, 2, \cdots, a$.

We denote by $< n_1, n_2, \cdots, n_s >$ and $[n_1, n_2, \cdots, n_s]$ the greatest common divisor and the smallest common multiple of s positive integers $n_1, \cdots, n_s$ respectively.

Theorem 1.3 (The dimensional property of index i_m) Let Ω be a open bounded μ-invariant neighborhood of θ in X_{2a}. If $m^a \neq 0 \pmod{p}$, then

$$i_m(\partial \Omega) = 2a,$$

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where

$$m = [|k_1|, |k_2|, \cdots, |k_a|].$$

Remark The geometrical index theory has been well known to be used to find out multiple solutions or subharmonic solutions to a variety of differential equations such as nonautonomous Hamiltonian systems, wave equations and elliptic systems, etc..For detail, one can refer to [1,2,3,4]. However, our Z_p geometrical index i_m can be exploited to obtain more periodic solutions or subharmonic solutions than the usual Z_p index theory appeared in current papers.

2 Applications of index i_m to FDE

Let

$$X = \{u \in L^\alpha([0, 2\pi p], R^N) : \int_0^{2\pi p} u(s)ds = 0\} \qquad (2.1)$$

where $\alpha > 1$. A linear isometric action μ of Z_p on Banach space X is defined by $\mu(u(t)) = u(t + 2\pi)$.

In the following, we denote by scm (l) the smallest common multiple of integers $1, 2, \cdots, l$, and for given $p > 1$,

$$n \stackrel{\triangle}{=} \max\{l \in N | [\text{scm }(l)]^{lN} \neq 0 (\text{mod } p)\}, \qquad (2.2)$$

$$m \stackrel{\triangle}{=} \text{scm }(n). \qquad (2.3)$$

For given closed μ-invariant subset of X, we define index $i_m(A)$ as (1.1). Consider the $2nN$-dimensional subspace X_{2nN} of X

$$X_{2nN} = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_n,$$

where

$$Y_j = \text{span}\{\sin \frac{jt}{p} e_k, \cos \frac{jt}{p} e_k, k = 1, 2, \cdots, N\}$$

where $e_1 = (1, 0, \cdots, 0), \cdots, e_N = (0, \cdots, 0, 1)$.

For any $u(t) \in X_{2nN}$, we may express $u(t)$ as

$$u(t) = \sum_{|j|=1}^n \xi_j e^{i(jt/p)}, \xi_j \in C^N, \xi_{-j} = \bar{\xi}_j, \qquad (2.4)$$

$\forall \varepsilon > 0, \eta > 0$, we take a subset of X_{2nN} as follows

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$$S_{n,\eta,\varepsilon} = \{u(t) \in X_{2nN} : u(t) = \sum_{|j|=1}^{n} \xi_j e^{i(jt/p)}, \xi_{-j} = \bar{\xi}_j, |\xi_1| = \eta, \sum_{j=2}^{n} |\xi_j| = \varepsilon\}.$$

$$(2.5)$$

In view of the conclusion of Theorem 1.3 and the meaning of m, n, we get

Theorem 2.1 $i_m(S_{n,\eta,\varepsilon}) = 2nN$.

Consider the following second-order functional differential equation:

$$\frac{d^2x}{dt^2} + (3 + \sin t)[x^{\frac{1}{3}}(t - 6\pi) + x^{\frac{1}{3}}(t - 12\pi)] = 0, \qquad (2.6)$$

and corresponding ordinary differential equation:

$$\frac{d^2x}{dt^2} + 2(3 + \sin t)x^{\frac{1}{3}} = 0, \qquad (2.7)$$

where $x \in R$. By classical technique as used in [3,4] and the index theory introduced in section 1, there exists at least 4 periodic solutions with minimal period 6π to (2.6) and (2.7).

Remark Our index i_m can be exploited to obtain more subharmonic solutions than those obtained in [3] and [4]. For example, according to the results of [3] and [4], for equation (2.7), the authors obtained only 2 and 1 periodic solutions with minimal period 6π, respectively. Another example, let $p = 3^{30}$, authors in [3] obtained only 4 subharmonic solutions of Hamiltonian systems. Under the same assumptions, our result is 28 subharmonic solutions.

References

1. K.C.Chang, Critical point theory and its applications. Science and Technical Press, Shanghai, China(1980).
2. J.Q.Liu, Acta Math. Sinica (New series) **5**, 193-196(1989).
3. J.Q.Liu and Z.Q.Wang, Nonlinear analysis (T.M.A.) **7**, 803-821(1993).
4. R.Michalek and G.Tarantello, J. Differential Equations **72**, 28-55(1988).

CRITERIA OF 3-DIMENSIONAL HOMOGENEOUS VECTOR FIELDS

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In this paper, a criterion for hyperbolicity and a criterion for existence of closed invariant cone of 3-dimensional homogeneous vector fields are obtained, as well as a bifurcation criterion of the saddle connection and a bifurcation criterion of the periodic orbits.

1 Introduction

In this paper, the criteria of invariant cone of 3-dimensional homogeneous vector fields is investigated. The relative background about homogeneous vector fields can be found in [1] $\sim$ [2]. Let $f(x) : R^3 \to R^3$ be a homogeneous vector field of degree k : i.e., $f(\alpha x) = \alpha^k f(x)$, and $f(x)$ be continuously differentiable. Let $r(x) = x^T x = |x|$, where $x \in R^3$, T denotes transpose of x, $|x|$ Eulcideean norm of x. Based on the homogeneity of f and r, we have $kf(x) = f'(x)x$, $r(x) = r'(x)x$, (so, $r'(x) = x^T/r$), and $r''(x)(x, \cdot) = 0$, where $f'(x)$ is the Jacobian matrix of f. The blowing-up $\bar{f}$ of f is defined by $\bar{f} = \tilde{f}/r^{k-1}$, where $\tilde{f}$ be defined by $\Phi_* \tilde{f} = f, \Phi : S^2 \times R \to R^3$ be defined by $(u, r) \to ru, S^2 = \{x \in R^3; r(x) = 1\}$ is the unit sphere in R^3. We can easily obtain

$$\bar{f}(u, r) = (f(u) - r'(u)f(u)u, rr'(u)f(u)). \qquad (1)$$

The vector field $h(u) := f(u) - r'(u)f(u)u$ is called the projection of f on the unit sphere. Since the vector field f is homogeneous, for any orbit $\Gamma \subset S^2$ of h, the surface $S(\Gamma) = \{tp; t \in R, p \in \Gamma\}$ is invariant by f, which is called invariant cone of f. If Γ is a periodic orbit, $S(\Gamma)$ is called a closed invariant cone. If Γ is a saddle connection (particularly, a homoclic loop), $S(\Gamma)$ is called a saddle connection invariant cone (particularly, a homoclinic closed invariant cone). If Γ is hyperbolic, $S(\Gamma)$ is called hyperbolic. If Γ is stable (unstable), $S(\Gamma)$ ia called stable (unstable).

Lemma 1 *Let $f(u_0) = \lambda_0 u_0$ for some $\lambda_0 \in R, u_0 \in S^2$, then:*

(1) u_0 is a singularity of h, and u_0 is an eigenvector of $h'(u_0)$, with eigenvalue 0.

(2) An eigenvalue of $f'(u_0)$ is $k\lambda_0$, with eigenvector u_0. Moreover, suppose that the eigenvalues of $f'(u_0)$ are $k\lambda_0, \lambda_1, \lambda_2$, with eigenvector $u_0, u_1, u_2,$

respectively, and linearly independent, then the eigenvalues of $h'(u_0)$ are $0, \lambda_1 - \lambda_0, \lambda_2 - \lambda_0$, with eigenvector $u_0, u_1 - r'(u_0)u_1u_0, u_2 - r'(u_0)u_2u_0$, respectively.

Proof. $f(u_0) = \lambda_0 u_0$ implies $h(u_0) = 0$. $h(u)$ can be rewritten as homogeneous form: $h(u) = r^2(u)f(u) - r'(u)f(u)u$. According to homogeneity we obtain $h'(u_0)u_0 = (k+2)h(u_0) = 0$, $f'(u_0)u_0 = kf(u_0) = k\lambda_0 u_0$ and

$$h'(u) = 2r(u)f(u)r'(u) + r^2(u)f'(u) - uf^T(u)r''(u) - ur'f'(u) - r'(u)f(u)I. \quad (2)$$

Thus $h'(u_0)u_1 = (\lambda_1 - \lambda_0)(u_1 - r'(u_0)u_1u_0)$. Using $h(u_0)u_0 = 0$, the second assertion can be obtained.

2   Main Results

In this section, we state our main results for the invariant cone of homogeneous vector field.

Theorem 14 *If the integral along a closed orbit Γ of h*

$$I = \int_0^\tau (\operatorname{tr} f'(u) - (k+2)r'(u)f(u))ds \quad (3)$$

is nonzero, where τ is the period of Γ, then when $I < 0 (> 0)$, $S(\Gamma)$ is stable (unstable) hyperbolic isolated invariant cone.

Proof. Let $\varphi(s)$ be a parametrization of the closed orbit Γ. Then φ is also the unit normal vector at point $\varphi(s) \in S^2$, and the unit tangent vector of Γ can be written as $\varphi'(s) = h'(\varphi)/|h(\varphi)|$. We take φ, φ' and $\psi := \varphi \times \varphi'$ (where $\times$ denotes the cross product of vectors) as the new orthogonal coordinate axes along Γ. Let the new coordinate of the point $u \in S^2$ be (s, n, m), where u is close to Γ, n, m are function of s. Then $u = \varphi + n\psi + m\varphi$. Thus $u' = \varphi' + n'\psi + n\psi' + m'\varphi + m\varphi'$. From $\psi^T \varphi' = 0$, $\psi^T \psi' = (|\psi|^2)'/2 = 0$, we get $\psi \cdot h(u) = \psi \cdot u' = n'$ where $\cdot$ denotes inner product of vectors. Thus

$$n' = \psi \cdot h(\varphi + n\psi + m\varphi) = \psi \cdot h'.(\varphi)\psi n + o(n) \quad (4)$$

From (2) and $r^{2'}(\varphi) = 2r(\varphi)r'(\varphi) = 2\varphi^T$, we get $\psi \cdot h'(\varphi)\psi = \psi \cdot f'(\varphi)\psi - r'(\varphi)f(\varphi)$. According to the orthogonal conditions of φ, φ', and ψ, we find

$$\psi \cdot f'(\varphi)\psi = \operatorname{tr} f'(\varphi) - (k+1)r'(\varphi)f(\varphi) - h^T(\varphi)f'(\varphi)h(\varphi)/|h(\varphi)|^2. \quad (5)$$

Using $r'(\varphi)h(\varphi) = \varphi^T h(\varphi) = h(\varphi) \cdot \varphi^T = 0$, we find

$$h^T(\varphi)f'(\varphi)h(\varphi)/|h(\varphi)|^2 = h^T(\varphi)h'(\varphi)h(\varphi)/|h(\varphi)|^2 + r'(\varphi)f(\varphi), \quad (6)$$

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and

$$\int_o^\tau h \cdot h'h\,ds = \int \frac{d|h|^2}{|h|^2} = 0 \tag{7}$$

Thus

$$\int_0^\tau \psi \cdot h'\psi\,ds = \int_0^\tau (\operatorname{tr}f'(\varphi) - (k+2)r'(\varphi)f(\varphi))ds \tag{8}$$

Based on (3) and (4), we can obtain Theorem 2.1.

Theorem 15 *If the expression* $\operatorname{tr}f'(\varphi) - (k+2)r'(\varphi)f(\varphi)$ *is not identically zero and does not change sign on a simply connected region* $D \subset S^2$, *then* $f(x)$ *has not closed invariant cone lying entirely in* $S(D)$.

Proof. Let $u \in S^2$, then $u \times h(u)$ is also a vector field on S^2, and $(u \times h(u)) \cdot u \equiv 0$. $\operatorname{curl}(u \times h(u)) = u\operatorname{div}h - h\operatorname{div}u + u'h - h'u$. From (2) we have

$$\operatorname{div}h(u) = 2r'(u)f(u) + \operatorname{div}f'(u) - u^T r''(u)f(u) - r'(u)f'(u)u - 3r'(u)f(u)$$
$$= 2r'f(u) + \operatorname{div}f'(u) - r'(u)f(u) - kr'(u)f(u) - 3r'(u)f(u)$$
$$= \operatorname{div}f'(u) - (k+2)r'(u)f(u).$$

Hence from $u \cdot h = 0, u \cdot u'h = u^T r''(u)h = r'(u)h = 0$, and $u \cdot h'u = 0$ we get

$$u \cdot (\operatorname{curl}(u \times h(u)) = \operatorname{div}h - u \cdot h\operatorname{div}u + u \cdot u'h - u \cdot h'u = \operatorname{div}h. \tag{9}$$

Suppose Γ is a closed orbit in D, then Γ is the boundary of a simply connected region $\Omega \subset D$. The vector $g \times h$ is orthogonal to tangent vector $h(u)$ on the boundary. Based on Stokes Theorem

$$\int_\Omega u \cdot [\operatorname{curl}(u \times h)]dA = \int_\Gamma h \cdot (u \times h)ds = 0 \tag{10}$$

which cannot be true in view of condition of Theorem 2.2.

Theorem 16 *If the integral along a saddle connection* Γ *of projection of* f,

$$\int_{-\infty}^{+\infty} \exp(- \int_0^s \operatorname{tr}f'(u) - (k+2)r'(u)f(u)ds)(u \times f(u)) \cdot p(u)ds \tag{11}$$

is nonzero, then homogeneous vector fields of degree k : $f(x) + \varepsilon p(x)$, $\varepsilon \neq 0$ *small, has no saddle connection invariant cone in a neighborhood of* $S(\Gamma)$.

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Proof. For the projection of $f(x) + \varepsilon p(x)$,

$$h_\varepsilon(u) := h(u) + \varepsilon h^*(u) := f(u) - u \cdot f(u)u + \varepsilon(p(u) - u \cdot p(u)u), \qquad (12)$$

the stable and unstable orbits of saddle, which lies in a neighborhood of Γ, can be expressed as follows, with uniform validity in the indicated time intervals,

$$x_\varepsilon^u = x_0(t - t_0) + \varepsilon x u_1^u(t, t_0) + O(\varepsilon^2), \ t \in (-\infty, t^0], \qquad (13)$$

$$x_\varepsilon^s = x_0(t - t_0) + \varepsilon x u_1^s(t, t_0) + O(\varepsilon^2), \ t \in (t_0, +\infty) \qquad (14)$$

where $x_0(t - t_0)$ is the expression of Γ.

$$d(t_0) := (\varphi \times \frac{h}{|h|}) \cdot (x_1^u - x_1^s) = \frac{1}{|h|}(\varphi \times h) \cdot (x_1^u - x_1^s)\varepsilon + O(\varepsilon^2) \qquad (15)$$

where $\varphi = x_0(t - t_0)$. From (13) we get

$$\frac{dx_\varepsilon^u}{dt} = h(x_0(t - t_0)) + \varepsilon(h'(x_0(t - t_0))x_1^u + h^*(x_0(t - t_0))) + O(\varepsilon^2). \qquad (16)$$

Hence

$$\frac{dx_1^u}{dt} = h'x_1^u + h^*. \qquad (17)$$

Analogous calculations yields

$$\frac{dx_1^s}{dt} = h'x_1^s + h^*. \qquad (18)$$

Defining $\Delta^u(t, t_0) := (\varphi \times h) \cdot x_1^u, \Delta^s(t, t_0) := (\varphi \times h) \cdot x_1^s, \Delta(t, t_0) := \Delta^u(t, t_0) - \Delta s(t, t_0)$. Then

$$d(t_0) = \varepsilon \frac{\Delta(t_0, t_0)}{|h|} + O(\varepsilon^2), \qquad (19)$$

$$\begin{aligned}
\frac{d\Delta^u(t, t_0)}{dt} &= \frac{d(\varphi \times h)}{dt} \cdot x_1^u + (\varphi \times h) \cdot \frac{dx_1^u}{dt} \\
&= (\varphi \times f'h) \cdot x_1^u + (\varphi \times h) \cdot f'x_1^u \\
&\quad -2(u \cdot f)(\varphi \times h) \cdot x_1^u + (\varphi \times h) \times h^* \\
&= (\mathrm{tr}f' - (k+2)u \times f)\Delta^u(t, t_0) + (u \times f) \cdot p,
\end{aligned}$$

$$\frac{d\Delta^s(t, t_0)}{dt} = (\mathrm{tr}f' - (k+2)u \cdot f)\Delta^s(t, t_0) + (u \times f) \cdot p. \qquad (20)$$

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Integrating and using Lemma 1.1, we can obtain

$$\Delta^u(t_0, t_0) = \int_{-\infty}^{t_0} \exp\left(\int_t^{t_0} (\operatorname{tr} f' - (k+2)u \cdot f) ds\right)(u \times f) \cdot p \, dt, \qquad (21)$$

$$\Delta^s(t_0, t_0) = -\int_{t_0}^{+\infty} \exp\left(-\int_{t_0}^{t} (\operatorname{tr} f' - (k+2)u \cdot f) ds\right)(u \times f) \cdot p \, dt. \qquad (22)$$

This proves the theorem 2.3.

The same idea can be applied to periodic orbit bifurcation.

Theorem 17 *If the integral along a periodic orbit* Γ *of projection* $f(u)$ *of* f

$$\int_0^{\tau} \exp\left(-\int_0^{s} (\operatorname{tr} f'(u) - (k+2)u \cdot f(u))) ds\right)(u \times f(u)) \cdot p(u) ds \qquad (23)$$

is nonzero, where τ *is the period of* Γ, *then the homogeneous vector fields* $f(x) + \varepsilon p(x)$ *of degree* $k, \varepsilon \neq 0$ *small, has no closed invariant cone, or has the only one closed invariant cone in a neighborhood of* $D(\Gamma)$.

References

1. K. P. Hadeler, *J. Differential Equations*, **95**, 183 (1992).
2. M. I. T. Camacho, *Trans. Amer. Soc.*, **268**, 79 (1981).
3. M. I. T. Camacho, *J. Differntial Equations*, **97**, 288(1992).
4. P. Bonckaert, F. Dumortier and S. V. Strien, *Ergod. Th. and Dynam. Sys.*, **9**, 281 (1989).
5. Y. He, *J. Guangxi Normal University* (Natural Science Edition), **14(3)**, 28 (1996).

OSCILLATIONS OF SECOND ORDER IMPULSIVE ADVANCED DIFFERENTIAL EQUATIONS

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Sufficient conditions are obtained for oscillations of all solutions of a class of second order impulsive advanced differential equations.

1 Introduction and preliminaries

The oscillatory theory of first order impulsive delay differential equations has been extensively developed during the past few years, see [4, 5]. However, there are not much concerning the oscillatory properties of the second order impulsive differential equations with or without delay, which is an important mathematical model of many evolution prosess, see [6, 7]. In this paper, we consider the second order impulsive advanced differential equation

$$\begin{cases} (r(t)x'(t))' + f(t, x(t+\tau)) = 0, t \geq t_0, t \neq t_k, k = 1, 2, .., \\ x(t_k^+) = I_k(x(t_k)), \qquad\qquad x'(t_k^+) = N_k(x'(t_k)), k = 1, 2, ..., \end{cases} \tag{1}$$

where $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, $\lim\limits_{k \to +\infty} t_k = +\infty$, τ is a positive constant.

Throughout this paper, the following conditions are assumed:

(i). $r \in C(R, (0, +\infty)), f \in C(R \times R, R), xf(t, x) > 0(x \neq 0)$; and $\dfrac{f(t, x)}{\psi(x)} \geq Q(t)(x \neq 0)$, where $Q \in C(R, [0, +\infty))$, and $x\psi(x) > 0(x \neq 0), \psi'(x) \geq 0$;

(ii). $I_k, N_k \in C(R, R)$ and there exist positive numbers $a_k, \overline{a}_k, b_k, \overline{b}_k$ such that

$$\overline{a}_k \leq \frac{I_k(x)}{x} \leq a_k, \quad \overline{b}_k \leq \frac{N_k(x)}{x} \leq b_k \ \ (x \neq 0, k = 1, 2, ...).$$

By a solution of Eq.(1) we mean a real valued function $x(t)$ defined on $[t_0, +\infty)$ which satisfies Eq.(1). A solution of Eq.(1) is said to be nonoscillatory

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if this solution is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

Lemma 2 [1]. *Assume that*

(a1). the sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ with $\lim\limits_{k \to +\infty} t_k = +\infty$;

(a2). $m \in PC'(R_+, R)$ is left continuous at t_k for $k = 1, 2, ...$;

(a3). for $k = 1, 2, ..., t \ge t_0$,

$$m'(t) \le p(t)m(t) + q(t), t \ne t_k, \tag{2}$$

$$m(t_k^+) \le d_k m(t_k) + b_k, \tag{3}$$

where $p, q \in C(R_+, R), d_k \ge 0$ and b_k are real constants.
Then,

$$
\begin{aligned}
m(t) \le m(t_0) \prod_{t_0 < t_k < t} d_k exp \left(\int_{t_0}^t p(s)ds \right) \\
+ \int_{t_0}^t \prod_{s < t_k < t} d_k exp \left(\int_s^t p(\sigma)d\sigma \right) q(s)ds \\
+ \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_j exp \left(\int_{t_k}^t p(s)ds \right) b_k.
\end{aligned}
\tag{4}
$$

Remark 1.1 If the inequalities (2) and (3) are reversed then in the conclusion the inequality (4) is also reversed.

2   Main results

Lemma 3 *Let $x(t)$ be a solution of Eq.(1). Assume that there exists some $T \ge t_0$ such that $x(t) > 0$ for $t \ge T$ and the following conditions hold:*

(iii). Conditions (i) and (ii) are valid.   (iv). $\lim\limits_{t \to +\infty} \int_{t_0}^t \frac{1}{r(s)} \prod_{t_0 < t_k < s} \frac{\overline{b}_k}{a_k} ds = +\infty$.

Then $x'(t) \ge 0$ for $t \in [T, t_l] \bigcup \left(\bigcup_{k=l}^{+\infty} (t_k, t_{k+1}] \right)$, where $l = \min\{k : t_k \ge T\}$.

Proof. At first, we claim that $x'(t_k) \ge 0$ holds for any $k \ge l$. Otherwise, then there exists some j such that $j \ge l$ and $x'(t_j) < 0$. From Eq.(1) and (ii), we have

$$x'(t_j^+) = N_j(x'(t_j)) \le \overline{b}_j x'(t_j) < 0.$$

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Set $x'(t_j^+) = -\alpha(\alpha > 0)$. By Eq.(1) and (i), for $t \in \bigcup\limits_{i=1}^{+\infty}(t_{j+i-1}, t_{j+i}]$ we get

$$(r(t)x'(t))' = -f(t, x(t+\tau)) \leq -Q(t)\psi(x(t+\tau)) \leq 0.$$

Hence, $r(t)x'(t)\cdot$is monotonically nonincreasing in $(t_{j+i-1}, t_{j+i}], i = 1, 2, \dots$. One can easily prove that, for any positive integer $n \geq 2$,

$$x'(t_{j+n}) \leq -\frac{r(t_j)}{r(t_{j+n})}\left(\prod_{i=1}^{n-1}\bar{b}_{j+i}\right)\alpha < 0.$$

Consider the following inequalities

$$(r(t)x'(t))' \leq 0, \quad t > t_j, t \neq t_k, k = j+1, j+2, \dots,$$

$$x'(t_k^+) \leq \bar{b}_k x'(t_k), \quad k = j+1, j+2, \dots .$$

From Lemma 1.1 it follows that

$$x'(t) \leq \frac{r(t_j)x'(t_j^+)}{r(t)}\prod_{t_j < t_k < t}\bar{b}_k.$$

Since $x(t_k^+) \leq a_k x(t_k)$ for $k = j+1, j+2, \dots$, by Lemma 1.1 we obtain

$$
\begin{aligned}
x(t) &\leq x(t_j^+)\prod_{t_j < t_k < t} a_k + \int_{t_j^+}^t \prod_{s < t_k < t} a_k \left(\frac{r(t_j)x'(t_j^+)}{r(s)}\prod_{t_j < t_k < s}\bar{b}_k\right) ds \\
&\leq \prod_{t_j < t_k < t} a_k \left[x(t_j^+) - \alpha r(t_j)\int_{t_j^+}^t \frac{1}{r(s)}\prod_{t_j < t_k < s}\frac{\bar{b}_k}{a_k} ds\right].
\end{aligned}
\tag{5}
$$

Since $x(t) > 0$ for $t \geq T$, the inequality (5) contradicts (iv) of Lemma 2.1. Therefore, $x'(t_k) \geq 0$ for $k \geq l$. The condition (ii) implies $x'(t_k^+) \geq \bar{b}_k x'(t_k) \geq 0$ for any $k \geq l$. Because $r(t)x'(t)$ is nonincreasing in $(t_k, t_{k+1}]$, it is clear that $x'(t) \geq \frac{r(t_{k+1})}{r(t)}x'(t_{k+1}) \geq 0$ for $t \in (t_k, t_{k+1}], k \geq l$. On the other hand, $x'(t) \geq \frac{r(t_l)}{r(t)}x'(t_l) \geq 0$ for any $t \in [T, t_l]$. Thus the proof of Lemma 2.1 is complete.

Remark 2.1 In the case that $x(t)$ is eventually negative, under the conditions (iii) and (iv), it can be proved similarly that

$$x'(t) \leq 0 \text{ for } t \in [T, t_l] \bigcup \left(\bigcup_{k=l}^{+\infty}(t_k, t_{k+1}]\right), \text{ where } l = \min\{k : t_k \geq T\}.$$

Theorem 18 *Assume that conditions (iii) and (iv) of Lemma 2.1 hold and there exists a positive integer k_0 such that $\overline{a}_k \geq 1$, for $k \geq k_0$. If*

$$\int_{\pm\epsilon}^{\pm\infty} \frac{du}{\psi(u)} < +\infty \qquad (6)$$

holds for some $\epsilon > 0$, and

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s)} \left(\lim_{t \to +\infty} \int_s^t \prod_{s < t_k < u} \frac{1}{b_k} Q(u) du \right) ds = +\infty. \qquad (7)$$

Then every solution of Eq.(1) oscillates.

Proof. Without loss of generality, we can assume $k_0 = 1$. Let $x(t)$ be a nonoscillatory solution of Eq.(1). Suppose that $x(t) > 0$ for $t \geq t_0$. By Lemma 2.1, we can find $x'(t) \geq 0$ for $t \geq t_0$. It is clear that $x'(t+\tau) \geq 0$ also for $t \geq t_0$. Since $\overline{a}_k \geq 1$ for $k = 1, 2, ...$, we get

$$x(t_0^+) \leq x(t_1) \leq x(t_1^+) \leq x(t_2) \leq x(t_2^+) \leq \dots .$$

Obviously, $x(t)$ is monotonically nondecreasing in $[t_0, +\infty)$. From Eq.(1) and (i) we have

$$(r(t)x'(t))' \leq -Q(t)\psi(x(t+\tau)), \quad t \geq t_0, t \neq t_k,$$

$$x'(t_k^+) \leq b_k x'(t_k), \quad k = 1, 2, \dots .$$

From Lemma 1.1 it follows that

$$r(t)x'(t) \leq r(s)x'(s) \prod_{s < t_k < t} b_k - \int_s^t \prod_{u < t_k < t} b_k Q(u)\psi(x(u+\tau))du, \quad t_0 \leq s \leq t.$$

From the above inequality, we have

$$x'(s) \geq \frac{1}{r(s)} \int_s^t \prod_{s < t_k < u} \frac{1}{b_k} Q(u)\psi(x(u+\tau))du. \qquad (8)$$

In view of (8), for $s \in (t_k, t_{k+1}], k = 1, 2, ...$, we get

$$\frac{x'(s)}{\psi(x(s+\tau))} \geq \frac{1}{r(s)} \int_s^t \prod_{s < t_k < u} \frac{1}{b_k} Q(u)du.$$

For $s \in (t_k, t_{k+1}], k = 1, 2, ...$, we obtain

$$\int_{t_k^+}^{t_{k+1}} \frac{x'(s)}{\psi(x(s+\tau))} ds \leq \int_{t_k^+}^{t_{k+1}} \frac{x'(s)}{\psi(x(s))} ds = \int_{x(t_k^+)}^{x(t_{k+1})} \frac{du}{\psi(u)}.$$

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Hence,

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s)} \left(\lim_{t \to +\infty} \int_s^t \prod_{s < t_k < u} \frac{1}{b_k} Q(u) du \right) ds$$

$$\leq \sum_{k=0}^{+\infty} \int_{x(t_k^+)}^{x(t_{k+1})} \frac{du}{\psi(u)} \leq \int_{x(t_0^+)}^{+\infty} \frac{du}{\psi(u)}. \tag{9}$$

The last inequality (9) contradicts Eq.(6),(7) of Theorem 2.1. That is, all solutions of Eq.(1) oscillate. The proof is complete.

From Theorem 2.1, we can easily obtain the following corollaries.

Corollary 2 *Assume that conditions (iii) and (iv) of Lemma 2.1 hold and there exists a positive integer k_0 such that $\bar{a}_k \geq 1$, $b_k \leq 1$ for $k \geq k_0$. If $\int_{\pm\epsilon}^{\pm\infty} \frac{du}{\psi(u)} < +\infty$ holds for some $\epsilon > 0$, and $\int_{t_0}^{+\infty} \frac{1}{r(s)} \int_s^{+\infty} Q(t) dt ds = +\infty$. Then every solution of Eq.(1) oscillates.*

Corollary 3 *Assume that conditions (iii) and (iv) of Lemma 2.1 hold and there exists a positive integer k_0 and a constant $\alpha > 0$ such that $\bar{a}_k \geq 1$, $\frac{1}{b_k} \geq t_{k+1}^\alpha$ for $k \geq k_0$. If $\int_{\pm\epsilon}^{\pm\infty} \frac{du}{\psi(u)} < +\infty$ holds for some $\epsilon > 0$, and*

$$\sum_{k=0}^{+\infty} [R(t_{k+1}) - R(t_k^+)] \int_{t_{k+1}^+}^{+\infty} t^\alpha Q(t) dt = +\infty, \text{ where } R(t) = \int_{t_0}^t \frac{ds}{r(s)}.$$

Then every solution of Eq.(1) oscillates.

Theorem 19 *Assume that conditions (iii) and (iv) of Lemma 2.1 hold and there exists a positive integer k_0 such that $\bar{a}_k \geq 1$ for $k \geq k_0$. Suppose that $\psi(ab) \geq \psi(a)\psi(b)$ for any $ab > 0, t_{k+1} - t_k = \tau$ for all $k = 1, 2, \dots$. If $\int_{\pm\epsilon}^{\pm\infty} \frac{du}{\psi(u)} < +\infty$ holds for some $\epsilon > 0$, and*

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s)} \left(\lim_{t \to +\infty} \int_s^t \prod_{s < t_k < u} \frac{1}{c_k} Q(u) du \right) ds = +\infty,$$

where $c_k = \dfrac{b_k}{\psi(\bar{a}_{k+1})}, k = 1, 2, 3, \dots$.

Then every solution of Eq.(1) oscillates.

The proof of Theorem 2.2 is analogous to that of Theorem 2.1 and is omitted.

Corollary 4 *Assume that conditions (iii) and (iv) of Lemma 2.1 hold and there exists a positive integer k_0 and a constant $\alpha > 0$ such that $\bar{a}_k \geq 1$, $\dfrac{1}{c_k} \geq t_{k+1}^\alpha$ for $k \geq k_0$, where $c_k = \dfrac{b_k^\sigma}{\psi(\bar{a}_{k+1})}, k = 1,2,3,\dots$. Suppose that $\psi(ab) \geq \psi(a)\psi(b)$ for any $ab > 0$, $t_{k+1} - t_k = \tau$ for all $k = 1,2,\dots$. If $\displaystyle\int_{\pm\epsilon}^{\pm\infty} \dfrac{du}{\psi(u)} < +\infty$ holds for some $\epsilon > 0$, and*

$$\sum_{k=0}^{+\infty} [R(t_{k+1}) - R(t_k^+)] \int_{t_{k+1}^+}^{+\infty} t^\alpha Q(t) dt = +\infty, \text{ where } R(t) = \int_{t_0}^{t} \frac{ds}{r(s)}.$$

Then every solution of Eq.(1) oscillates.

1. V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *World Scientific, Singapore,* 1989.
2. I. Gyori, G. Ladas, *Clarendon Press, Oxford,* 1991.
3. L. H. Erbe, Q. K. Kong, B. G. Zhang, *Marcel Dekker, Inc,* 1995.
4. K.Gopalsamy, B. G. Zhang, *J. Math. Anal. Appl.* **139**, 110–122 (1989)
5. D. Alexander, D. Michael, *J. Math. Anal. Appl.* **206**, 254–269 (1997)
6. D. Bainov, M. Dimitrova, *Appl. Anal.* **64** 57–67 (1997)
7. Y. S. Chen, W. Z. Feng, *J. Math. Anal. Appl.* **210** 150–169 (1997)

THE SOBOLEV EXPONENTS OF NONLINEAR WAVE EQUATIONS IN HIGH SPACE DIMENSIONS

Q. L. HU

This paper deals with the Sobolev exponents for the equations $u_{tt} - \Delta u = K(u)(Du)^\alpha (k \in Z^+, \parallel K(u) \parallel_{H^s} \leq C \parallel u \parallel_{H^s}^k, \rho = |\alpha| \geq 2, x \in R^n, n \geq 4)$ under the usual initial data assumptions $u(0,x) \in H^s(R^n)$ and $u_t(0,x) \in H^{s-1}(R^n)$. The Sobolev exponent $s(n,p)=\max\{n/2, (n/2-1)(\rho-3)/(\rho-1)+2\}$ is obtained. The interesting aspect of the result is that $s(n,\rho) = n/2$ if $2 \leq \rho \leq n$. The results acquired in [1][2] are extended to high space dimensions.

1 Introduction

Ponce and Sideris discussed the minimal Sobolev regularity of the following three dimensional nonlinear wave equations

$$\begin{cases} u_{tt} - \Delta u = u^k(Du)^\alpha, x \in R^3, t \geq 0, \\ u(0,x) = f(x), u_t(0,x) = g(x), x \in R^3, \end{cases} \tag{1}$$

where $|\alpha| = \rho \geq 2, k \in Z^+$.

For problem (1), the Sobolev exponent is reduced to $s(\rho) = max\{2, (5\rho - 7)/(2\rho - 2)\}$ in [1]. The number $s(\rho)$ lies in the interval $[2,5/2] \subset [3/2,5/2]$. Here we know $s(2) = 2$, this result is one of the main results in [2]. Also we find $s(3) = 2$.

The nature question is that if $x \in R^n$, whether the Sobolev exponent for the equation $u_{tt} - \Delta u = u^k(Du)^\alpha(k \in Z^+, \rho = |\alpha| \geq 2, x \in R^n, n \geq 3)$ lies in the interval $[n/2, (n+2)/2]$. In this paper we will answer this problem. The Sobolev exponent obtained for the equation $u_{tt} - \Delta u = K(u)(Du)^\alpha(k \in Z^+, \parallel K(u) \parallel_{H^s} \leq C \parallel u \parallel_{H^s}^k, \rho = |\alpha| \geq 2, x \in R^n, n \geq 4)$ is $s(n,\rho) = \{n/2, (n/2 - 1)(\rho - 3)/(\rho - 1) + 2\}$ under the usual initial value assumptions $u(0,x) \in H^s(R^n)$ and $u_t(0,x) \in H^{s-1}(R^n)$. The number $s(n,\rho) = max\{n/2, (n/2 - 1)(\rho - 3)/(\rho - 1) + 2\}$ truly lies in the interval $[n/2, (n+2)/2]$. The interesting aspect of the result is that $s(n,\rho) = n/2$ if $2 \leq \rho \leq n$. This means that we obtain the Sobolev exponent for the equation $u_{tt} - \Delta u = u^k(Du)^\alpha(x \in R^n, n \geq 4, k \in Z^+, 2 \leq |\alpha| \leq n)$ is $n/2$ under the restrictions $u(0,x) \in H^s(R^n)$ and $u_t(0,x) \in H^{s-1}(R^n)$. In particular, if $n = 4$ we know the Sobolev exponent is 2.

The results acquired in this paper can be regarded as extensions of results in [1] in high space dimensions.

For simplicity, we will denote by C any constants appearing in our paper. Now we give some lemmas.

Lemma 1 Suppose that $u^0(t,x) = V'(t)f_1(x) + V(t)f_2(x)$ and $V(t) = (-\Delta)^{-1/2} \sin t(-\Delta)^{1/2}$, then for any $q > 2$, we have

$$(\int_{-\infty}^\infty \parallel (-\Delta)^{1/q}u^0(t) \parallel_{L^q(R^n)}^r dt)^{1/r} \leq C(\parallel (-\Delta)^{1/2}f_1 \parallel_{L^2(R^n)} + \parallel f_2 \parallel_{L^2(R^n)}),$$

where $r = 2q/(q-2)$.

The proof can be found in [1] Page 171.

Lemma 2 If $s' > n/q, q > 0$, then

$$\parallel u \parallel_{L^\infty(R^n)} \leq C \parallel (-\Delta)^{s'/2}u \parallel_{L^q(R^n)}.$$

The proof can be found in [2] Page 32.

Lemma 3 Suppose that $(s-2)q > n-2, q > 2, s > n/2$ and $\sigma = 1/q + s/2$, it follows that

$$\parallel Du \parallel_{L^\infty(R^n)} \leq C \parallel u \parallel_{H^s(R^n)} + C \parallel (-\Delta)^{\sigma-1/2}u \parallel_{L^q(R^n)}.$$

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The proof is similar to that of Lemma 4 in [1], we omit it.

2   Main theorem

In this section, we consider the following problem

$$\begin{cases} u_{tt} - \Delta u = K(u)(Du)^\alpha, x \in R^n, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), x \in R^n, \end{cases} \tag{2}$$

where $\| K(u) \|_{H^s} \leq C \| u \|_{H^s}^k, k \in Z^+, \rho = |\alpha| \geq 2$.

The main theorem of this paper can be described as follows.

Theorem 1 Suppose that $n \geq 4, s > \max \{n/2, (n/2-1)(\rho-3)/(\rho-1) + 2\}, u_0(x) \in H^s(R^n)$ and $u_1 \in H^{s-1}(R^n)$. Then there exists a $T > 0$ depending on s and $\| u_0(x) \|_{H^s(R^n)} + \| u_1(x) \|_{H^{s-1}(R^n)}$ such that problem (2) has a unique solution $u(t,x)$ satisfying

$$u(t,x) \in C([0,T], H^s(R^n)) \cap C^1([0,T], H^{s-1}(R^n)),$$

and

$$\int_0^r \| (-\Delta)^{\sigma-1/2} u(t,\cdot) \|_{L^q(R^n)}^T dt \leq \infty,$$

for any $\sigma = s/2 + 1/q, (s-2)q > n - 2$ and $r = 2q/(q-2)$.

Proof The method used here is quite similar to that of [1]. It is obvious to notice that the theorem is reasonable for any $s > (n+2)/2$. Now we assume that s satisfies $n/2 < s \leq (n+2)/2$, then we know $q > 2$ from the assumption $(s-2)q > n - 2$. Let

$$\| u \|_{H^s} = \| u(t,x) \|_{H^s(R^n)}$$

and

$$\| u(t) \| = \| u \|_{H^s} + \| u_t \|_{H^{s-1}} + (\int_0^t \| (-\Delta)^{\sigma-1/2} u(\tau,\cdot) \|_{L^q(R^n)}^r d\tau)^{1/r},$$

where $r = 2q/(q-2), \sigma = s/2 + 1/q$, we write

$$\| u \|_T = \sup_{t \in [0,T]} \| u(t) \|,$$

and

$$X_T^B = \{u| \| u \|_T < B\}.$$

We define

$$\Lambda u = u^0(t,x) + \int_0^t V(t-\tau)G(u,Du)d\tau. \tag{3}$$

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where $G(u, Du) = K(u)(Du)^{\alpha}, k \in Z^+, \rho = |\alpha| \geq 2, x \in R^n, u^0(t,x) = V'(t)u_0(x) + V(t)u_1(x)$ and $V(t) = (-\Delta)^{-1/2} \sin t(-\Delta)^{1/2}$. We know the integral operator Λ is equivalent to problem (2). It will show that for appropriate choice of T and B the operator Λ is a contraction of X_T^B into itself.

Using the usual estimates for the linear propagator $V(t)$ and $V'(t)$ in (3), it follows from Plancherel's theorem, that

$$\| \Lambda u(t) \|_{H^s(R^n)} \leq C \| u_0 \|_{H^s} + C(1+t) \| u_1 \|_{H^{s-1}}$$

$$+ C \int_0^t (t-\tau) \| G(u, Du)(\tau) \|_{L^2(R^n)} \, d\tau$$

$$+ C \int_0^t \| (-\Delta)^{(s-1)/2} G(u, Du) \|_{L^2(R^n)} \, d\tau.$$

$$(4)$$

Since $H^{\sigma'}(R^n) \cdot H^{s_1}(R^n) \subset H^{\sigma'}(R^n)$ for any $s_1 > n/2, \sigma' \leq s_1$, it follows that

$$\| (-\Delta)^{(s-1)/2} G(u, Du) \|_{L^2(R^n)} \leq C \| u \|_{H^s(R^n)}^{k+1} \| Du \|_{L^\infty(R^n)}^{\rho-1},$$

and

$$\sup_{t \in [0,T]} \| \Lambda u(t) \|_{H^s(R^n)} \leq C \| u_0 \|_{H^s}$$

$$+ C(1+T) \| u_1 \|_{H^{s-1}} + CT^2 \sup_{t \in [0,T]} \| u(t) \|_{H^s(R^n)}^{k+\rho}$$

$$+ C \sup_{t \in [0,T]} \| u(t) \|_{H^s}^{k+1} \int_0^T \| Du \|_{L^\infty(R^n)}^{\rho-1} \, d\tau.$$

$$(5)$$

It follows from $(s-2)q > n-2, q > 2$ and $s > (n/2-1)(\rho-3)/(\rho-1) + 2$ that $r/(\rho-1) > 1$, namely, $r > \rho - 1$. Using the Holder inequality, we have

$$\sup_{t \in [0,T]} \| \Lambda u(t) \|_{H^s(R^n)} \leq C \| u_0 \|_{H^s}$$

$$+ C(1+T) \| u_1 \|_{H^{s-1}} + C(T+T^2) \sup_{t \in [0,T]} \| u(t) \|_{H^s(R^n)}^{k+\rho}$$

$$+ CT^{(r+1-\rho)/r} \sup_{t \in [0,T]} \| u(t) \|_{H^s}^{k+1} \left(\int_0^T \| (-\Delta)^{\sigma-1/2} u(\tau) \|_{L^q(R^n)}^r \, d\tau \right)^{\rho-1)/r}$$

$$= I.$$

$$(6)$$

By the same method as above, it also can get

$$\| (\Lambda u)_t \|_{H^{s-1}} < I.$$

$$(7)$$

Since

$$(\int_0^T \| (-\Delta) \wedge^{\sigma-1/2} u(t) \|_{L^q(R^n)}^r \, dt)^{1/r} \leq C(\int_{-\infty}^\infty \| (-\Delta)^{\sigma-1/2} u^0 \|_{L^q(R^n)}^r \, dt)^{1/r}$$

$$+ (\int_0^T \| (-\Delta)^{\sigma-1/2} \int_0^T V(t-\tau) G(u, Du)(\tau) d\tau \|_{L^q(R^n)}^r \, dt)^{1/r}$$

$$= M_1 + M_2. \tag{8}$$

It follows from Lemma 1 that

$$M_1 \leq C(\| u_0 \|_{H^s} + \| u_1 \|_{H^{s-1}}). \tag{9}$$

By the same estimates as in [2], we have

$$M_2 \leq C \int_0^T (\int_0^T \| (-\Delta)^{\sigma-1/2} V(t-\tau) G(u, Du)(\tau) \|_{L^q}^r \, dt)^{1/r} d\tau.$$

Noticing that $V(t-\tau) = V(t)V'(\tau) - V'(t)V(\tau)$, Lemma 1, and the fact that $V'(\tau)$ and $(-\Delta)^{1/2} V(\tau)$ are bounded in $L^2(R^n)$ to get the following

$$M_2 \leq CT \sup_{t \in [0,T]} \| u(t) \|_{H^s(R^n)}^{k+\rho} + CT^{(r+1-\rho)/r} \sup_{t \in [0,T]} \| u(t) \|_{H^s}^{k+1}$$

$$(\int_0^T \| (-\Delta)^{\sigma-1/2} u(\tau) \|_{L^q(R^n)}^r \, d\tau)^{(\rho-1)/r} \tag{10}$$

Choosing $m = (r + 1 - \rho)/r$, we know $m > 0$ from $r > \rho - 1$. By (6)-(10) we have

$$\| \Lambda u \|_T \leq C\{\| u_0 \|_{H^s} + (1+T) \| u_1 \|_{H^{s-1}} + (T + T^2 + T^m) \| u \|_T^{k+\rho}\}.$$

Letting $B = 2C(1+T)(\| u_0 \|_{H^s} + \| u_1 \|_{H^{s-1}})$, we choose T sufficiently small such that

$$C(T + T^2 + T^m)B^{k+\rho-1} \leq 1/2,$$

then

$$\| \Lambda u \|_T \leq B.$$

In a similar way, it can be shown that under same assumptions on B and T, the operator Λ is a contraction on X_T^B. Thus, there exists a unique fixed point of Λ which satisfies the integral equation (3). This completes the proof of the theorem.

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References

1. G.Ponce and T.C.Sideris, Comm. PDE, **18**, 169 (1993).
2. M. Taylor, Pseduo-differential operators,Princeton University Press. 1981.

BUNDARY VALUE PROBLEMS FOR SINGULAR SECOND-ORDER FDES

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Positive solutions to the singular boundary value problem of FDES are obtained by applying the Schauder fixed point theorem. The result improves an existence theorem due to Erbe and Kong (1994).

1 Introduction

In a paper [1], Erbe and Kong have studied the boundary value problem for a singular second-order functional differential equation,

$$\begin{cases} y'' = -f(x, y(w(x))), & 0 < x < 1, \\ \alpha y(x) - \beta y'(x) = \xi(x), & a \le x \le 0, \\ \gamma y(x) + \delta y'(x) = \eta(x), & 1 \le x \le b, \end{cases} \tag{1.1}$$

under the following assumptions:

(A_1) $\alpha, \beta, \gamma, \delta$ are nonnegative constants, with $\rho := \delta\alpha + \gamma\beta + \gamma\alpha > 0$.

(A_2) $w(x)$ is a continuous function defined on [0,1] and satisfying

$$p = \inf\{w(x); 0 \le x \le 1\} < 1 \text{ and } q = \sup\{w(x); 0 \le x \le 1\} > 0.$$

Hence the set $E := \{x \in [0,1] : 0 \le w(x) \le 1\}$ is compact and mes$E > 0$.

(A_3) $\xi(x)$ and $\eta(x)$ are continuous functions defined on $[a, 0]$ and $[1, b]$, respectively, where $a := \min\{0, p\}$ and $b := \max\{1, q\}$. Moreover, $\xi(0) = \eta(1) = 0$, $\xi(x) \ge 0$ for $\beta = 0$, $\int_x^0 e^{-\frac{\alpha}{\beta}s}\xi(s)ds \ge 0$ for $\beta > 0$, $\eta(x) \ge 0$ for $\delta = 0$, and $\int_1^x e^{\frac{\gamma}{\delta}s}\eta(s)ds \ge 0$ for $\delta > 0$.

(A_4) $f(x, y) : (0, 1) \times (0, \infty) \to (0, \infty)$ is continuous and integrable on [0,1] for each fixed $y \in (0, \infty)$.

(A_5) $f(x, y)$ is decreasing in y for each fixed x.

(A_6) $\lim_{y \to 0+} f(x,y) = \infty$ uniformly on compact subsets of (0,1).

(A_7) $\lim_{y \to \infty} f(x,y) = 0$ uniformly on compact subsets of (0,1).

(A_8) For all $\theta > 0$

$$0 < \int_0^1 f(t, \theta g_1(w(t)))dt < \infty,$$

$$g_1(x) := \begin{cases} x - a, & a \leq x \leq \frac{1}{2}, \\ b - x, & \frac{1}{2} \leq x \leq b. \end{cases}$$

They affirm that if $(A_1) - (A_8)$ hold then the BVP (1.1) has at least one positive solution in $C[a,b] \cap C^2(0,1)$, which is Theorem 2.2 in [1], and give only the proof for the case $\delta\beta > 0$, which is based on an application of a fixed point theorem due to [2] for mappings that are decreasing with respect a cone in a Banach space.

In this paper, we study the BVP(1.1) with the aim of improving Theorem 2.2 in [1] for the case $\beta\delta > 0$. The hypotheses we adopt are as follows:

$(H_1) - (H_4)$ The hypothesis are the same as $(A_1) - (A_4)$.

(H_5) There exists an $\varepsilon > 0$ such that $f(x,y)$ is nonincreasing in $y \leq \varepsilon$ for almost all $x \in (0,1)$.

(H_6) $0 < \int_E h(t)f(t,\varepsilon)dt < \infty$, where E is determined by (H_2).

(H_7) $\lim_{y \to \infty} f(x,y)/y = 0$ uniformly on compact subsets of (0,1).

The main results of the present paper are as follows.

Theorem 1. Assume that $(H_1) - (H_7)$ hold. Then there exists a $\theta^* > 0$ such that

$$y(x) \geq \theta^* \quad \text{on } [a,b]$$

for all solutions, $y(x)$, to the BVP(1.1).

Theorem 2. Assume that $(H_1) - (H_8)$ hold. Then the BVP(1.1) has at least one positive solution.

We say that a function $y(x)$ is a solution to the BVP(1.1), if it satisfies the following conditions:

(i) $y(x)$ is continuous and nonnegative on $[a,b]$,

(ii) $y(x) = y_a(x)$ on $[a,0]$, where $y_a(x) : [a,0] \to [0,\infty)$ is defined by

$$y_a(x) := e^{\frac{\alpha}{\beta}x} \left(\frac{1}{\beta} \int_x^0 e^{-\frac{\alpha}{\beta}s} \xi(s)ds + y(0) \right) \text{ for } \beta > 0. \tag{1.3}$$

(iii) $y(x) = y_b(x)$ on $[1,b]$, where $y_b(x) : [1,b] \to [0,\infty)$ is defined by

$$y_b(x) := e^{-\frac{\gamma}{\delta}x} \left(\frac{1}{\delta} \int_1^x e^{\frac{\gamma}{\delta}s} \eta(s)ds + e^{\frac{\gamma}{\delta}} y(1) \right) \text{ for } \delta > 0. \tag{1.4}$$

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(iv) $y''(x) = -f(x, y(w(x)))$ for all $x \in (0, 1)$.

Furthermore, a solution $y(x)$ is said to be positive, if $y(x) > 0$ in (0,1).

It is obvious that Theorem 2 corrects and improves Theorem 2.2 in [1] for the case $\beta\delta > 0$.

2 Proof of Main Results

Let $y(x)$ be a solutioin to the BVP(1.1). Then it can be represented as

$$y(x) = \begin{cases} y_a(x), & a \le x \le 0, \\ \int_0^1 G(x,t)f(t, y(w(t)))dt, & 0 \le x \le 1, \\ y_b(x), & 1 \le x \le b, \end{cases} \tag{2.1}$$

where $y_a(x)$ and $y_b(x)$ are defined by (1.3) and (1.4), respectively, and

$$G(x,t) := \begin{cases} (\delta + \gamma - \gamma x)(\beta + \alpha t)/\rho, & 0 \le t \le x \le 1, \\ (\delta + \gamma - \gamma t)(\beta + \alpha x)/\rho, & 0 \le x \le t \le 1, \end{cases}$$

is the Green's function.

It is easy to see that there exist two positive numbers $\lambda < 1$ and B such that

$$\lambda B \le G(x,t) \le B \text{ on } [0,1] \times [0,1]. \tag{2.2}$$

From (2.1) and (2.2), we have, for every solution to the BVP(1.1)

$$\begin{cases} ||y|| \le B \int_0^1 f(t, y(w(t)))dt, \\ y(x) \ge \lambda ||y|| \text{ on } [0,1], \end{cases} \tag{2.3}$$

where $||y|| = \sup\{|y(x)|; 0 \le x \le 1\}$.

Proof of Theorem 1. We first claim that there exists a $\theta^* > 0$ such that $y(x) \ge \theta^*$ on [0,1] for all solutions, $y(x)$, to the BVP(1.1).

Suppose to contrary that the claim is false. This implies that there exists a sequence $\{y_m(x)\}$ of solutions to the BVP(1.1) such that $\lim_{m\to\infty} ||y_m|| = 0$. Without loss of generality, we may assume that

$$\varepsilon \ge ||y_m|| \ge ||y_{m+1}|| \text{ for all } m \ge 1. \tag{2.4}$$

From (2.1), (2.2), (2.3), (2.4), (H_5) and (H_6), it follows that

$$\begin{aligned} y_m(\frac{1}{2}) &\ge \lambda B \int_E f(t, y_m(w(t)))dt \\ &\ge \lambda B \int_E f(t, ||y_m||)dt \\ &\ge \lambda B \int_E h(t)f(t, \varepsilon)dt > 0, \end{aligned}$$

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which contradicts the assumption that $\lim_{m\to\infty} \|y_m\| = 0$ and hence the claim is true. Theorem 1 follows from the claim and the definition of solutions (providing θ^* is suitable small).

Proof of Theorem 2.

By applying the Schauder fixed point Theorem, we can prove BVP(1.1) has a positive solution .

References

1. L. H. Erbe and Qingkai Kong, *J. Comput. Appl. Math.*, **53**, 377 (1994).
2. J. A. Gatica, V.Oliker and P. Waltman, *J. Differential Equations*, **79**, 62 (1989).
3. S. Taliaferro, *Nonlinear Anal.*, **3**, 897 (1979).

ASYMPTOTIC BEHAVIOR AND EXISTENCE OF POSITIVE SOLUTIONS FOR HIGHER ORDER NEUTRAL DIFFERENCE EQUATIONS

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Consider the discrete Lasota-Wazewska model

$$\Delta^d(x_n - cx_{n-m}) + p_n x_{n-k} = 0, \quad n = N, N+1, N+2, \cdots$$

where c and p_n are real numbers, K, m are positive integers with $m < k$, and δ denotes the forward difference operator: $\delta u_n = u_{n+1} - u_n$, Some sufficient conditions under which such an equation has a bounded and eventually positive solution which tends to zero as $n \to \infty$ are obtained. So my result in [6] is generalized.

1 Introduction

Consider the neutral difference equation

$$\delta(x_n - Cx_{n-m}) + P_n x_{n-k} = 0 \quad n = N, N+1, N+2, \cdots \tag{1}$$

where C and P_n are real number, k, m are positive integers with $m < k$, and δ denotes the forward difference operator $\delta U_n = U_{n+1} - U_n$

The problem on the existence of positive solutions of Eq.(1) has been studied recently by several authors, some results have obtained, here we refer to [1-3]. The asymptotic behavior and the existence of positive solution of Eq.(1) are studied in [4-6]. Our aim in this paper is to discuss the higher order difference equation:

$$\delta^d(x_n - Cx_{n-m}) + P_n x_{n-k} = 0 \quad n = N, N+1, N+2, \cdots \qquad (2)$$

Some sufficient conditions under which Eq.(2) has a bounded and eventually posit ive solution which tends to zero as $n \to \infty$ have obtained.

By a solution of Eq.(2), we mean a sequence x_n which is defined of $n \geq N - K$, and satisfies Eq. (2) for all $n \geq N$. Clearly if the real number $a_0, a_1, \cdots, a_K$ are given, the Eq.(2) has a unique solution x_n satisfying the initial conditions

$$x_{N-i} = a_i \quad i = 0, 1, 2, \cdots, k \qquad (3)$$

A solution x_n of Eq.(2) is said to be eventually positive, if there exists a positive integer N_0 such that $x_n > 0$ for all $n \geq N_0$.

Some lemmas and Main Results

qquad In this paper, we obtain following results:

Theorem 1 Assume that

$(H_1)C > 0$, and $P_n < 0$ for $n \geq N$;

(H_2) there exists a constant $q > 0$ such that for sufficiently large n,

$$d_n = \frac{1}{C}e^{-qm} - \frac{1}{C}\frac{1}{A_{d-1}^{d-1}}\sum_{s=n+m}^{\infty} A_{s-n+d-1}^{d-1} P_s e^{q(n-s+k)} \leq 1 \qquad (4)$$

$(H_3)d$ is an odd number or

$(H_1')0 < C < 1$, and $p_n > 0$ for $n \geq N$

(H_2') there exists a constant $q > 0$ such that for sufficiently large n,

$$d_n' = Ce^{qm} + \frac{1}{A_{d-1}^{d-1}}\sum_{s=n}^{\infty} a_{s-n+d-1}^{d-1} P_s e^{q(n-s+k)} \leq 1 \qquad (5)$$

$(H_3')d$ is an odd number Then Eq.(2) has a bounded and eventually positive solution x_n satisfying $\lim_{n\to\infty} x_n = 0$. When $A_n^d = n(n-1)\cdots(n-d+1), A_1^0 = 1$.

Theorem 2 Assume that

$(H_3)C > 0$, and $P_n > 0$ for $n \geq N$;

(H_4) there exists a constant $q > 0$ such that for sufficiently large n,

$$d_n^* = \frac{1}{C}e^{-qm} + \frac{1}{C}\frac{1}{A_{d-1}^{d-1}}\sum_{s=n}^{d-1}p_s e^{q(n-s+k)} \le 1 \tag{6}$$

$(H_5)d$ is an even number or

$(H_3')0 < C < 1$, and $P_n < 0$ for $n \ge N$;

(H_4') there exists a constant $q > 0$ such that for sufficiently large n,

$$d_n^{*'} = Ce^{qm} - \frac{1}{A_{d-1}^{d-1}}\sum_{s=n+m}^{\infty} A_{s-n+d-1}^{d-1}p_s e^{q(n-s+k)} \le 1 \tag{7}$$

$(H_5')d$ is an even number Then Eq.(2) has a bounded and eventually positive solution x_n satisfying $\lim_{n\to\infty}x_n = 0$. Where $A_n^d = n(n=1)\cdots(n-d+1), A_n^0 = 1$.

Lemma 1 b_n is a sequence, $n = 1,2,\cdots,y_n = \frac{1}{A_{d-1}^{d-1}}\sum_{l=n}^{\infty}A_{l-n+d-1}^{d-1}b_l$, then that

$$\sum_{s=n}^{\infty}y_s = \frac{1}{A_d^d}\sum_{l=n}^{\infty}A_{l-n+d}^d b_l \tag{8}$$

$$\delta_n = -\frac{1}{A_{d-2}^{d-2}}\sum_{l=n}^{\infty}A_{l-n+d-2}^{d-2}b_l \tag{9}$$

if d is an odd,

$$\delta^d y_n = -b_n \tag{10}$$

if d is an even,

$$\delta^d y_n = b_n \tag{11}$$

Proof Since $\sum_{s=n}^{l} C_{l-s+d-1}^{d-1} = C_{l-n+d}^d$, we have

$$\sum_{s=n}^{\infty}y_s = \frac{1}{A_{d-1}^{d-1}}\sum_{s=n}^{\infty}\sum_{l=s}^{\infty}A_{l-s+d-1}^{d-1}b_l = \frac{1}{A_{d-1}^{d-1}}\sum_{s=n}^{\infty}\sum_{s=n}^{l}A_{l-s+d-1}^{d-1}b_l$$

$$= \sum_{l=n}^{\infty}(\sum_{s=n}lC_{l-s+d-1}^{d-1})b_l = \sum_{l=n}^{\infty}C_{l-n+d}^d b_l = \frac{1}{A_d^d}\sum_{l-n+d}^{\infty}A_{l-n+d}^d b_l.$$

Hence, (2.6) is true.

Let $Z_s = \frac{1}{A_{d-2}^{d-2}}\sum_{l=s}^{\infty}A_{l-s+d-2}^{d-2}b_l$, By (2.6), we have

$$\sum_{s=n}^{\infty}Z_s = \frac{1}{A_{d-1}^{d-1}}\sum_{l=n}^{\infty}A_{l-s+d-1}^{d-1}b_l = y_n$$

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Hence, $\delta_n = \sum_{s=n+1}^{\infty} Z_s - \sum_{s=n}^{\infty} Z_s = -Z_n = -\frac{1}{A_{d-2}^{d-2}} \sum_{t=n}^{\infty} A_{l-n+d-2}^{d-2} b_l$. By (2.7) again and again, We have

$$\delta^{d-1} y_n = (-1)^{d-1} \sum_{l=n}^{\infty} b_l$$

if d is an odd, then $\delta^{d-1} y_n = \sum_{l=n}^{\infty} b_l, \delta^d y_n = -b_n$, if d is an even, then $\delta^{d-1} y_n = -\sum_{l=n}^{\infty} b_l, \delta^d y_n = b_n$

Lemma 2 (Krasnoselskii[7]) Assume that X is a Banach space and S is a bounded, Closed and convex subset of X, Let operators $T_1, T_2 : S \to X$ satisfy following conditions:

(i)$T_1 X + T_2 y \in S$ for any $x, y \in S$;

(ii)T_1 is a strict contraction, that is, there exists a constant λ with $0 \leq \lambda < 1$ such that $\|T_1 x - T_2 y\| \leq \lambda \|x - y\|$ for all $x, y \in S$;

(iii)T_2 is completely. Continuous, that is, T_2 is continuous and compact. Then $T = T_1 + T_2$ has a fixed point in S.

2 Proof of Main Results

Proof of theorem 1 if there exists a positive integers N_0 such that $d_n \equiv 1$ in (4) for all $n \geq N_0$, then for all $n \geq N_0$, we have

$$* \frac{1}{C} e^{-q(n+m)} - \frac{1}{C} \frac{1}{A_{d-1}^{d-1}} \sum_{s=n+m}^{\infty} A_{s-n+d-1}^{d-1} P_s e^{-q(s-k)} \tag{12}$$

$$= e^{-qn} (\frac{1}{C} e^{-qm} - \frac{1}{C} \frac{1}{A_{d-1}^{d-1}} \sum_{s=n+m}^{\inf ty} A_{s-n+d-1}^{d-1} P_s e^{q(n-s+k)} \tag{13}$$

$*$

$= e^{-qn} d_n = e^{-qn}$, it follow that

$$e^{-qn} - C e^{-q(n+m)} = \frac{1}{A_{d-1}^{d-1}} \sum_{s=n}^{\infty} A_{s-n+d-1}^{d-1} p_s e^{-q(s-k)}$$

for all $n \geq N_0 + m$, by (2.8) which implies that

$$\delta^d (e^{-qn} - C e^{-q(n-m)}) + P_n e^{-q(n-k)} = 0$$

for all $n \geq N_0 + m$, and hence that sequence e^{-qn} satisfies Eq.(2) for a ll $n \geq N_0 + m$, on the other hand, it is easy to see that there exists a solution

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$x_n(n \geq N - k)$ of Eq.(2) such that $x_n = e^{-qn}$ for all $n \geq N_0 + m$. Obviously, such solution of Eq.(2) is bounded eventually positive, and $\lim_{n \to \infty} x_n = 0$.

We next assume that there exists a positive integer $N^* > N$ such that $N^* - K \geq N$, and

$$d_{N^*} = \frac{1}{C} e^{-qm} - \frac{1}{C} \frac{1}{A_{d-1}^{d-1}} \sum_{s=N^*+m}^{\infty} A_{s-n+d-1}^{d-1} p_s e^{q(N^*-s+k)} < 1 \qquad (14)$$

and $d_n \leq 1$ in (4).

Consider the Banach space X of all bounded real sequences $y = y_n$ where $n \geq N$, with supnorm, $\|y\| = \sup_{n \geq N} |y_n|$, we define a subset S in X as

$$S = y \in X: \ 0 \leq y_n \leq 1, n \geq N.$$

Clearly, S is a bounded, closed and convex subset of X. Now we define a operator $T: S \to X$ as $Ty_n = T_1 y_n + T_2 y_n$ where

$$T_1 y_n = \begin{cases} \frac{1}{C} e^{-qm} y_{n+m} & n \geq N^* \\ T_1 y_{N^*} - 1 + e^{h(N^*-n)} & N \leq n \leq N^* \end{cases} \qquad (15)$$

$$T_2 y_n = \begin{cases} -\frac{1}{C} \frac{1}{A_{d-1}^{d-1}} \sum_{s=n+m}^{\infty} A_{s-n+d-1}^{d-1} P_s e^{q(n-s+k)} y_{s-k} & n \geq N^* \\ T_2 y_{N^*} & N \leq n \leq N^* \end{cases} \qquad (16)$$

in which $h = \frac{\ln(2-d_{N^*})}{N^*-N}$, we shall show that $T_1 W + T_2 Z \in S$ for any $w, z \in S$. In fact, for any $W = W_n, Z = Z_n \in S$, we have

$$T_1 W_n + T_2 Z_n = \frac{1}{C} e^{-qm} W_{n+m} - \frac{1}{C} \frac{1}{A_{d-1}^{d-1}} \sum_{s=n+m}^{\infty} A_{s-n+d-1}^{d-1} P_s e^{q(n-s+k)} Z_{s-k}$$

$$\leq \frac{1}{C} e^{-qm} - \frac{1}{C} \frac{1}{A_{d-1}^{d-1}} \sum_{s=n+m}^{\infty} A_{s-n+d-1}^{d-1} P_s e^{q(n-s+k)}$$

$$\leq 1 \ \forall n \geq N^*$$

$$T_1 W_n + T_z Z_n = T_1 W_{N^*} - 1 + e^{h(N^*-n)} + T_2 Z_{N^*}$$

$$= \frac{1}{C} e^{-qm} W_{N^*+m}$$

$$- \frac{1}{C} \frac{1}{A_{d-1}^{d-1}} \sum_{s=N^*+m}^{infty} A_{s-n+d-1}^{d-1} P_s e^{q(N^*-s+k)} Z_{s-k}$$

$$-1 + (2 - d_{N^*})^{\frac{N^*-n}{N^*-N}}$$

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$$\leq \frac{1}{C}e^{-qm} - \frac{1}{C}\frac{1}{A_{d-1}^{d-1}} \sum_{s=N^*+m}^{\inf ty} A_{s-n+d-1}^{d-1} P_s e^{q(N^*-s+k)}$$
$$-1 + (2 - d_{N^*}) = 1 \quad \text{for} \quad N \leq n \leq N^*.$$

It is easy to see that $T_1 W_n + T_2 Z_n \geq 0$ for all $n \geq N$, and $T_1 W + T_2 Z \in S$ for any $W, Z \in S$. Again by (H_1) and (4), It is certain that $0 < \frac{1}{C}e^{-qm} < 1$, It follow s that T_1 is a strict contraction. We next show that T_2 is completely continuous. Clearly, T_2 is continuous. So it suffices to show that T_2 is a compact operator. From (4) we know that there exists a constant $M > 0$ such that

$$-\frac{1}{C}\frac{1}{A_{d-1}^{d-1}} \sum_{s=n+m}^{\infty} A_{s-n+d-1}^{d-1} P_s e^{q(n-s+k)} \leq M \quad \forall n \geq N^*$$

Thus, it follows form (2) that $T_2 y_n \leq M$ for any $y = y_n \in S$ and $n \neq qN$. Consider any sequence $T_2 y^{(j)} (j = 1, 2, \cdots)$ of $T_2 S$, where $T_2 y^{(j)} = T_2 y_n^{(j)} (n \geq N)$ and $y_n^{(j)} \in S$, since $|T_2 y_n^{(j)}| \geq M$ for all $n \geq N$, and $j = 1, 2, \cdots, T_2 y_n^{(j)}$ is uniformly bounded for $j = 1, 2, \cdots$, By diagonal rule, we may pick out a convergent subsequence $T_2 y_n^{(n_j)}$, L et $T_2 y_n^{(n_j)} \to T_2 y_n^{(0)}$ as $j \to \infty$, Then it follows that $T_2 y^{(n_j)} \to T_2 y^{(0)}$ as $j \to \infty$. Which implies that $\overline{T_2 S}$ is a compact set of X. and hence T_2 is a completely continuous. Therefore, By lemma 2, $\dot{T}$ has a fixed point $y = y_n \in S$. i.e

$$y_n = \begin{cases} \frac{1}{C}e^{-qm}y_{n+m} - \frac{1}{C}\frac{1}{A_{d-1}^{d-1}} \sum_{s=n+m}^{\infty} A_{s-n+d-1}^{d-1} P_s e^{q(n-s+k)} y_{s-k} & n \geq N^* \\ y_{N^*} - 1 + e^{h(N^*-n)} & N \leq n \leq N^* \end{cases}$$
$$(17)$$

Since $y_n \geq 0$ for $n \geq N$, then $y_n \geq e^{h(N^*-n)} - 1 > 0$ for $N \leq n \leq N^*$. By (H_1) and (2.13), we have $y_n \geq -\frac{1}{C}P_{n+m}e^{q(-m+k)}y_{n+m-k}$ for $n \geq N^*$. Notice that $m < k$, it follows from above that $y_n > 0$ for $n \geq N^*$, Hence, we have $y_n > 0$ for $n \geq N$.

Let

$$V_n = y_n e^{-qn} \tag{18}$$

Then (2.13) implies that

$$V_n = \frac{1}{C}V_{n+m} - \frac{1}{C}\frac{1}{A_{d-1}^{d-1}} \sum_{s=n+m}^{\infty} A_{s-n+d-1}^{d-1} P_s U_{s-k} \, for \, n \geq N^*$$

It follows that

$$V_n - CV_{n-m} = \frac{1}{A_{d-1}^{d-1}} \sum_{s=n}^{\infty} A_{s-n+d-1}^{d-1} P_s V_{s-k} \, for \, n \geq N^* + m$$

by (2.8), which implies that $\delta^d(V_n - CV_{n-m}) + P_nV_{n-k} = 0$ for $n \geq N^* + m$. Therefore, the sequence V_n satisfies Eq.(2) for all $n \geq N^* + m$, Again by Eq.(2), it is easy to see that there exists a solution $x_n (n \geq N - k)$ of Eq.(2) such that $x_n = V_n$ for all $n \geq N^* + m$, which together (2.14) implies that such solution of Eq.(2) is bounded, eventually positive, and $\lim_{n\to\infty} x_n = 0$. Assume that $(H_1'), (H_2'), (H_3')$ is true, By using the similar method, we can complete its proof. Thus the proof of Theorem 1 is now completed.

Proof of Theorem 2 if there exists a positive integer N_0 such that $d_n^* = 1$ in (2.5) for all $n \geq N_0$, then by an argument similar to that in the proof of Theorem 1, we can assert that Eq.(2) exists a bounded and eventually positive solution satisfying $\lim_{n\to\infty} x_n = 0$ such that $x_n = e^{-qn}$ for all $n \geq N_0$. Therefore, we next assume that there exists a positive integer $N^* > N$, such that $N^* - k \geq N$, and

$$d_{N^*}^* = \frac{1}{C}e^{-qm} + \frac{1}{C}\frac{1}{A_{d-1}^{d-1}}\sum_{s=n}^{\in fty} A_{s-n+d-1}^{d-1} P_s e^{q(n-s+k)} < 1$$

and $d_n^* \leq 1$ in (2.5)

Let X and S be defined as in the proof Theorem 1. Define $T : S \to X$ as $Ty_n = T_1y_n + T_2y_n$ for $y = \{y_n\} \in S$, where

$$T_1y_n = \begin{cases} \frac{1}{C}e^{-qm}y_{n-m} & n \geq N^* \\ T_1y_{N^*} - 1 + e^{h^*(N^*-n)} & N \leq n \leq N^* \end{cases}$$

$$T_2y_n = \begin{cases} \frac{1}{C}\frac{1}{A_{d-1}^{d-1}}\sum_{s=n}^{\infty} A_{s-n+d-1}^{d-1} P_s e^{q(n-s+k)}y_{s-k} & n \geq N^* \\ T_2y_{N^*} & N \leq n \leq N^* \end{cases}$$

in which $h^* = \frac{\ln(2-d_{N^*}^*)}{N^*-N}$, by using the similar argument as in the proof of Theorem 1, we can easily show that T has a fixed point $y = \{y_n\} \in S$. Set $V_n = y_ne^{-qn}$. Again by an analogous argument as in the proof of Theorem 1, we can assure that this case Eq. (2) has a bounded and eventually positive solution $\{x_n\}$ satisfying $\lim_{n\to\infty} x_n = 0$. Such that $x_n = V_n$ for all $n \geq N^* + m$. This completes the proof of Theorem 2.

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References

1. M.Wazewska-Czyzewska and A.Lasota, Mathematical problems of the dynamics of t he red blood cells Systems, Annals of the Polish Mathematical Society, Seires III, Applied Mathematics **6**, 23-40 (1976).
2. O.Arino and M.Kimmel, Stability analysis of models of cell productio n systems, Mathematical Modelling **17**, 1269-1300 (1986).
3. M.R.S.Kulenovic and G.Ladas, Linearized oscillation in population dy namies, Bulletin of Mathematical Bilolgy **49**, 515-527 (1987).
4. M.R.S.Kulenovic, Global attractivity in population dynamics, Computers Math.,Applic **18**, 925-928 (1989).
5. I.Gyore and G.Lades, Oscillation theory of delay differential equsti ons with applications,Clarendon Press.Oxford,1991.
6. V.LJ.Kocic and G.Ladas, Oscillation and global attrativity in a disc ret e model of Nichoson's blowflies, Applicable Analysis **38**, 1-31 (1990).
7. Joseph W.H.So and J.S.Yu,Global Attractivity and Uniform Persistence i n Nicho lson's Blowflies, Differential Equations and Dynamical Syste ms **2**, 11-18 (1994).
8. Li Jingwen, Global Attractivity in the Discrete Nicholson's Blowflie s, Ann.of Diff.Eqs. **12**, 173-182 (1996).
9. V.L.J.Kocic and G.Ladas, Global attractivity in nonlinear delay diff er ence equations, Proc.Amer.Math.Soc. **115**, 1083-1088 (1992).
10. CH.G.Philos,Oscillations in a nonautonmous delay Logistic difference equation, Proceediongs of the Edinburgh Mathematical Society **35**, 1 21-131 (1992).
11. Ming-Po Chen and J.S.Yu, Oscillation and Global Attractivity in the Discrete Lasota-Wazewaska Model. Soochow J.Math , to appear (1999).
12. Li Jingwen,Asymptotic Behavior of a Delay Differential Model, J.Biomath **9**, 91-95 (1994).
13. Li Jingwen, Global Attractivity of Delay Differential Model with De via ting Argument, J.Biomath **10**, 122-129 (1995).

THE FUNCTION-CONTROLLABILITY OF THE NONLINEAR CONTROL SYSTEMS WITH STATE DELAY AND CONTROL DELAY

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In this paper, we discuss the function-controllability of the nonlinear control systems with state delay and control delay. Give some criteria for the function-controllability of such systems and their first variation systems.

1 Introduction

The phenomena of state delay and control delay are widespread phenomena in many practice systems, such as economic,biological and physiological systems.To more accurately design,analyze and control such system, up to now there many scholars have studied time delay. But all of their results are just for the systems with state delays, or the systems with control delays.For the systems which have the state delays and control delays at the same times, there is no any effective achievement.

Considering nonlinear control system:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-\tau), u(t), u(t-h)), & t \geq t_0, \\ x(t) = \varphi(t), & t_0 - \tau \leq t \leq t_0, \\ u(t) = \psi(t), & t_0 - h \leq t \leq t_0, \end{cases} \quad (1)$$

where $x(t) \in R^n$ is a state vector; $u(t) \in R^m$ is a control vector, $u(\cdot)$ is an admissible control(that is, it is contained in the square integrable functions L^2 on every finite interval); $\tau > 0, h > 0$ are time delays; and $\varphi(t)$ is the initial state function, $\psi(t)$ is the initial control function.

Let $\mathcal{B}$ be a Banach-space defined in $[t_0 - \tau, t_0]$ composed with n-dimension continuous vector-valued functions,the norm be

$$||\varphi|| = \max_{t \in [t_0 - \tau, t_0]} |\varphi|, \quad \varphi \in \mathcal{B}.$$

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Definition 1.1 Let

$$A(t) = \frac{\partial f(t,0,0,0,0)}{\partial x(t)} \triangleq \frac{\partial f}{\partial x(t)}\Big|_{x(t)\equiv x(t-\tau)\equiv u(t)\equiv u(t-h)\equiv 0},$$

$$B(t) = \frac{\partial f(t,0,0,0,0)}{\partial x(t-\tau)} \triangleq \frac{\partial f}{\partial x(t-\tau)}\Big|_{x(t)\equiv x(t-\tau)\equiv u(t)\equiv u(t-h)\equiv 0},$$

$$C(t) = \frac{\partial f(t,0,0,0,0)}{\partial u(t)} \triangleq \frac{\partial f}{\partial u(t)}\Big|_{x(t)\equiv x(t-\tau)\equiv u(t)\equiv u(t-h)\equiv 0},$$

$$D(t) = \frac{\partial f(t,0,0,0,0)}{\partial u(t-h)} \triangleq \frac{\partial f}{\partial u(t-h)}\Big|_{x(t)\equiv x(t-\tau)\equiv u(t)\equiv u(t-h)\equiv 0},$$

we call system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + C(t)u(t) + D(t)u(t-h), \ t \geq t_0, \\ x(t) = \varphi(t), \qquad\qquad t_0 - \tau \leq t \leq t_0, \\ u(t) = \psi(t), \qquad\qquad t_0 - h \leq t \leq t_0, \end{cases} \tag{2}$$

the first variation of system (1).

Let $\mathcal{K}$ be an abstract normed linear space of functions defined on $[t_0 - \tau, t_0]$.Then we have

Definition 1.2 Let $\alpha \in \mathcal{K}$;and $x^0(\cdot, t_0, \varphi_\alpha, u_\alpha), (\varphi_\alpha \in \mathcal{B})$ be the trajectory of system (1) which satisfies that when $t \in [t_1, t_1 + \tau]$, $x^0(t, t_0, \varphi_\alpha, u_\alpha) \doteq \alpha(t - t_1 + t_0 - \tau)$, and if

$$A(t) = \frac{\partial f(t, x^0(t,t_0,\varphi_\alpha,u_\alpha), x^0(t-\tau,t_0,\varphi_\alpha,u_\alpha), u_\alpha(t), u_\alpha(t-h))}{\partial x(t)},$$

$$B(t) = \frac{\partial f(t, x^0(t,t_0,\varphi_\alpha,u_\alpha), x^0(t-\tau,t_0,\varphi_\alpha,u_\alpha), u_\alpha(t), u_\alpha(t-h))}{\partial x(t-\tau)},$$

$$C(t) = \frac{\partial f(t, x^0(t,t_0,\varphi_\alpha,u_\alpha), x^0(t-\tau,t_0,\varphi_\alpha,u_\alpha), u_\alpha(t), u_\alpha(t-h))}{\partial u(t)},$$

$$D(t) = \frac{\partial f(t, x^0(t,t_0,\varphi_\alpha,u_\alpha), x^0(t-\tau,t_0,\varphi_\alpha,u_\alpha), u_\alpha(t), u_\alpha(t-h))}{\partial u(t-h)},$$

we call system (2) the first variation of system (1) about $x^0(\cdot, t_0, \varphi_\alpha, u_\alpha)$.

Definition 1.3 A system (1) is controllable to a functions $\alpha(\cdot) \in \mathcal{K}$ with respect to the space of initial functions $\mathcal{B}$, if for any given $\varphi \in \mathcal{B}$,there exist a time t_1, $t_0 < t_1 < \infty$,and an admissible control segment $u_{[t_0-h,t_1+\tau]}$ such that $x(t; t_0, \varphi, u) = \alpha(t - t_1 + t_0 - \tau), t \in [t_1, t_1 + \tau]$, where $x(t; t_0, \varphi, u)$ is the solution of (1),starting at time t_0, with state initial function φ and control u. If $\alpha(\cdot) \equiv 0$,then the system is controllable to the origin.

Definition 1.4 A system (1) is locally controllable to a functions $\alpha \in \mathcal{K}$ with respect to the space of initial functions $\mathcal{B}$, if given any initial time t_0 and a trajectory $x^0(\cdot, t_0, \varphi_\alpha, u_\alpha)$, $\varphi_\alpha \in \mathcal{B}$, u_α admissible, such that,for some

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time $t_1 > t_0, x(t; t_0, \varphi, u) = \alpha(t - t_1 + t_0 - \tau)$, for all $t \in [t_1, t_1 + \tau]$, then there is a neighborhood $N(\varphi_\alpha)$ of the initial function φ_α such that for each $\varphi \in N(\varphi_\alpha)$ there exists an admissible control u^* defined on $[t_0, t_1 + \tau]$ such that $x(t; t_0, \varphi_\alpha, u^*) = \alpha(t - t_1 + t_0 - \tau)$ for all $t \in [t_1, t_1 + \tau]$. If $\alpha(\cdot) \equiv 0$, then the system is locally controllable to the origin with respect to $\mathcal{B}$.

In this paper, we discuss the function-controllability of the nonlinear control systems with state delay and control delay. Give some criteria for the function-controllability of such systems and their first variation systems.

2 The Function-controllability of System (2)

We know that the solution of system (2) at time t_1 can be written as

$$x(t_1) = x(t_1, \varphi) + \int_{t_0}^{t_1} X(t_1, s)[C(s)u(s) + D(s)u(s - h)]ds. \tag{3}$$

Where $x(t, \varphi) = X(t, t_0)\varphi(t_0) + \int_{t_0 - \tau}^{t_0} X(t, s + \tau)B(s + \tau)\varphi(s)ds$, and X(t, s) satisfies

$$\begin{cases} \frac{\partial}{\partial t}X(t,s) = A(t)X(t,s) + B(t)X(t - \tau, s), \text{for} (t,s) \in [s, t_1] \times [t_0, t_1], \\ X(t,s) = \begin{cases} I, & \text{for} \quad t = s, \\ 0, & \text{for} \quad (t,s) \in [t_0 - \tau, s) \times [t_0, t_1], \end{cases} \end{cases} \tag{4}$$

From (3), by simple calculation we have

$$\begin{aligned} x(t_1) = M(t_1, t_0, \varphi, \psi) &+ \int_{t_0}^{t_1 - h}[X(t_1, s)C(s) + X(t_1, s + h)D(s + h)]u(s)ds \\ &+ \int_{t_1 - h}^{t_1} X(t_1, s)C(s)u(s)ds. \end{aligned} \tag{5}$$

There $M(t_1, t_0, \varphi, \psi) = x(t_1, \varphi) + \int_{t_0 - h}^{t_0} X(t_1, s + h)D(s + h)\psi(s)ds$. Now, let

$$\begin{aligned} W(t_0, t_1) = &\int_{t_0}^{t_1 - h}[X(t_1, s)C(s) + X(t_1, s + h)D(s + h)][C^T(s)X^T(t_1, s) \\ &+ D^T(s + h)X^T(t_1, s + h)]ds \\ &+ \int_{t_1 - h}^{t_1} X(t_1, s)C(s)C^T(s)X^T(t_1, s)ds, \end{aligned}$$

we have

Theorem 2.1 For system (2), if there exists $t_1 > t_0$, such that

$$\text{rank}(W(t_0, t_1)) = n, \tag{6}$$

then there exists an admissible control which results in the solution having a zerocrossing in finite time.

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Proof In (4),take

$$u^*(s) = \begin{cases} [C^T(s)X^T(t_1,s) + D^T(s+h)X^T(t_1,s+h)] \\ \qquad\qquad\qquad \times W^{-1}(t_0,t_1)M(t_0,t_1,\varphi,\psi), \ s \in [t_0,t_1-h), \\ C^T(s)X^T(t_1,s)W^{-1}(t_0,t_1)M(t_0,t_1,\varphi,\psi), \qquad s \in [t_1-h,t_1], \end{cases}$$

we have $x(t_1) = 0$.

Theorem 2.2 System (2) is controllable to the origin with respect to $\mathcal{B}$ if it satisfies

(i) there exists $t_1 > t_0$ such that (6) holds;

(ii) for $\varphi \in \mathcal{B}$,with t_1 as in (6) and for some admissible control $u_{[t_0-h,t_1]}$ such that $x(t_1,t_0,\varphi,u_{[t_0-h,t_1]}) = 0$, the equation

$$B(t)x(t-\tau) + C(t)u(t) + D(t)u(t-h) = 0 \tag{7}$$

has an admissible solution $u(\cdot)$ on the interval $(t_1,t_1+\tau)$.

Proof By Theorem 2.1 we have that for any $\varphi \in \mathcal{B}$ there exists $u_{[t_0-h,t_1]}$ such that $x(t_1,t_0,\varphi,u_{[t_0-h,t_1]}) = 0$. If (7) holds,then over interval $(t_1,t_1+\tau)$, (2) becomes

$$\begin{cases} \dot{x}(t) = A(t)x(t), \ t \in (t_1,t_1+\tau) \\ x(t_1) = 0 \end{cases}. \tag{8}$$

It follows by the uniqueness theorem for ordinary differential equations that $x(t) = 0$ for all $t \in [t_1,t_1+\tau]$.

Similarly we can prove

Theorem 2.3 System (2) is controllable to a functions $\alpha(\cdot) \in \mathcal{K}$ with respect to the space of initial functions $\mathcal{B}$ if it satisfies

(i) there exists $t_1 > t_0$ such that (6) holds;

(ii) for $\varphi \in \mathcal{B}$,with t_1 as in (6) and for some admissible control $u_{[t_0-h,t_1]}$ such that $x(t_1,t_0,\varphi,u_{[t_0-h,t_1]}) = \alpha(t_1+t_0-\tau)$,the equation

$$\begin{aligned} B(t)x(t-\tau) + C(t)u(t) + D(t)u(t-h) - \tfrac{d}{dt}\alpha(t-t_1+t_0-\tau) \\ + A(t)\alpha(t-t_1+t_0-\tau) = 0 \end{aligned} \tag{9}$$

has an admissible solution $u(\cdot)$ on the interval $(t_1,t_1+\tau)$.

3 The Function-controllability of System (1)

Now we discuss the function-controllability of the nonlinear control system with state delay and control delay (1).

Theorem 3.1 System (1) is locally controllable to the origin with respect to $\mathcal{B}$ if its first variation (2) satisfies

(i) there exists $t_1 > t_0$ such that (6) holds;

(ii) for $\varphi \in \mathcal{B}$,with t_1 as in (6) and for some admissible control $u_{[t_0-h,t_1]}$ such that $x(t_1, t_0, \varphi, u_{[t_0-h,t_1]}) = 0$,the equation

$$B(t)x(t-\tau) + C(t)u(t) + D(t)u(t-h) = 0$$

has an admissible solution $u(\cdot)$ on the interval $(t_1, t_1 + \tau)$.

Proof We introduce a parameter ξ into the control u and define the control u^ξ on $[t_0, t_1 + \tau]$ as follows:

When $t_0 \leq t \leq t_1 - h$, $u^\xi(t) = u(t, \xi) = [C^T(t)X^T(t_1, t) + D^T(t + h)X^T(t_1, t+h)]\xi$,

When $t_1 - h < t \leq t_1$, $u^\xi = u(t, \xi) = C^T(t)X^T(t_1, t)\xi$,

When $t_1 < t \leq t_1 + \tau$,control u^ξ satisfies equation

$B(t)x(t - \tau, t_0, 0^{\mathcal{B}}, u^\xi) + C(t)u(t) + D(t)u(t - h) = 0$,

here $0^{\mathcal{B}}$ is the zero-subset of space $\mathcal{B}$. We still let $u^\xi = u(t, \xi)$.

It is obviously that when $t \in [t_0, t_1]$,$u(t, 0) = u^0(t) = 0$,and if $\varphi \equiv 0, \psi \equiv 0$,we have that on $[t_0 - \tau, t_1]$, $x(t, t_0, 0^{\mathcal{B}}, u^0) = 0$.

Let

$$J(t) = \frac{\partial x(t, t_0, 0^{\mathcal{B}}, u^\xi)}{\partial \xi}|_{\xi=0}.$$

For $\varphi \equiv 0$,from (1) we have

$$x(t, t_0, 0^{\mathcal{B}}, u^\xi) = \int_{t_0}^t f(s, x(s), x(s - \tau, u^\xi(s), u^\xi(s - h))ds, \qquad t_0 \leq t \leq t_1 + \tau.$$

Taking partial derivative about ξ, we have

$$J(t) = \int_{t_0}^t (A(s)J(s) + B(s)J(s - \tau) + C(s)\frac{\partial u(s)}{\partial \xi}|_{\xi=0}$$
$$+ D(s)\frac{\partial u(s - h)}{\partial \xi}|_{\xi=0})ds.$$

Differentiating gives

$$\dot{J}(t) = A(t)J(t) + B(t)J(t - \tau) + C(t)\frac{\partial u(t, 0)}{\partial \xi} + D(t)\frac{\partial u(t - h, 0)}{\partial \xi}. \qquad (10)$$

That is when $t \in (t_1 - h, t_1]$,the solution of equation (10) is

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$$J(t) = \int_{t_0}^{t-h} [X(t,s)C(s) + X(t,s+h)D(s+h)] \frac{\partial u(s,0)}{\partial \xi} ds$$

$$+ \int_{t-h}^{t} X(t,s)C(s) \frac{\partial u(s,0)}{\partial \xi} ds$$

$$= \int_{t_0}^{t-h} [X(t,s)C(s) + X(t,s+h)D(s+h)]$$

$$\times [C^T(s)X^T(t,s) + D^T(s+h)X^T(t,s+h)]ds$$

$$+ \int_{t-h}^{t} X(t,s)C(s)C^T(s)X^T(t,s)ds.$$

From condition (i), we have $\det(J(t_1)) \neq 0$. Moreover, on the interval $(t_1, t_1 + \tau)$, (10) is

$$\dot{J}(t) = A(t)J(t), \tag{11}$$

so that $J(t)$ is a fundamental matrix solution for (11). It follows that $\det(J(t)) \neq 0$ for $t \in [t_1, t_1 + \tau]$.

By the definition of $J(t)$ and the implicit function theorem, above facts show that one may solve $x(t, t_0, \varphi, \xi) = 0$, $t_1 \leq t \leq t_1 + \tau$, for ξ as a function of φ.

Similar to the proof of Theorem 3.1, we can prove

Theorem 3.2 System (1) is locally controllable to a functions $\alpha(\cdot) \in \mathcal{K}$ with respect to the space of initial functions $\mathcal{B}$ if its first variation (2) about the trajectory $x(\cdot, t_0, \varphi_\alpha, u_\alpha)$ as defined in Definition 1.4 satisfies

(i) for $t_1 > t_0$ defined in Definition 1.4, (6) holds;

(ii) for $\varphi \in \mathcal{B}$, with t_1 as above and for some admissible control $u_{[t_0-h,t_1]}$ such that $x(t_1, t_0, \varphi, u_{[t_0-h,t_1]}) = \alpha(t_1 + t_0 - \tau)$, the equation

$$B(t)x(t-\tau) + C(t)u(t) + D(t)u(t-h) - \frac{d}{dt}\alpha(t - t_1 + t_0 - \tau)$$
$$+ A(t)\alpha(t - t_1 + t_0 - \tau) = 0$$

has an admissible solution $u(\cdot)$ on the interval $(t_1, t_1 + \tau)$.

References

1. J. Hale, Theory of Functional Differential Equations (Springer Verlay, 1992).
2. Zuxiu Zheng, Theory of Functional Differential Equations (Anhei Education Press, 1994).
3. Wei Jiang, The Degenerate Differential Systems with Delay (Anhei University Press, 1998).
4. Wei Jiang, Journal of Mathematical Study **12**, 55-60(1996).
5. B.Zmood Ronald, SIAM.J. Control **12**, 609-623(1974).
6. GuoJun Ji, The Doctor Thesis of Southeast University (1998).

STABILITY OF NONLINEAR ORDINARY DIFFERENTIAL SYSTEMS

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Stability and instability criteria are found for nonlinear systems of ordinary differential equations by the methods involving the Lozinskii measures of matrices. The results obtained are different from those by the Liapunov function method and by the variation of constants formula, and they are relatively easy to apply.

1 Introduction and preliminaries

It is well-known that the stability properties of the system of linear differential equations

$$x' = Ax \qquad (1)$$

where $A \in \mathbb{C}^{n \times n}$ is a constant $n \times n$ matrix, is completely determined by the Jordan canonical form of A, or the eigenvalue structure of A. However, if $A = A(t)$ is a variable matrix, the eigenvalue criterion fails. Among various approaches such as the Liapunov function analysis and the Lasalle invariance principle, the Lozinskii measure method has been used for stability of the non-autonomous linear system (1.1), and several results on stability have been found, see [1,3,4]. However, the Lozinskii measure method has not been extensively applied in the literature to the stability of nonlinear differential systems. In this paper, we will establish stability and instability criteria for nonlinear differential systems utilizing the Lozinskii measures. Our new results are different from those by other classical approaches such as the Liapunov function method and the variation of constants formula, and they are relatively easy to apply. Furthermore, they improve some recent work on stability, see [2] and the references therein.

Throughout this paper we let $\mathbb{R}$ be the set of all real numbers, $\mathbb{C}$ the set of all complex numbers, and $\mathbb{R}_+ = [0, \infty)$. Let $\mathbb{X}, \mathbb{Y}$ be two Banach spaces. By $C(\mathbb{X}, \mathbb{Y}), C^1(\mathbb{X}, \mathbb{Y})$, and $L_{loc}(\mathbb{X}, \mathbb{Y})$ we denote the collections of all operators mapping $\mathbb{X}$ into $\mathbb{Y}$ which are continuous, continuously differentiable, and locally integrable over $\mathbb{X}$, respectively. By $C^1_*(\mathbb{X}, \mathbb{Y})$ we denote the collection of all operators which are in $C(\mathbb{X}, \mathbb{Y})$ and are continuously differentiable except possibly on a closed subset of $\mathbb{X}$ with a zero measure.

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In order to establish our stability criteria for nonlinear systems we need the following notations and definitions for matrices.

For any $n \times n$ matrix A we denote by $\lambda_i(A)$, $i = 1, \cdots, n$, the eigenvalues of A satisfying the ordering

$$\Re\lambda_1(A) \geq \Re\lambda_2(A) \geq \cdots \geq \Re\lambda_n(A).$$

Definition 1.1 *(see [4]) We define* $|A|_i = \sup\limits_{x \in \mathbb{C}^n, x \neq 0} \dfrac{|Ax|_i}{|x|_i}$, $i = 1, 2, \cdots, \infty$, *where* $x = (x_1, \cdots, x_n)^T$,

$$|x|_i = (\sum_{j=1}^{n} |x_j|^i)^{\frac{1}{i}}, \; i < \infty, \; and \; |x|_\infty = \max_{1 \leq j \leq n}\{|x_j|\}.$$

The Lozinskii measures $\mu_i(A)$ *and* $\nu_i(A)$ *are defined by*

$$\mu_i(A) = \lim_{h \to 0^+} \frac{|I + hA|_i - 1}{h}, \quad \nu_i(A) = -\mu_i(-A), \quad i = 1, 2, \cdots, \infty.$$

It has been shown that $\mu_i(A)$ *and* $\nu_i(A), i = 1, 2, \cdots, \infty$, *exist for any* $n \times n$ *matrix* A *and can be explicitly evaluated for* $i = 1, 2, \infty$:

$$\mu_1(A) = \sup_j(\Re a_{jj} + \sum_{i, i \neq j} |a_{ij}|), \quad \nu_1(A) = \inf_j(\Re a_{jj} - \sum_{i, i \neq j} |a_{ij}|),$$

$$\mu_2(A) = \lambda_1(\frac{1}{2}(A + A^*)), \quad \nu_2(A) = \lambda_n(\frac{1}{2}(A + A^*)), \tag{2}$$

$$\mu_\infty(A) = \sup_i(\Re a_{ii} + \sum_{j, j \neq i} |a_{ij}|), \quad \nu_\infty(A) = \inf_i(\Re a_{ii} - \sum_{j, j \neq i} |a_{ij}|),$$

where A^* *is the complex conjugate transpose of* A.

In the following, without specification, we denote by $\mu(A)$ *and* $\nu(A)$ *any pair of* $\mu_i(A)$ *and* $\nu_i(A)$, $i = 1, 2, \cdots, \infty$. *The following properties of* μ *and* ν *are well known: for any* $A, B \in \mathbb{C}^{n \times n}$ *and any Lozinskii measures* μ *and* ν *we have*

(i) $\mu(A + B) \leq \mu(A) + \mu(B), \quad \nu(A + B) \geq \nu(A) + \nu(B)$;

(ii) $\mu(\alpha A) = \alpha\mu(A), \quad \nu(\alpha A) = \alpha\nu(A), \quad \alpha > 0,$

(iii) $-|A| \leq \nu(A) \leq \Re\lambda_n(A) \leq \Re\lambda_1(A) \leq \mu(A) \leq |A|.$

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2 Stability by first approximation

Consider

$$x' = A(t)x + f(t, x) \tag{1}$$

where $A \in C(\mathbb{R}_+, \mathbb{C}^{n \times n})$, $f \in C(\mathbb{R}_+ \times \mathbb{C}^n, \mathbb{C}^n)$.

The next two theorems are for the non-uniform stability and instability of Eq. (2.1) which are difficult to derive from the variation of constants formula.

Theorem 1 *Assume there exists a nonsingular $C \in C^1_*(\mathbb{R}_+, \mathbb{C}^{n \times n})$ such that C^{-1} is bounded for $t \geq 0$, and there exists $\sigma > 0$ and $\varphi \in L_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ such that*

$$(i) \quad |C(t)f(t, x)| \leq \varphi(t)|C(t)x| \quad \text{for } |x| \leq \sigma, \ t \in \mathbb{R}_+ \tag{2}$$

$$(ii) \quad \int_0^\infty [\mu(C'C^{-1} + CAC^{-1}) + \varphi] = -\infty. \tag{3}$$

Then Eq. (2.1) is asymptotically stable.

Proof: Let x be a solution of (2.1) and let $r(t) = C(t)x(t)$. Then for $|x| \leq \sigma$, $t \in \mathbb{R}_+$ a.e. from (2.2)

$$\begin{aligned}
D^+|r(t)| &= \lim_{h \to 0^+} \frac{1}{h} \left[|Cx + h(Cx)'| - |Cx| \right] \\
&= \lim_{h \to 0^+} \frac{1}{h} \left[|Cx + h(C'x + CAx + Cf)| - |Cx| \right] \\
&\leq \lim_{h \to 0^+} \frac{1}{h} \left\{ |Cx| \left[|I + h(C'C^{-1} + CAC^{-1})| - 1 \right] + h|Cf| \right\} \\
&\leq |Cx| \lim_{h \to 0^+} \frac{1}{h} \left[|I + h(C'C^{-1} + CAC^{-1})| - 1 \right] + \varphi|Cx| \\
&= |Cx| \left[\mu(C'C^{-1} + CAC^{-1}) + \varphi \right] \\
&= |r(t)| \left[\mu(C'C^{-1} + CAC^{-1}) + \varphi \right](t).
\end{aligned}$$

If $|x(t_0)| \leq \sigma$, then from (2.3)

$$|r(t)| \leq |r(t_0)| \exp \int_{t_0}^t \left[\mu(C'C^{-1} + CAC^{-1}) + \varphi \right] \to 0, \quad \text{as } t \to \infty.$$

By the boundedness of C^{-1} we see

$$|x(t)| = |C^{-1}(t)r(t)| \leq |C^{-1}(t)||r(t)| \to 0, \quad \text{as } t \to \infty.$$

This implies that (2.1) is asymptotically stable. ∎

Theorem 2 *Assume there exist a nonsingular $C \in C_*^1(\mathbb{R}_+, \mathbb{C}^{n \times n})$, $\sigma > 0$ and $\varphi \in L_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ such that (2.2) holds, and for some $t_0 \in \mathbb{R}_+$*

$$\limsup_{t \to \infty} |C(t)|^{-1} \exp \int_{t_0}^t [\nu(C'C^{-1} + CAC^{-1}) - \varphi] = \infty. \qquad (4)$$

Then Eq. (2.1) is unstable.

Proof: Assume the contrary, i.e., (2.1) is stable. Then there exists a solution $x(t)$ of (2.1) satisfying $|x(t)| \leq \sigma$ for $t \in \mathbb{R}_+$. Similar to the proof of Theorem 2.1 we can show that for $t \geq t_0 \geq 0$

$$|(Cx)(t_0)| \leq |(Cx)(t)| \exp \int_t^{t_0} [\nu(C'C^{-1} + CAC^{-1}) - \varphi].$$

That is

$$|(Cx)(t)| \geq |(Cx)(t_0)| \exp \int_{t_0}^t [\nu(C'C^{-1} + CAC^{-1}) - \varphi].$$

Then

$$|x(t)| \geq |(Cx)(t_0)||C(t)|^{-1} \exp \int_{t_0}^t [\nu(C'C^{-1} + CAC^{-1}) - \varphi].$$

From this and (2.4) we see that $|x(t)|$ is unbounded, contradicting the assumption. ∎

As a special case of Eq. (2.1) we consider equations of the form

$$x' = A(t)x + B(t)g(x) \qquad (5)$$

where $A, B \in C(\mathbb{R}_+, \mathbb{C}^{n \times n})$, $g \in C^1(\mathbb{C}^n, \mathbb{C}^n)$ satisfying $g(0) = 0$, $\dfrac{\partial g_i(0)}{\partial x_j} = 0$ for i, $j = 1, ..., n$, $i \neq j$. Denote $G(x) = \text{diag}\{\frac{g_1(x)}{x_1}, ..., \frac{g_n(x)}{x_n}\}$, $x_i \neq 0$, $i = 1, ..., n$, and

$$G_0 = \text{diag} \left\{ \frac{\partial g_1(0)}{\partial x_1}, ..., \frac{\partial g_n(0)}{\partial x_n} \right\}.$$

It is easy to see that $g(x) = G(x)x$ and $\lim_{x \to 0} G(x) = G_0$. The following results are directly from theorems 2.1 and 2.2.

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Corollary 1 *Assume there exist a nonsingular constant $C \in \mathbb{C}^{n \times n}$ and a $\delta > 0$ such that*

$$\int_0^\infty \left[\mu(C(A + BG_0)C^{-1}) + \delta|B| \right] = -\infty. \tag{6}$$

Then Eq. (2.5) is asymptotically stable.

Corollary 2 *Assume there exist a nonsingular constant $C \in \mathbb{C}^{n \times n}$ and a $\delta > 0$ such that*

$$\limsup_{t \to \infty} \int_0^t \left[\nu(C(A + BG_0)C^{-1}) - \delta|B| \right] = \infty. \tag{7}$$

Then Eq. (2.5) is unstable.

Remark 2.1 In (2.6), $\mu\left(C(A + BG_0)C^{-1}\right)$ can be replaced by $\mu(CAC^{-1}) + \mu(CBG_0C^{-1})$, and in (2.7), $\nu\left(C(A + BG_0)C^{-1}\right)$ can be replaced by $\nu(CAC^{-1}) + \nu(CBG_0C^{-1})$. Some special forms of the Lozinskii measures are

(i) for $A = \text{diag}(a_1, ..., a_n)$ and $C = \text{diag}(c_1, ..., c_n)$

$$\mu(CAC^{-1}) = \max_{1 \le i \le n} \{\Re a_i\}, \qquad \nu(CAC^{-1}) = \min_{1 \le i \le n} \{\Re a_i\};$$

(ii) for $C = I$

$$\mu_1(CBG_0C^{-1}) = \mu_1(BG_0) = \max_j \left\{ \frac{\partial g_j(0)}{\partial x_j} \left(\Re b_{jj} + \sum_{i, i \ne j} |b_{ij}| \right) \right\};$$

(iii) for $C = G_0$

$$\mu_\infty(CBG_0C^{-1}) = \mu_\infty(G_0B) = \max_i \left\{ \frac{\partial g_i(0)}{\partial x_i} \left(\Re b_{ii} + \sum_{j, j \ne i} |b_{ij}| \right) \right\}.$$

3 Stability of general nonlinear equations

In this section we investigate the equation

$$x' = \sum_{k=1}^m A_k(t) g_k(x) \tag{1}$$

where $A_k \in C(\mathbb{R}_+, \mathbb{R}^{n \times n})$, $g_k = (g_k^{(1)}, ..., g_k^{(n)})^T \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, ..., m$, and

$$x_i g_k^{(i)}(x) > 0 \quad \text{for } x_i \ne 0, \; i = 1, ..., n, \; k = 1, ..., m. \tag{2}$$

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Obviously, (3.2) implies that

$$x_i = 0 \text{ for some } i \iff g_k^{(i)}(x) = 0 \text{ for the same } i, \quad k = 1, ..., m. \quad (3)$$

All results in this section involve only the Lozinskii measures μ_1 and ν_1, corresponding to the $\ell_1 - norm$ in $\mathbb{R}^n$: $|x| = \sum_{i=1}^{n} |x_i|$. Recall that for a real matrix A

$$\mu_1(A) = \sup_j(a_{ij} + \sum_{i, i \neq j} |a_{ij}|), \quad \nu_1(A) = \inf_j(a_{jj} + \sum_{i, i \neq j} |a_{ij}|).$$

The following lemma will be used in the proofs.

Lemma 1 *Let $x, y, z \in \mathbb{R}^n$ be such that*

(i) $x_i y_i \geq 0$, $i = 1, ..., n$;

(ii) $x_i = 0$ for some $i \iff y_i = 0$ for the same i.

Then for $\alpha > 0$ and small $h > 0$, $|x|_1 = \alpha |y|_1$ implies that

$$|x + hz|_1 = \alpha |y + \frac{h}{\alpha} z|_1.$$

Proof: Let $I_1 = \{i \in \{1, ..., n\} : x_i \neq 0\}$ and $I_2 = \{i \in \{1, ..., n\} : x_i = 0\}$. From the conditions we see that $y_i = 0$ for $i \in I_2$ and $\operatorname{sgn} y_i = \operatorname{sgn} x_i$ for $i \in I_1$, and

$$\sum_{i \in I_1} x_i \operatorname{sgn} x_i = \alpha \sum_{i \in I_1} y_i \operatorname{sgn} y_i.$$

For $h > 0$ small enough

$$|x + hz|_1 = \sum_{i=1}^{n} |x_i + hz_i| = \sum_{i \in I_1} (x_i + hz_i) \operatorname{sgn} x_i + \sum_{i \in I_2} hz_i \operatorname{sgn} z_i$$

$$= \alpha \left\lceil \sum_{i \in I_1} (y_i + \frac{h}{\alpha} z_i) \operatorname{sgn} y_i + \sum_{i \in I_2} \frac{h}{\alpha} z_i \operatorname{sgn} z_i \right\rceil$$

$$= \alpha \sum_{i=1}^{n} |y_i + \frac{h}{\alpha} z_i| = \alpha |y + \frac{h}{\alpha} z|_1.$$

The proof is complete. ∎

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Theorem 1 *Assume there exist $C = \text{diag}(c, ..., c_n)$ and $D = \text{diag}(d_1, ..., d_n)$ where $c_i > 0$ and $d_i > 0$ for $i = 1, ..., n$, such that $\mu_1(CA_k(t)D) \leq 0$, for $t \geq 0$ and $k = 1, ..., m$. Then Eq. (3.1) is uniformly stable.*

Proof: Choose a Liapunov function $V(x) = |Cx|_1$. Then the Dini-derivative of V along any solution x of (3.1) satisfies that

$$D^+V(x)|_{(6.1)} = \lim_{h \to 0^+} \frac{1}{h}\left[|Cx + hCx'|_1 - |Cx|_1\right]$$

$$= \lim_{h \to 0^+} \frac{1}{h}\left[|Cx + h\sum_{k=1}^m CA_kg_k(x)|_1 - |Cx|_1\right] \tag{4}$$

$$= \lim_{h \to 0^+} \frac{1}{h}\left[|mCx + h\sum_{k=1}^m CA_kg_k(x)| - m|Cx|_1\right]$$

$$\leq \sum_{k=1}^m \lim_{h \to 0^+} \frac{1}{h}\left[|Cx + h(CA_kD)D^{-1}g_k(x)|_1 - |Cx|_1\right]. \tag{5}$$

Note that (3.2) and (3.3) imply that conditions i) and ii) of Lemma 3.1 are satisfied with x and y replaced by Cx and $D^{-1}g_k(x)$, $k = 1, ..., m$, respectively. Define functions $\alpha_k, k = 1, ..., m$, by

$$\alpha_k(t) = \begin{cases} |Cx(t)|_1/|D^{-1}g_k(x(t))|_1, & \text{if } x(t) \neq 0 \\ 1, & \text{if } x(t) = 0. \end{cases} \tag{6}$$

Then by Lemma 3.1 we have that for $k = 1, ..., m$

$$|Cx + h(CA_kD)D^{-1}g_k(x)|_1 = \alpha_k|D^{-1}g_k(x) + \frac{h}{\alpha_k}(CA_kD)D^{-1}g_k(x)|_1$$

$$\leq \alpha_k|I + \frac{h}{\alpha_k}(CA_kD)|_1 |D^{-1}g_k(x)|_1. \tag{7}$$

Substitute (3.6) and (3.7) into (3.5) we obtain that

$$D^+V(x)|_{(3.1)} \leq \sum_{k=1}^m \lim_{h \to 0^+} \frac{\alpha_k}{h}\left[|I + \frac{h}{\alpha_k}(CA_kD)|_1 - 1\right]|D^{-1}g_k(x)|_1$$

$$= \sum_{k=1}^m \mu_1(CA_kD)|D^{-1}g_k(x)|_1 \leq 0.$$

Therefore, Eq. (3.1) is uniformly stable. ∎

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Theorem 2 *Let $A_k(t) \equiv A_k \in \mathbb{R}^{n \times n}$, $k = 1, ..., m$, are constant matrices. Assume there exist $C = \text{diag}(c_1, .., c_n)$ and $D = \text{diag}(d_1, ..., d_n)$, where $c_i > 0, d_i > 0$, $i = 1, ..., n$, such that*

(i) $\mu_1(CA_kD) \le 0, k = 1, ..., m$, *and*

(ii) $\mu_1(CA_kD) < 0$ *for some* $k \in \{1, ..., m\}$.

Then Eq. (3.1) is uniformly asymptotically stable.

Proof: Choose $V(x) = |Cx|_1$. Then similar to the proof of Theorem 3.1 we see that $D^+V(x)|_{(3.1)} < 0$. This concludes the theorem. ∎

Theorem 3 *Assume there exist $C = \text{diag}(c_1, ..., c_n)$ and $g = (g^{(1)}, ..., g^{(n)})^T \in C(\mathbb{R}^n, \mathbb{R}^n)$ satisfying that $c_i > 0$, $i = 1, ..., n$, and*

$$x_i g^{(i)}(x) > 0 \quad \text{for } x_i \ne 0, \ i = 1, ..., n. \tag{8}$$

Denote

$$H_k(x) = \begin{cases} \text{diag}\left(g_k^{(1)}(x)/g^{(1)}(x), ..., g_k^{(n)}(x)/g^{(n)}(x) \right), & \text{if } x \ne 0 \\ 1, & \text{if } x = 0. \end{cases}$$

Then Eq. (3.1) is uniformly stable provided

$$\mu_1\left(\sum_{k=1}^{m} CA_k(t)H_k(x) \right) \le 0 \tag{9}$$

for $t \in \mathbb{R}_+$, $|x| < \delta$ for some $\delta > 0$.

Proof: (3.8) implies that conditions i) and ii) of Lemma 3.1 are satisfied with x and y replaced by Cx and $g(x)$. Define a function α by

$$\alpha(t) = \begin{cases} |Cx(t)|_1/|g(x(t))|_1 & \text{if } x(t) \ne 0 \\ 1, & \text{if } x(t) = 0. \end{cases}$$

Then by Lemma 3.1 we have that

$$|Cx + h \sum_{k=1}^{m} CA_k g_k(x)|_1 = \alpha|g(x) + \frac{h}{\alpha} \sum_{k=1}^{m} CA_k H_k(x)g(x)|_1$$

$$\le \alpha|g(x)|_1 \ |I + \frac{h}{\alpha} \sum_{k=1}^{m} CA_k H_k(x)|_1.$$

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From (3.1)

$$D^+V(x)|_{(3.1)} \leq \lim_{h \to 0^+} \frac{\alpha}{h} \left\lceil |I + \frac{h}{\alpha} \sum_{k=1}^{m} CA_k H_k(x)|_1 - 1 \right\rceil |g(x)|_1$$

$$= \mu_1 \left(\sum_{k=1}^{m} CA_k(t)G_k(x) \right) |g(x)|_1 \leq 0.$$

Therefore, (3.1) is uniformly stable. ∎

Theorem 4 *Let $A_k(t) \equiv A_k \in \mathbb{R}^{n \times n}$, $k = 1, ..., m$, are constant matrices. In addition to the conditions of Theorem 3.3 assume*

$$\mu_1 \left(\sum_{k=1}^{m} CA_k H_k(x) \right) < 0 \tag{10}$$

for $0 < |x| < \delta$ for some $\delta > 0$. Then Eq. (3.1) is uniformly asymptotically stable.

The proof is omitted.

Remark 3.1 In Theorems 3.3 and 3.4, conditions (3.9) and (3.10) can be replaced by

$$\sum_{k=1}^{n} \mu_1 \left(CA_k(t)H_k(x) \right) \leq 0$$

and

$$\sum_{k=1}^{n} \mu_1 \left(CA_k H_k(x) \right) < 0,$$

respectively. Note that we do not require the nonpositivity of each $\mu_1 \left(CA_k(t)H_k(x) \right)$ individually.

As a special case of (3.1) we consider the equation

$$x' = A(t)x + \sum_{k=1}^{m} A_k(t)g_k(x) \tag{11}$$

where $A, A_k \in C(\mathbb{R}_+, \mathbb{R}^{n \times n})$, g_k satisfies (3.2) and (3.3), $k = 1, ..., m$.

Theorem 5 *Assume there exist $C = \text{diag}(c_1, ..., c_n)$ and $D = \text{diag}(d_1, ..., d_n)$ where $c_i > 0$, $d_i > 0$, $i = 1, ..., n$, such that*

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(i) $\mu_1 (CA_k(t)D) \le 0$ for $t \ge 0$, $k = 1, ..., m$ and

$$\int_0^\infty \mu_1 \left(CA(s)C^{-1} \right) ds = -\infty,$$

then Eq. (3.11) is asymptotically stable;

(ii) $\mu_1 (CA_k(t)D) \le 0$ for $t \ge 0$, $k = 1, ..., m$, and $\mu_1 \left(CA(t)C^{-1} \right) \le -\alpha < 0$, $t \ge 0$, then Eq. (3.11) is uniformly asymptotically stable;

(iii) $\nu_1 (CA_k(t)D) \ge 0$ for $t \ge 0$, $k = 1, ..., m$, and

$$\limsup_{t \to \infty} \int_0^t \nu_1 \left(CA(s)C^{-1} \right) ds = \infty.$$

Then Eq. (3.11) is unstable.

Proof:

(i) Let $V(x) = |Cx|_1$. Then similar to the proof of Theorem 3.1 we see that

$$D^+ V(x)|_{(3.11)} \le \mu_1(CAC^{-1})|Cx|_1 + \sum_{i=1}^m \mu_1(CA_kD)|D^{-1}g_k(x)|_1$$

$$\le \mu_1 \left(CA(t)C^{-1} \right) V(x).$$

Then for $t \ge t_0 \ge 0$

$$V(x(t)) \le V(x(t_0)) \exp \int_{t_0}^t \mu_1 \left(CA(s)C^{-1} \right) ds \to 0 \quad \text{as } t \to \infty.$$

This implies that (3.11) is asymptotically stable.

(ii) Under the condition we have $D^+ V(x)|_{(3.11)} \le -\alpha V(x)$. The conclusion is then obvious.

(iii) The proof is similar to the proof of Theorem 2.2 and hence is omitted.

∎

Finally we remark that all the above stability criteria can be extended to the more general form of equations

$$y' = B(y) \sum_{k=1}^m A_k(t)f_k(y) \tag{12}$$

where $B(y) = \text{diag} \left(b_1(y_1), ..., b_n(y_n) \right)$, $b_i(y_i) > 0$ for $y_i \in \mathbb{R}$, $i = 1, ..., n$,

$A_k \in C(\mathbb{R}^n, \mathbb{R}^{n \times n})$, $f_k = \left(f_k^{(1)}, ..., f_k^{(n)} \right)^T \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, ..., m$,

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and

$$y_i f_k^{(i)}(y) > 0 \ \text{ for } y_i \neq 0, \ i = 1, ..., n, \ k = 1, ..., m.$$

In fact, if we make the substitution $x_i = \int_o^y \dfrac{ds}{b_i(s)}$, then (3.12) becomes (3.1) with $g_k(x) = f_k(y(x))$. It is easy to see that $g_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ and satisfies (3.2) for $k = 1, ..., m$. Therefore, Eq. (3.11) has exactly the same stability properties as Eq. (3.1). We omit the details here.

References

1. W. A. Coppel, "Stability and Asymptotic Behavior of Differential Equations", Heath Mathematical Monographs, D.C. Heath and Company, Boston, 1965.
2. X. X. Liao, Stability of general ecological systems and neural networks systems. World Congress of Nonlinear Analysts '92, Vol. I–IV (Tampa, FL, 1992), 1325–1340, de Gruyter, Berlin, 1996.
3. S. M. Lozinskii, *Error estimates for numerical integration of ordinary differential equations*, Izv. Vyssh. Uchebn. Zaved. Mat. 5 (1958), 52-90.
4. M. Vidyasagar, "Nonlinear Systems Analysis", Prentice Hall, Inc., Englewood cliffs, New Iersey, 1978.

THE GLOBAL AND LOCAL C^2-SOLUTIONS FOR THE WAVE EQUATION IN THREE SPACE DIMENSIONS

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In this paper we study the global and local C^2-solutions for the semilinear wave equations without spherical symmetry in three space dimensions.

1 Introduction

This paper deals with the Cauchy problem for semilinear wave equations with small data in three space dimensions of the following form

$$\begin{cases} u_{tt} - \Delta u = G(u_t, Du), & x \in R^3, \ t \geq 0, \\ u(x,0) = \epsilon f(x), \ u_t(x,0) = \epsilon g(x), \end{cases} \tag{1}$$

where u is a real valued unknown function, ϵ is a positive parameter.

John[1] has shown in the case $G = |u_t|^2$ that the classical solutions of (1) blow up at finite time under certain conditions on the Cauchy data. We can derived from his proof that the blow-up result obtained in ref. 1 is still valid for $1 < p \leq 2$ in the case $G = |u_t|^p$. Ref. 2 obtained that the global C^2 solutions of (1) with spherical symmetry in the case $G = |u_t|^p$ $(p > 2)$ without the assumption that the support of $f(x)$ and $g(x)$ is compact. Also ref. 3 has studied that the global existence of solutions in weighted L^∞- space for problem (1) with the spherical symmetry by assuming that G, f and g satisfy some hypotheses. The spherical symmetry means that the unknown $u(x,t)$ is the function of variables (r,t) where $r = |x|$. For more general nonlinear term $G(u_t, Du)$, ref. 4 has shown the global C^2 solution existence and nonexistence of (1) with the spherical symmetry by removing the compactness assumption of the support of initial data.

The aim of this paper is that we study (1) instead of spherical symmetry. Actually we assume that $f(x)$, $g(x) \in C^3(R^3)$ satisfy the following

(i) $|D^\alpha f(x)|$, $|D^\alpha g(x)| \leq C_\alpha/(1 + |x|)^{1+k}$,

where α is an arbitrary multi-integer, C_α and k are positive constants. In fact we can assume $0 < k < 1$ throughout in this paper, since for any $k' \geq 1$, the inequality $(1 + |x|)^{-k'-1} \leq (1 + |x|)^{-k-1}$ $(0 < k < 1)$ holds.

(ii) $G(v, w) \in C^2(R^2)$,

(iii) There exist $p > 3$ and $A > 0$ such that

$$|\partial_v^{\alpha_1} \partial_u^{\alpha_2} G(v, u)| \leq A(|v|^{p-|\alpha|} + |u|^{p-|\alpha|}), \quad |\alpha| = \alpha_1 + \alpha_2 \leq 2$$

for all $(v, u) \in R^2$ with $|v|, |u| \leq 1$.

(iv) For $|\alpha| = 2$, there exists a positive constant B satisfying

$$|\partial_v^{\alpha_1} \partial_w^{\alpha_2} G(v, w) - \partial_v^{\alpha_1} \partial_w^{\alpha_2} G(v', w')| \leq$$

$$B\{(|v| + |v'|)^{\mu_0}|v - v'|^{\mu_1} + (|w| + |w'|)^{\mu_0}|w - w'|^{\mu_1}\}$$

for all $(v, w), (v', w') \in R^2$ with $|v|, |w|, |v'|, |w'| \leq 1$, where

$$(\mu_0, \mu_1) = (p - 3, 1), \qquad p \geq 3.$$

Note that assumptions admit the nonlinear terms $|Du|^p$, $|u_t|^p$ and $u_t|u_t|^{p-1}$.

Under the hypotheses (i)-(iv) and $p > 3$, $k > 2/(p-1)$, we shall show that the global small C^2 solution of (1) exist. In additoin, we shall obtain that if $p > 3$, $0 < k < 2/(p-1)$ and the assumption (i)-(iv) holds, then for small $\epsilon > 0$, the lifespan T_ϵ of corresponding local solution exceeds the number $c\epsilon^{-(p-1)/(2-k(p-1))}$, where c is a positive constant which is independent of ϵ.

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We should note that the results obtained in this paper are in some sense weaker than those we are supposed to get, because ref. 4 has shown that the global small C^2 solutions of (1) with spherical symmetry exist for $p > 2$ and $k > 1/(p-1)$, the lifespan T_ϵ of corresponding local solution for the spherical symmetric version of (1) exceeds the number $c\epsilon^{-(p-1)/(1-k(p-1))}$. Whether the global C^2 solutions of (1) exist for $p > 2$, $k > 1/(p-1)$, and the local C^2 solutions are available for $p > 2$ and $0 < k < 1/(p-1)$ still remains open.

It is very meaningful to compare results in this paper with those for the equation

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & x \in R^3, \quad t \geq 0. \\ u(x,0) = \epsilon\varphi(x), \quad u_t(x,0) = \epsilon\psi(x), \end{cases} \tag{2}$$

where $p > 1$. When initial data are suitably smooth and compact support, the existence result by ref. 6 and nonexistence results by ref. 6 -7, together with Keller's comparison theorem (see ref. 8), lead to the fact that problem (2) with $p > 1 + \sqrt{2}$ has global small solutions and that $1 < p < 1 + \sqrt{2}$ has blow up solutions for the data satisfying some additional positive conditions no matter how small $\epsilon > 0$ is. Also ref. 9 obtained the more refined results than ref. 6-7. Moreover, ref. 5 has shown that if the support of the Cauchy data is not compact, then there are two cases, blow-up or global existence, according to the decay of the data as $|x|$ tend to infinite evev though $p > 1 + \sqrt{2}$. More precisely, ref. 5 proved the existence of global small solution to (2) with $p > 1 + \sqrt{2}$ for the data satisfying

$$|D^\alpha\varphi(x)|, \quad |D^\beta\psi(x)| = O(|x|^{-1-k}) \qquad (|\alpha| \leq 3, |\beta| \leq 2) \quad |x| \to \infty \tag{3}$$

with $k > 2/(p-1)$. He has also shown the nonexistence of global solutions to (2) with the data satisfying $\varphi(x) = 0$, $\psi(x) \geq 1/(1+|x|)^{1+k}$ with $0 < k < 2/(p-1)$, even if $p > 1 + \sqrt{2}$.

For simplicity, we still denote by C any positive constant appearing in our paper, which may depend on k, p, C_α, A and B, but never depends on ϵ.

2 Lemmas

In order to prove the existence and uniqueness in the classical sense of C^2 for problem (1), by ref. 1 we know that the equivalent integral equation for (1) has the following form

$$\begin{aligned} u(x,t) &= \epsilon\{\tfrac{\partial}{\partial t}[\tfrac{t}{4\pi} \int_{|\omega|=1} f(x+t\omega)d\sigma_\omega] + \tfrac{t}{4\pi} \int_{|\omega|=1} g(x+t\omega)d\sigma_\omega\} \\ &\quad + \tfrac{\epsilon}{4\pi} \int_0^t (t-\tau) \int_{|\omega|=1} G(u_\tau, Du)(\tau, x+(t-\tau)\omega)d\sigma_\omega d\tau \\ &= u^0(t,x) + v^0(t,x), \end{aligned} \tag{4}$$

where ω is a unit vector in R^3 and $d\sigma_\omega$ is an area element on a sphere of radius 1. Let J_k be given by

$$J_k = \begin{cases} (t,x), & (t,x) \in [0,\infty) \times R^3, \quad k > 2/(p-1), \\ (t,x), & (t,x) \in [0,T] \times R^3, \quad 0 < k < 2/(p-1). \end{cases}$$

We define $C^2(J_k)$ be the space of all real-valued and twice continuously differentiable functions W on J_k with norm $\|.\|_{J_k}$ given by

$$\|W\|_{J_k} = \sup_{(t,x) \in J_k} [(1+t+|x|)^k \|W(t,x)\|] < \infty,$$

where

$$\|W(t,x)\| = \sum_{0 \leq j+j_1+j_2+j_3 \leq 2} \left| \frac{\partial^{j+j_1+j_2+j_3} W(t,x)}{\partial t^j \partial x_1^{j_1} \partial x_2^{j_2} \partial x_3^{j_3}} \right|.$$

By the definition of space $C^2(J_k)$, we know that $C^2(J_k)$ is a Banach space, and for any $u \in C^2(J_k)$, $\|u\|_{J_k}$ is bounded. We shall use the iteration scheme called a variant contraction mapping theorem to construct solutions of (1) in the space $C^2(J_k)$.

Now we introduce the following two lemmas which can be found in ref. 5.

Lemma 1 *If $0 < k < 1$, then*

$$\frac{\epsilon t}{4\pi} \int_{|\omega|=1} \frac{d\sigma_\omega}{(1+|x+t\omega|)^{1+k}} \leq \frac{C\epsilon}{(1+t+|x|)^k},$$

$$\frac{\epsilon}{4\pi} \int_{|\omega|=1} \frac{d\sigma_\omega}{(1+|x+t\omega|)^{1+k}} \leq \frac{C\epsilon}{(1+t+|x|)^k}.$$

Lemma 2 *Suppose that $f(x)$, $g(x)$ satisfy (i), then*

$$\|u^0(t,x)\| \leq \frac{C\epsilon}{(1+t+|x|)^k} \quad (0 < k < 1). \tag{5}$$

Lemma 3 *Suppose that the function $K(x,t)$ take following form*

$$K(x,t) = \frac{1}{4\pi} \int_0^t (t-\tau) \int_{|\omega|=1} G(u_\tau^0, Du^0)(\tau, x+(t-\tau)\omega) d\sigma_\omega d\tau,$$

where G satisfies assumptions (ii)-(iv), $u_\tau^0 = u_t^0(x,t)|_{t=\tau}$. Then

$$\|K(x,t)\| \leq \begin{cases} \frac{C\epsilon^p}{(1+t+|x|)^k}, & k > 2/(p-1), \\ \frac{C\epsilon^p(1+t)^{2-k(p-1)}}{(1+t+|x|)^k}, & 0 < k < 2/(p-1). \end{cases}$$

The proof of Lemma 3 is similar to that of ref. 7, we omit it.

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3 Main results

Theorem 1 *Suppose that $p > 3$, $k > 2/(p-1)$ and the initial value $f(x)$, $g(x)$ satisfy assumptions (i). Then there exists $\epsilon_0 > 0$ such that for any ϵ with $0 < \epsilon < \epsilon_0$ the problem (1) have a unique global solution $u(x,t)$ which is twice continuous differentiable with respect to any $(x,t) \in R^3 \times R^+$.*

Theorem 2 *Suppose that $p > 3$, $0 < k < 2/(p-1)$ and the initial data satisfies (i). Then there exists ϵ_0, for any ϵ with $0 < \epsilon < \epsilon_0$, the problem (1) has a unique local C^2 solution $u(x,t)$. Furthermore, for all the ϵ with $0 < \epsilon < \epsilon_0$, the lifespan T_ϵ of the corresponding solution exceeds the number $C\epsilon^{-(p-1)/(2-kp+k)}$, where C is a positive constant independent of ϵ.*

In order to prove above theorems, we define

$$u_n(t,x) = u^0(x,t)$$
$$+ \frac{1}{4\pi} \int_0^t (t-\tau) \int_{|\omega|=1} G((u_\tau)_{n-1}, Du)_{n-1})(\tau, x+(t-\tau)\omega) d\sigma_\omega d\tau,$$

where $u_0(x,t) = u^0(x,t)$, $(u_\tau)_{n-1} = \partial_t u_{n-1}(x,t)|_{t=\tau}$.

By Lemma 2, we can choose ϵ be sufficiently small such that

$$\|u^0(x,t)\| \le \epsilon C \le 1/2.$$

By Lemma 2 and 3, we know

$$\|u_1(x,t)\| \le \begin{cases} \frac{\epsilon C}{(1+t+|x|)^k} + \frac{C\epsilon^p}{(1+t+|x|)^k}, & k > 2/(p-1), \\ \frac{\epsilon C}{(1+t+|x|)^k} + \frac{C\epsilon^p(1+T)^{2-k(p-1)}}{(1+t+|x|)^k}, \\ \qquad\qquad 0 < k < 2/(p-1), \quad 0 \le t \le T. \end{cases} \tag{6}$$

In fact, if $0 < k < 2/(p-1)$, we can choose $T > 0$ and small positive ϵ so that the inequality

$$C\epsilon^{p-1}(1+T)^{2-k(p-1)} \le 1/2 \tag{7}$$

holds. From (6), we can choose ϵ be sufficiently small such that for $k > 2/(p-1)$, $p > 3$ that

$$\|u_1(x,t)\|_{J_k} \le C\epsilon + C\epsilon^p \le 1/2.$$

Therefore, we can choose sufficiently small $\epsilon > 0$ such that for both cases $k > 2/(p-1)$ and $0 < k < 2/(p-1)$ that

$$\|u_1\|_{J_k} \le C\epsilon + \epsilon/2 \le 1/2. \tag{8}$$

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By Lemma 3, we have

$$\|u_1(x,t) - u_0(x,t)\| \leq \begin{cases} \frac{C\epsilon^p}{(1+t+|x|)^k}, & k > 2/(p-1), \\ \frac{C\epsilon^p(1+T)^{2-k(p-1)}}{(1+t+|x|)^k}, & 0 < k < 2/(p-1), 0 \leq t \leq T \end{cases}$$

$$\leq C\epsilon + \epsilon/2 \leq 1/2. \tag{9}$$

Similar to the estimates as above, by induction and Lemma 3, we have

$$\|u_n\|_{J_k} \leq C\epsilon + \epsilon/2 \leq 1/2. \tag{10}$$

By repeating essentially the same argument as in Lemma 3, (7)-(10), assumption (i)-(iv), and the definition of norm $\| \cdot \|_{J_k}$, it follows from the same idea of ref. 5 that

$$\|u_{n+1} - u_n\|_{J_k} \leq C(\frac{1}{2})^n + Cn(\frac{1}{2})^n + Cn^2(\frac{1}{2})^{n-1} + C(\frac{1}{2})^{(p-2)n}. \tag{11}$$

We know $\{u_n\}$ is a Cauchy sequence in $C^2(J_k)$, then there exists a function $u(x,t) \in C^2(J_k)$ such that $u_n(x,t)$ is uniformly convergent to $u(x,t)$ in $C^2(J_k)$. If $k > 2/(p-1)$, we complete the proof of Theorem 1. If $0 < k < 2/(p-1)$, we notice the inequality (7), we complete the proof of Theorem 2.

References

1. F. John, *Comm. Pure Appl. Math.*, **34**, 29 (1981).
2. H. Takamura, *Comm. in PDE*, **17**, 189 (1992).
3. T. Sideris, *Comm. in PDE*, **8**, 1291 (1983).
4. K. Hidano, *Nonlinear Analysis*, **26**, 941 (1995).
5. F. Asakura, *Comm. in PDE*, **11**, 1459 (1986).
6. F. John, *Manuscripta Math.*, **28**, 235 (1979).
7. J. Schaeffer, *Proc. R. Soc. Edinb*, **101**, 31 (1985).
8. J. Keller, *Comm. Pure Appl. Math.*, **10**, 523(1959).
9. H. Lindblad, *Comm. in PDE*, **15**, 757 (1990).
10. R. Guenther and L. John, *Partial differential equations of mathematical physics and integral equations*, (Prentice-Hall, New Jersey, 1988)

THE BLOW-UP RATE FOR A SYSTEM OF SEMILINEAR HEAT EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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In this paper we prove that $\max\limits_{x \in [0,1]} u(x,t)$ (resp. $\max\limits_{x \in [0,1]} v(x,t)$) goes to infinity like $(T-t)^{\alpha_1/2}$ (resp. $(T-t)^{\alpha_2/2}$), where $\alpha_i < 0$ are the solutions of $(P-Id)(\alpha_1, \alpha_2)^t = (-1, -1)^t$.

1 Introduction

In this paper we consider the blow-up rate for the following system of semilinear heat equations with nonlinear boundary conditions:

$$
\begin{cases}
u_t = u_{xx} + u^{l_1}, & v_t = v_{xx} + v^{l_2}, & (x,t) \in (0,1) \times (0,T), \\
u_x(0,t) = 0, & v_x(0,t) = 0, & t \in (0,T), \\
u_x(1,t) = (u^{p_{11}} v^{p_{12}})(1,t), & v_x(1,t) = (u^{p_{21}} v^{p_{22}})(1,t), & t \in (0,t), \\
u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in (0,1).
\end{cases}
\tag{1}
$$

Here $0 < l_1 \leq 1, 0 < l_2 \leq 1$ and the matrix $P = (p_{ij})$ satisfies the following assumption

(A). $P = (p_{ij})$ is a matrix with non-negative entries such that

$$\max\{p_{11}, p_{22}\} < 1, \quad \det(P - Id) < 0.$$

Under this hypothesis there exists a unique vector (α_1, α_2) with $\alpha_i < 0$ such that

$$
(P - Id) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.
\tag{2}
$$

Here, without loss of generality we assume that $\alpha_1 \leq \alpha_2 < 0$. Further, we suppose that l_1, l_2 satisfy the following hypothesis:

(B). $\qquad\qquad l_1 \geq [\alpha_1 - (1 - l_2)\alpha_2]/\alpha_1$.

We also suppose that the initial data satisfy the following conditions

(C). $u_0(x)$, $v_0(x) \in C^3([0,1])$, $u_0''' \geq 0$, $u_0'' \geq 0$, $u_0' \geq 0$, $v_0''' \geq 0$, $v_0'' \geq 0$, $v_0' \geq 0$, $u_0(x) \geq 1$, and $v_0(x) \geq 1$ for any $x \in (0,1)$.

Under Condition (C), by the minimum principle we have $u(x,t) \geq 1$ and $v(x,t) \geq 1$ for any $(x,t) \in [0,1] \times [0,T)$. Under hypothesis (A) it is proved in [10] that the solution $(u(x,t), v(x,t))$ of (1) blows up in finite time T. As $t \to T$ we have

$$\limsup_{t \to T}\{\|u(.,t)\|_{L^\infty([0,1])} + \|v(.,t)\|_{L^\infty([0,1])}\} = +\infty.$$

We can also prove that both functions $u(x,t)$ and $v(x,t)$ go to infinity as $t \to T$.

Over the past two decades blow-up problem of the solutions for nonlinear parabolic equations with nonlinear boundary conditions has deserved a great deal of interest (see ref. 2, 3, 5, 7, 10-12). For this kind of problems, some of those results close related to ours are as follows. In ref. 1, 9, 11 considered the following system:

$$\begin{cases} u_t = \Delta u, & v_t = \Delta v, & (x,t) \in B_1(0) \times (0,T), \\ \frac{\partial u}{\partial n} = u^{p_{11}} v^{p_{12}}, & \frac{\partial v}{\partial n} = u^{p_{21}} v^{p_{22}}, & (x,t) \in \partial B_1(0) \times (0,T), \\ u(x,0) = u_0(x) > 0, & v(x,0) = v_0(x) > 0, & x \in B_1(0), \end{cases} \quad (3)$$

where the matrix $P = (p_{ij})$ satisfies hypothesis (A), the inital functions $u_0, v_0 \in C^3(\bar{B}_1(0))$ are radially symmetric and satisfy the boundary conditions, and the first three derivatives of $u_0(r)$, $v_0(r)(r = \|x\|)$ are non-negative. In ref. 11 the author proved that there exists positive constants c and C such that

$$c \leq \max_{x \in B_R(0)} u(x,t)(T-t)^{-\alpha_1/2} \leq C \text{ for } 0 < t < T,$$
$$c \leq \max_{x \in B_R(0)} v(x,t)(T-t)^{-\alpha_2/2} \leq C \text{ for } 0 < t < T,$$

where α_1 and α_2 are given by (2).

In the paper, by a modification of the method given in [11], we establish the following results.

Theorem 1.1 *If assumptions (A)-(C) hold, then the solution $(u(x,t), v(x,t))$ of (1.1) blows up at finite time T and there exists positive constants c and C such that*

$$c \leq \max_{x \in [0,1]} u(x,t)(T-t)^{-\alpha_1/2} \leq C \text{ for } 0 < t < T,$$
$$c \leq \max_{x \in [0,1]} v(x,t)(T-t)^{-\alpha_2/2} \leq C \text{ for } 0 < t < T,$$

where $\alpha_i (i = 1,2)$ are given by (2).

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Theorem 1.2 *If the assumptions (A), (B) and (C) hold, then for any $r \in [0, 1)$ there exists a constant $C = C(r)$ such that*

$$\max_{x \in [0, r]} u(x, t) < C, \qquad t \in [0, T),$$

$$\max_{x \in [0, r]} v(x, t) < C, \qquad t \in [0, T),$$

(i.e. the blow-up set is localized in boundary $x = 1$).

2 Auxiliary Propositions

In this section, we state some propositions that play an important role in Sections 3. We begin with some results of ref. 9 (also see ref. 4, 6 and ref. 10).
Proposition 2.1 *Let z be the positive solution of the problem*

$$
\begin{aligned}
z_t &= z_{xx}, & (x, t) &\in (0, 1) \times (0, T), \\
z_x(0, t) &= 0, \quad z_x(1, t) = z^k(1, t), & t &\in (0, T), \\
z(x, 0) &= z_0(x) > 0, & x &\in \Omega,
\end{aligned}
$$

where $k > 1$, $z_0 \in C^3$ satisfies the inequalities $z_0' \geq 0$, $z_0'' \geq 0$, $z_0''' \geq 0$ and boundary conditions. Then there exist positive constants c and C such that

$$c \leq \max_{x \in [0, 1]} u(x, t)(T - t)^\alpha = u(1, t)(T - t)^\alpha \leq C, \ \text{for} \ 0 < t < T,$$

where $\alpha = 1/(2(k - 1))$.
Proposition 2.2 *Let $w(x, t)$ be the positive solution of the problem*

$$
\begin{aligned}
w_t &= w_{xx} + w^l, & \text{in} \quad (0, 1) \times (0, T), \\
w_x(0, t) &= 0, \quad w_x(1, t) = w^q(1, t), & t &\in (0.T), \\
w(x, 0) &= w_0(x) > 0, & \text{in} \quad [0, 1],
\end{aligned}
$$

where $l > 0$, $q > 0$, $\max\{l, q\} > 1$, the initial data $w_0(x)$ satisfies the inequalities $w_0'' + w_0^l \geq 0$ and $w_0' \geq 0$, and T is the blow-up time. Then blow-up occurs only at $x = 1$ and there exist positive constants c and C such that

$$c \leq \max_{x \in [0, 1]} w(x, t)(T - t)^\alpha = w(1, t)(T - t)^\alpha \leq C \ \text{for} \ 0 < t < T,$$

where $\alpha = 1/(l - 1)$ if $l \geq 2q - 1$, $\alpha = 1/(2(q - 1))$ if $l < 2q - 1$, T is blow-up time.

By Proposition 2.1, 2.2 and mimimum principle, we can prove a result that has independent interest for a single equation.

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Proposition 2.3 *Let $U(x,t)$ be the positive solution of the problem*

$$
\begin{aligned}
&U_t = U_{xx} + U^{\tilde{l}}, && \text{in} \quad (0,1) \times (0,T),\\
&U_x(0,t) = 0, \quad U_x(1,t) = C_0 \frac{U^r(1,t)}{(T-t)^s}, && t \in (0.T),\\
&U(x,0) = U_0(x), && \text{in} \quad (0,1),
\end{aligned}
$$

where $0 < \tilde{l} \le 1$, $s > 1/2$, $0 < r < 1$, and the initial data $U_0 \in C^3$. Then $U(x,t)$ blows up as $t \to T$ and

$$
\tilde{c} \le \max_{x \in [0,1]} U(x,t)(T-t)^\beta \le \tilde{C}, \quad t \in (0,T),
$$

where $\beta = (s - 1/2)/(1 - r)$.

3 Blow-up Rate for the System

In this section, we prove Theorem 1.1 and 1.2. To this end, we start with a result on comparison for the functions $u(x,t)$ and $v^\gamma(x,t)$ (where (u,v) is the solution of (1)). The proof of this result is similar to that of Lemma 3.1 in [10].

Lemma 3.1 *Under assumptions (A), (B) and (C), there exists a constant $C > 0$ such that $Cu \ge v^{\alpha_1/\alpha_2}$, where (u,v) is the solution of (1).*

   Proof of Theorem 1.1 We begin with $v(x,t)$. By Lemma 3.1, we obtain

$$
\begin{cases}
v_t = v_{xx} + v^{l_2} \ge v_{xx} & \text{in} \quad (0,1) \times (0,T),\\
v_x(0,t) = 0, \quad v_x(1,t) = u^{p_{21}}(1,t)v^{p_{22}}(1,t) \ge cv^{p_1}(1,t), \\
v(x,0) = v_0(x), & \text{in} \quad (0,1),
\end{cases}
\tag{4}
$$

where $p_1 = \frac{\alpha_1}{\alpha_2}p_{21} + p_{22} = 1 - \frac{1}{\alpha_2} > 1$. By Proposition 2.1, we can conclude that there exists a constant c_1 such that

$$
v(x,y) \ge \frac{c_1}{(T-t)^{1/(2(p_1-1))}}.
$$

But $1/(2(p_1 - 1)) = -\alpha_2/2$, then we have

$$
\max_{x \in [0,1]} v(x,t) \ge c_1(T-t)^{\alpha_2/2} \qquad (t \to T).
\tag{5}
$$

From (5) we get

$$
\begin{cases}
u_t = u_{xx} + u^{l_1}, & \text{in} \quad (0,1) \times (0,T),\\
u_x(0,t) = 0, \quad u_x(1,t) = u^{p_{11}}(1,t)v^{p_{12}}(1,t) \ge c_2 \frac{u^{r_1}(1,t)}{(T-t)^{s_1}}, \\
u(x,0) = u_0(x), & \text{in} \quad (0,1),
\end{cases}
\tag{6}
$$

where $0 < L_1 \leq 1$, $r_1 = P_{11}$ and $s_1 = -(\alpha_2 p_{12})/2$. By hypothesis (A), we have $s_1 > 1/2$ and $r_1 < 1$. Therefore, by Proposition 2.3, we obtain

$$u(x,t) \geq c_3(T-t)^{-(s_1-1/2)/(1-r_1)}.$$

We remark that

$$\frac{s_1 - 1/2}{1 - r_1} = -\alpha_1/2$$

and we have obtained the lower bound for $u(x,t)$,

$$\max_{x \in [0,1]} u(x,t) \geq c_3(T-t)^{\alpha_1/2} \qquad (t \to T). \qquad (7)$$

Next we prove the reverse inequalities in Theorem 1.1. Now we start with $u(x,t)$. By Lemma 3.1, we have

$$\begin{aligned} u_t &= u_{xx} + u^{l_1}, && \text{in} \quad (0,1) \times (0,T), \\ u_x(0,t) &= 0, \quad u_x(1,t) = u^{p_{11}}(1,t)v^{p_{12}}(1,t) \leq C_1 u^{p_2}(1,t), \\ u(x,0) &= u_0(x), && \text{in} \quad (0,1), \end{aligned}$$

where $0 < l_1 \leq 1$ and $p_2 = \{\alpha_1 p_{11} + \alpha_2 p_{12}\}/\alpha_1 = -1/\alpha_1 + 1 > 1$. Thus, by Proposition 2.2, we get

$$\max_{x \in [0,1]} u(x,t) \leq \frac{C_2}{(T-t)^{1/(2(p_2-1))}} \qquad (t \to T).$$

But $1/(2(p_2 - 1)) = -\alpha_1/2$, hence

$$\max_{x \in [0,1]} u(x,t) \leq \frac{C_2}{(T-t)^{-\alpha_1/2}} \qquad (t \to T). \qquad (8)$$

By the above estimate for $u(x,t)$, we have

$$\begin{cases} v_t = v_{xx} + v^{l_2} & \text{in} \quad (0,1) \times (0,T), \\ v_x(0,t) = 0, \quad v_x(1,t) = u^{p_{21}}(1,t)v^{p_{22}}(1,t) \leq C_3 \frac{v^{r_2}(1,t)}{(T-t)^{s_2}}, \\ v(x,0) = v_0(x), & \text{in} \quad (0,1), \end{cases} \qquad (9)$$

where $0 < l_2 \leq 1$, $r_2 = p_{22}$ and $s_2 = (-\alpha_1 p_{21})/2$. Using again assumption (A), we get $s_2 > 1/2$ and $r_2 < 1$. Thus, by Proposition 2.3 we obtain

$$\max_{x \in [0,1]} v(x,t) \leq \frac{C_4}{(T-t)^{(s_2-1/2)/(1-r_2)}}.$$

We observe that $(s_2 - 1/2)/(1 - r_2) = (-\alpha_2)/2$. Therefore, we have

$$\max_{x \in [0,1]} v(x,t) \leq \frac{C_5}{(T-t)^{-\alpha_2/2}} \qquad (t \to T). \qquad (10)$$

Combing this with (5)(7)(8)(10), we complete the proof of Theorem 1.1. The proof of Theorem 1.2 is similar to the proof of Theorem 1.1, we omit.

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References

1. K. Deng, *ZAMP*, **46**, 110 (1995).
2. Yu. V. Egorov and V. A. Kondratiev, in *Comptes Rendus Acad. Sci. Paris. Serie I*, **322**, 55 (1996).
3. Yu. V. Egorov and V. A. Kondratiev, in *Differential Equations, Asymptotic Analysis and Mathematical Physics*, ed. M. Demuth and B. W. Schulze. (Akademie-Verlag, Berlin, Serie: Math. Research **100**, 77, 1997).
4. M. Fila and P. Quittner, *Math. Meth. in the Appl. Sci.*, **14**, 197 (1991).
5. J. L. Gómez, V. Marquez and N. Wolanski, *J. Diff. Equ.*, **92**, 384 (1991).
6. B. Hu and H. M. Yin, *Trans. Amer. Math. Soc.*, **346**, 117 (1995).
7. H. A. Levine and L. E. Payne, *Proc. Amer. Math. Soc.*, **46**, 277 (1974).
8. Z. G. Lin and M. X. Wang, The blow-up properties of solutions to semilinear heat equations with nonlinear boundary conditions, *to appear J. Math. Anal. Appl.*.
9. Z. G. Lin and C. H. Xie, The blow-up rate for a system of heat equations with nonlinear boundary conditions, *to appear Nonlinear Anal. TMA.*.
10. C. L. Mu, The blow-up rate for a strongly coupled system of semilinear heat equations with nonlinear boundary conditions, *to appear Appl. Math. Lett.*.
11. J. D. Rossi, *Math. Mech. in the Appl. Sci.*, **20**, 1 (1997).
12. M. X. Wang, *Nonlinear equations of parabolic type*, (Science Press, Beijing, 1993) (in Chinese).

INVARIANCE AND INTEGRABILITY OF DIFFERENTIAL EQUATION $w' = \sum_{i=0}^n a_i(z)w^i / \sum_{i=0}^m b_i(z)w^i$ UNDER FRACTIONAL TRANSFORMATION

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The invariance of differential equation $w' = \sum_{i=0}^{n} a_i(z) w^i / \sum_{i=0}^{m} b_i(z) w^i$ under fractional transformation group are given. According to the invariance, a method to judge the integrability under such transformations is presented.

1 Introduction

Many problems, both in practice and in theory, require the researchers to find the exact solution or to judge the integrability of some given differential equations. Though the theory of Lie group, the theory of monodromy group and the theory of differential algebra have played important roles in studying the integrability of differential equations, it is still a difficult task to tell the system is integrable or not except some special case. one of the authors have investigated the integrability of equation $w' = \sum_{i=0}^{n} a_i(z) w^i$ under linear transformation group by the invariance[1]. This paper will study the more general equations:

$$w' = \frac{\sum_{i=0}^{n} a_i(z) w^i}{\sum_{j=0}^{m} b_j(z) w^j}, \quad n, m \in N, \tag{1}$$

where $a_i, b_j : D \mapsto C, a_i(z), b_j(z), i = 0, 1, \cdots, n, j = 0, 1, \cdots, m$, are analytic in D, $a_n(z), b_m(z) \neq 0$ for all $z \in D$, and D is a connected open subset of the complex plane. The equations of this form include a large class of important equations, i.e. Riccati equation, Abel equation, and polynomial system. As the special case, when $m = 0$, the relationship between invariance and integrability of it under linear transformation has been studied[1]. It is easy to see that the form of Eq. (1) is invariant under fractional transformation,

$$T(p, q, r, s) : \qquad w = \frac{p(z)W + q(z)}{r(z)W + s(z)} \tag{2}$$

where

$$Det(z) = p(z)s(z) - q(z)r(z) \neq 0 \quad \forall z \in D. \tag{3}$$

(Without loss of generality, we will assume that $Det(z) = 1$ hereinafter.) Further more, we will assume that $n = m + 2$, since under simple transformation, the equation can take the form that $n = m + 2$.

2 Invariance and Integrability under Fractional Transformations

Let $q_i(z), (i = 1, 2, \cdots, t)$ are distinct roots of $b(z, w) = 0$, with multiplicity k_i respectively. Then t is invariant under transformation Eq. (2). While $t \geq 3$, denote by $R_{k,l}$ ($k \neq l$) the dimension of linear space generated by functions

$$Q_{k,l}^i(z) = \frac{q_i(z) - q_k(z)}{q_i(z) - q_l(z)} \quad (i = 1, 2, \cdots, t, i \neq k, l).$$

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Lemma 2 $R_{k,l}$ *is independent of the choice of* k,l *(note: we will denote* $R_{k,l}$ *by* R*).*

Theorem 6 R *is invariant under fractional transformation* T.

Let

$$q(z) = q_1(z), \ r(z) = -\frac{1}{q_1(z) - q_3(z)}, \ p(z) = \frac{(q_1(z) - q_2(z))(q_3(z) - q_1(z))}{q_2(z) - q_3(z)}$$

Then, under transformation

$$w = TW = \frac{p(z)W}{r(z)p(z)W + 1} + q(z), \tag{4}$$

the denominator of right hand side of Eq. (1) converted into $B(z, W) = W^{k_1}(W - 1)^{k_2} \prod_{i=4}^{t}(W - \bar{q}(z))^{k_i}$, where $\bar{q}(z) = Q_{1,3}^i / Q_{1,3}^2$. If $r = 1$, $\bar{q}(z)$ are constants, and $B(z, W)$ is exact the functions of W.

While $t = 1$ or $t = 2$, it is easy to find transformation such that Eq. (1) can be converted into the equation expressed as

$$\frac{dw}{dz} = \frac{\sum_{i=0}^{n} a_i(z)w^i}{B(w)}, \tag{5}$$

where $B(w)$ is polynomial of w, with degree $n - 2$. And the discuss tell us that the condition under which Eq. (1) can be changed into the equation as Eq. (5) under fractional transformation is invariant.

Now, consider equation Eq. (5). If $B(w) = (w - c)^{n-2}$ (c be constant), let $W = 1/(w - c)$, the equation W satisfying take the form:

$$\frac{dW}{dz} = \sum_{i=1}^{n} a_i(z)W^i, \tag{6}$$

which have been discussed [1]. The results can be summed up as following:

Theorem 7 *For equation Eq. (6), these is a linear transformation*

$$W = p(z)u + q(z),$$

where $p(z)$ *and* $q(z)$ *can be obtained from* $a_i(z), (i = 0, 1, \cdots, n)$ *through finite algebra operations, such that Eq. (6) can be changed into the standard form:*

$$u' = A(z)(u^n + u^k + \sum_{i=0}^{k-1} A_i(z)u^i) \quad (0 \le k < n). \tag{7}$$

$A_i(z), i = 0, 1, \cdots, k - 1$ *and* $A(z)$, *looked as functions of coefficients* $a_i(z), i = 0, 1, \cdots, n$, *are invariant under linear transformation. And Eq. (6) is integrable under linear transformation if and only if either*

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1. Eq. (7) is a Bernoulli's equation; or

2. $A_i(z), (i = 0, 1, \cdots, k-1)$ are all constants.

When $B(w) \neq (w-c)^{n-2}$, rewrite Eq. (5) as

$$w' = \sum_{i=1}^{l} A_i(z)g_i(w), \tag{8}$$

where $A_i(z), (i = 1, \cdots, l)$ are linear independent relating to constants. Then this form is invariant under transform

$$w = TW = \frac{pw+q}{rW+s} \quad (ps - qr = 1, p, q, r, s \text{ are constants}). \tag{9}$$

Denote the set of all transformations Eq. (9) by G_0. We have

Theorem 8 *If $g_i(w), (i = 1, \cdots, l)$ can be spanned into a Lie algebra $\{g_1(w), \cdots, g_{l'}(w)\}$ of dimension $l'(l' \geq l)$ with Lie bracket*

$$[g_i(\omega), g_j(\omega)] = g_i'(\omega)g_j(\omega) - g_i(\omega)g_j'(\omega) = \sum_{k=1}^{l'} C_{ij}^k g_k(\omega),$$

$$(i, j = 1, 2, \cdots, l')$$

where C_{ij}^k are constants. Then

1). the structure constants C_{ij}^k are invariant under transformations in G_0.

2). $l' \leq 3$, and the equation can be integrated in finite steps while $l' \leq 2$. When $l' = 3$, the equation can be changed into Riccati equation. (Note: In this case, when the coefficients are rational functions, we can judged the integrability by finding its algebraic solution[3], which can be realize by computer[4]).

The results of this paper give a method to estimate whether the equation Eq. (1) is integrable under fractional transformations. This method can be realized automatically by programming under the software environment of *Mathematica* system.

References

1. Guan Ke-ying, Ma Ling, Zhang Yu-ming, *Invariant of Differential Equation $w' = \sum_{i=0}^{n} a_i(z)w^i$ under the Linear Transformation Group and its Integrability*, CSIAM'96, Fudan University Press, shanghai, pp 17-20.

2. Peter J.Olver *Application of Lie Group to Differential Equations,* Springer-Verlag, Berlin, 1986.

3. Waston.G.N *A treastisc on the theory of Bessel function.* Cambridge, 1952.

4. Lei Jinzhi, On Research of Algebra Curve Solution of Riccati Equation, baccalaureate thesis, BUAA, 1995

ON THE NON-CONSTANT PERIODIC ORBITS OF CUBIC KOLMOGOROV SYSTEMS OF A CLASS IN R³

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In this paper,we consider the Kolmogorov system which are the interaction of three species with the different intrinsic growth rates. We obtain the sufficient conditions to the existence of the non-constant periodic orbits of a Kolmogorov system in R³.

1 Introduction

Systems of nonlinear differential equations of the form

$$\frac{dx_i}{dt} = x_i F_i(x_1, x_2, \ldots, x_n), \; i = 1, 2, \ldots, n \tag{1}$$

are frequently used to modelling the interaction of species occupying the same ecological niche.

When $n = 2$,the differential equations modelling the interaction of two species have been studied extensively,system of the form

$$\frac{dx_1}{dt} = x_1 F_1(x, y), \frac{dx_2}{dt} = x_2 F_2(x, y) \tag{2}$$

is known as Kolmogorov systems.In the classical Lotka-Volterra-Gause model, F_1 and F_2 are linear.

When $n = 3$,May and Leonard [1]were the first to study the symmetric case of the Lotka-Volterra models, they showed that there exists a unique interior equilibrium which is a saddle point and they also give an example of L-V system with a heteroclinic cycle;A.Gaunersdorfer [6] studied the time averages for heteroclinic attractors of L-V system,the time averages fail to converge for

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almost all initial conditions, but spiral closed and closed to the boundary of a polygon;[2] studied the three dimensional competitive L-V systems with no periodic orbits;[4] discussed on the asymmetric May-Leonard model of three competing species;[3] studied for general Kolmogorov system with same intrinsic rates which has non-constant periodic orbits.

In this paper ,we shall discuss the non-constant periodic orbit of a class of Kolmogorov systems with different intrinsic growth rates.

Consider cubic Kolmogorov systems which is the weak coupled as follows

$$
\begin{aligned}
\frac{du}{dt} &= u[A_1 + A_2u + A_3v + A_4u^2 + A_5uv + A_6v^2], \\
\frac{dv}{dt} &= v[B_1 + B_2u + B_3v + B_4u^2 + B_5uv + B_6v^2], \\
\frac{dz}{dt} &= z(C_1 + C_2u + C_3v + C_4u^2 + C_5uv + C_6v^2 - C_7z^2).
\end{aligned} \tag{3}
$$

2 The Main Results

Without loss of generality,let $C_7 = 1$.The induced field of the system (3) will be obtained as the following the system (4)

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_1(1 + a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \\
&\quad a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2) \\
&\equiv x_1(1 + H_1(x)), \\
\frac{dx_2}{dt} &= x_2(1 + b_{11}x_1^2 + b_{12}x_1x_2 + \\
&\quad b_{13}x_1x_3 + b_{22}x_2^2 + b_{23}x_2x_3 + b_{33}x_3^2) \\
&\equiv x_2(1 + H_2(x)), \\
\frac{dx_3}{dt} &= x_3(1 + c_{11}x_3^2 + c_{12}x_1x_2 + \\
&\quad c_{13}x_1x_3 + c_{22}x_2^2 + c_{23}x_2 + c_{33}x_3^2) \\
&\equiv x_3(1 + H_3(x)).
\end{aligned} \tag{4}
$$

Where $a_{11} = -C_1$, $a_{12} = -C_2$, $a_{13} = -C_3$,
$a_{22} = -C_4$, $a_{23} = -C_5$, $a_{33} = -C_6$,
$b_{11} = A_1 - C_1$, $b_{12} = A_2 - C_2$, $b_{13} = A_3 - C_3$,
$b_{22} = A_4 - C_4$, $b_{23} = A_5 - C_5$, $b_{33} = A_6 - C_6$,
$c_{11} = B_1 - C_1$, $c_{12} = B_2 - C_2$, $c_{13} = B_3 - C_3$,
$c_{22} = B_4 - C_4$, $c_{23} = B_5 - C_5$, $c_{33} = B_6 - C_6$.

Proposition 1 If (4) has a non-constant periodic orbit when $x_1 \neq 0$,then (3) has the non-constant period orbit when $u \neq 0$.

$H_T(x)$ is a tangent vector field of homogeneous vector field

$$
H(x) = (x_1H_1(x), x_2H_2(x), x_3H_3(x), -x_4^3),
$$

According to Poincare central projection,we have

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Proposition 2 The flow of $H_T(x)$ on S^3_+ is topological equivalence to the flow of vector field defined by (4) in $\Pi_4(x_4 = 1)$.

Consider the cubic Kolmogorov system in two-dimensions:

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_1(r + ax_1 + bx_2 + cx_1^2 + ex_1x_2 + fx_2^2) \\
&\equiv P(x_1, x_2), \\
\frac{dx_2}{dt} &= x_2(s + gx_1 + hx_2 + ix_1^2 + jx_1x_2 + kx_2^2) \\
&\equiv P(x_1, x_2),
\end{aligned}
\tag{5}
$$

where $r = a_{33} - c_{33}, a = a_{13} - c_{13}, b = a_{23} - c_{23}, c = a_{11} - c_{11},$
$e = a_{12} - c_{12}, f = a_{22} - c_{22}, s = b_{33} - c_{33}, g = b_{13} - c_{13},$
$h = b_{23} - c_{23}, i = b_{11} - c_{11}, j = b_{12} - c_{12}, k = b_{22} - c_{22}.$

Theorem 1 Suppose that a critical point (u, v) in the first quadrant is a center (or focus) type,

$(i). r + au + bv + cu^2 + euv + fv^2 \equiv 0,$
$s + gu + hv + iu^2 + juv + kv^2 \equiv 0;$

$(ii). u(a + 2cu + ev) + v(h + jv + 2kv) \equiv 0, u, v \neq 0$
and one of the following sets of conditions holds:

1. $-2cgr + air + acs = 0,$
   $ahr - ghr - abs + ahs = 0,$
   $-(egr) + ajr - gjr + ajs = 0,$
   $akr - 2gkr - afs + 2aks = 0,$
   $-(gr) + as \neq 0.$

2. $-(ahr) + ghr + abs - ahs = 0,$
   $-2chr + hir + 2bcs - chs = 0,$
   $-(ehr) + bes - ehs + hjs = 0,$
   $-(hkr) - fhs + 2bks = 0,$
   $-(hr) + bs \neq 0.$

3. $egr - ajr + gjr - ajs = 0,$
   $eir - 2cjr + ijr + ces - cjs = 0,$
   $ehr - bes + ehs - bjs = 0,$
   $ekr - jkr - efs - fjs + 2eks = 0,$
   $-(jr) + es \neq 0.$

4. $-(akr) + 2gkr + afs - 2aks = 0,$
   $-(ckr) + ikr + cfs - cks = 0,$
   $hkr + fhs - 2bks = 0,$
   $-(ekr) + jkr + efs + fjs - 2eks = 0,$
   $-(kr) + fs \neq 0.$

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5. $egr - ajr + gjr - ajs = 0,$
   $-(ceg) + acj + cgj - aij = 0,$
   $beg - egh - abj + bgj = 0,$
   $efg - afj + fgj - 2egk + ajk = 0,$
   $-(eg) + aj \neq 0.$

6. $-(ahr) + ghr + abs - ahs = 0,$
   $-2bcg + ach + cgh + abi - ahi = 0,$
   $-(beg) + egh + abj - bgj = 0,$
   $-(afh) + fgh + abk - 2bgk + ahk = 0,$
   $-(bg) + ah \neq 0.$

7. $-(akr) + 2gkr + afs - 2aks = 0,$
   $-2cfg + afi + ack + 2cgk - 2aik = 0,$
   $afh - fgh - abk + 2bgk - ahk = 0,$
   $-(efg) + afj - fgj + 2egk - ajk = 0,$
   $-(fg) + ak \neq 0.$

8. $2chr - hir - 2bcs + chs = 0,$
   $-2bcg + ach + cgh + abi - ahi = 0,$
   $ceh + bei - ehi - 2bcj + bij = 0,$
   $2cfh - fhi - 2bck - chk + 2bik = 0,$
   $-(ch) + bi \neq 0.$

9. $eir - 2cjr + ijr + ces - cjs = 0,$
   $ceg - acj - cgj + aij = 0,$
   $ceh + bei - ehi - 2bcj + bij = 0,$
   $egi - 2cfj + fij + cek - 2eik + cjk = 0,$
   $-(ei) + cj \neq 0.$

10. $-(ckr) + ikr + cfs - cks = 0,$
    $2cfg - afi - ack - 2cgk + 2aik = 0,$
    $2cfh - fhi - 2bck - chk + 2bik = 0,$
    $-(efi) + 2cfj - fij - cek + 2eik - cjk = 0,$
    $-(fi) + ck \neq 0.$

11. $ehr - bes + ehs - bjs = 0,$
    $-(beg) + egh + abj - bgj = 0,$
    $-(ceh) - bei + ehi + 2bcj - bij = 0,$
    $efh - bek - ehk + bjk = 0,$
    $-(eh) + bj \neq 0.$

12. $hkr + fhs - 2bks = 0,$
    $-(afh) + fgh + abk - 2bgk + ahk = 0,$
    $-2cfh + fhi + 2bck + chk - 2bik = 0,$
    $-(efh) + bek + ehk - bjk = 0,$
    $-(fh) + bk \neq 0.$

13. $-(ekr)+jkr+efs+fjs-2eks=0,$
  $ffeg-afj+fgj-2egk+ajk=0,$
  $efi-2cfj+fij+cek-2eik+cjk=0,$
  $efh-bek-ehk+bjk=0,$
  $-(fj)+ek\neq0.$

Then the system (5) has the non-constant periodic solutions.

Theorem 2. If the vector field $(x_1H_1(x),x_2H_2(x),x_3H_3(x))$ has the invariant closed cones, then there exists the invariant closed cones of the vector field $H(x)$.

If (5) has the non-constant periodic orbits, then $(x_1H_1(x),x_2H_2(x),x_3H_3(x))$ has the invariant closed cones. Thus, we have

Theorem 3. If the conditions in Theorem 1 hold, there exist non-constant periodic solutions of system (3).

Acknowledgments

References

1. R.May and w.j.Leonard,Nonlinear Aspects of Compitition between Species, SIAM J.Appl.Math.29 (1975) 243-275.
2. P.Van Den Driessche and M.L.Zeeman,Three Dimentional Competitive Lotka-Volterra Systems with no Periodic orbits, SIAM J.Appl. Math. V58(1996),No.1,227-234.
3. Zhang Xingan, Liang Zhaojun,The Period Solutions of the Non-linear three Species model,Chinese Science Bulletin,40 (1995), NO.21,1934--1937.
4. C.W.Chi,S.B.Hsu and L.I.Wu,On the asymptotic May-Leonard model of three competing species,SIAM,J.Appl.Math.58 (1998),no.1,211-226.
5. Tamiyuki.Uno and Kenzi Odani,On a Lotka-Volterra model Which can be projicted to a sphere,Non-linear Analysis,Theory,Methods & Applications, vol.30 (1997),no.3,1405-1410.
6. A,Gasnersdorfer,Tine Averages for Heteroclic Attractors, SIAM,J.Appl.Math.52 (1992),no.5,1476-1489.
7. Ye Yanqian,Theory of Limit Cycles,Shang Hai:Shanghai sciense and technical Publishing House,1984.
8. N.G.Lloyd et al,Limit Cycles of a Cubic Kolmogorov System, Appl.Math.Lett,9 (1996), no.1,15-18.
9. Dai Guoren,Wo Shonglin,The Integral Line And Closed Trajectory of the System E_3, Acta Mathematica Scientia,V.9 (1989),251-261.

GLOBAL ATTRACTIVITY IN DELAYED HOPFIELD NEURAL NETWORK MODEL WITH VARIABLE COEFFICIENTS AND DELAYED

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By using the Lyapurov function ,some stability criteria are obtained for delayed Hopfield neural network model

$$x'(t) = -c_i(t)x_i(t) + \sum_{i=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{i=1}^{n} b_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) + I_i(t)$$

with variable coefficients and delays.

1. **Introduction** In paper [1], Hopfield first proposed neural network model for n neurons

$$c_i u'(t) = -\frac{u_i}{R_i} + \sum_{i=1}^{n} T_{ij} f_j(u_j(t)) + I_i(t), \tag{1.1}$$

$i = 1, 2, \cdots, n$ with an electrical circuit implementation. Even since, there has been increasing interest in the potential application of the dynamics of artificial neural networks in signal and image processing (see, [2,3]), they have studied the system (1.1) with delay

$$c_i u'(t) = -\frac{u_i}{R_i} + \sum_{i=1}^{n} a_{ij} f_j(u_j(t - \tau)) + I_i(t), \tag{1.2}$$

The global attractivity of system (1.1) or (1.2) is of great importance for both practical and theoretical purposes, it has been the major concern of most authors dealing with (1.1) and (1.2), and their generalizations, for example,

$$u'(t) = -b_i u_i(t) + \sum_{i=1}^{n} a_{ij} g_j(u_j(t - \tau_{ij})) + I_i(t), \tag{1.3}$$

and

$$x'(t) = -c_i x_i(t) + \sum_{i=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{i=1}^{n} b_{ij} f_j(x_j(t - r_j)) + I_i(t), \tag{1.4}$$

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$i, j = 1, 2, \cdots, n.$ where $c_i > 0$, $c_i, a_{ij}, b_{ij}, I_i, r_j$ all are constants.

Most authors have been obtained some results on the stability of delayed cellular neural networks, (see,[4-6]). Recently, Jinde Cao and Dongming Zhou[16]has considered the stability of the delayed cellular neutral network model (1.4). The purpose of this paper is to derive some sufficient conditions for the global asymptotic stability of the following delay system

$$x'(t) = -c_i(t)x_i(t) + \sum_{i=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{i=1}^{n} b_{ij}f_j(x_j(t-\tau_{ij})) + I_i(t), \qquad (1.5)$$

by using the Lyapunov function method and some analysis technique.

Where n is the number of units in a neural network, $x_i(t)$ are the state of the ith unit at time t , $f_j(x_j(t))$ denotes the output of the jth unit at time t,a_{ij} denotes the strength of the jth unit on the ith unit at time t, b_{ij} denotes the strength of the jth unit on the ith unit at time $t - \tau_{ij}(t)$, $I_i(t)$ are the extend bias on the ith unit at time t , $\tau_{ij}(t)$ correspond to the transmission delay, along the axon of the jth unit , $c_i(t)$ represents the rate with which the ith unit will reset it potential to the resting state in isolation when disconnected from the network and extend inputs at time t This work is motivated by an important significance in both theory and application, Hopfield-type neutral (1.1) and (1.5) have attracted the attention of scientific workers. Let $R^+ = [0, +\infty)$, $R = (-\infty, +\infty)$, $t_{-1} = min_{1 \leq i,j \leq n}\{-\tau_{ij}\}$, and

$$x_i(t) = \phi_i(t), t \in [t_{-1}, 0], \phi_i \in C([t_{-1}, 0], R)), i = 1, 2, \cdots, n \qquad (1.6)$$

is initial condition of Eq.(1.5) ,we use the notion $x_\phi(t)$ or $x(t)$ represents the solution of Eq.(1.5)-(1.6), Throughout this paper, we suppose that:

(C1): $a_{ij}(t), I_i(t), c_i(t)$ are continuous, bounded for $t \geq 0$, b_{ij}, τ_{ij} are constants, and
$$liminf_{t \to \infty} c_i(t) > 0, for i, j = 1, 2, \cdots, n,$$

(C2): $f_j(x)$ is continuous, $f_j(0) = 0$ and satisfies

$$0 < inf_{x \in R} f_j(x) \leq sup_{x \in R} f_j(x) < +\infty, j = 1, 2, \cdots, n$$

(C3): There exists a positive number L_j, for any $x, y \in R$,

$$|f_j(x) - f_j(y)| \leq L_j|x - y|.$$

(C4) The system (1.5) have unique equilibrium $x_i = N_i$.

Let
$$y_i(s) = x_i(t) - N_i, F_j(s) = f_j(s + N_j) - f_j(N_j),$$

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then Eq.(1.5) can easily change to

$$(y_i(t) + \sum_{j=1}^{n} b_{ij} \int_{t-\tau_{ij}}^{t} F_j(y_j(s))ds)' = -c_i(t)y_i(t) + \sum_{j=1}^{n}(a_{ij}(t) + b_{ij})F_j(y_j(t)),$$

(1.7)

2. Main Results

Lemma Assume that (C1),(C2) hold, then all the solutions of the Eq. (1.5) is bounded.

Proof We observe that all the solutions of the Eq. (1.5) satisfy the following differential inequalities

$$-c_i(t)x_i(t) - \alpha_i \leq x_i'(t) \leq -c_i(t)x_i(t) + \alpha_i,$$

(2.1)

where $\alpha_i = sup_{t\in R^+}(\sum_{j=1}^{n}((|a_{ij}(t)| + |b_{ij}|)sup_{x\in R}|f_j(x)|) + I_i(t)|)$. Using the condition (C_1), we can easily complete the proof.

Theorem In addition to (C1)-(C4), assume further that (H1): for $i = 1, 2, \cdots, n$,

$$liminf_{t\to+\infty}(2c_i(t) - p_i(t) - Q_i(t) - R_i(t)) > 0,$$

(2.2)

then for any solution of Eq. (1.5) $x(t) = (x_1(t), \cdots, x_n(t))^T$, we have $\lim_{t\to+\infty}|x_i(t) - N_i| = 0, i = 1, 2, \cdots, n.$ where

$$P_i(t) = |c_i(t)| \sum_{j=1}^{n} |b_{ij}|\tau_{ij} + \sum_{j=1}^{n}(|a_{ij}(t)| + |b_{ij}|),$$

$$Q_i(t) = (1 + \sum_{j=1}^{n} |b_{ij}|\tau_{ij})L_i^2\sum_{j=1}^{n}(|a_{ij}(t)| + |b_{ij}|),$$

$$R_i(t) = \sum_{j=1}^{n}\{L_i^2|b_{ji}| \int_{t-\tau_{ji}}^{t} [|c_j(u + \tau_{ji})| + \sum_{k=1}^{n}(|a_{jk}(u + \tau_{jk})| + |b_{jk}|)]du\},$$

Proof We consider the Lyapunov functional as following

$$v_1 = \sum_{j=1}^{n}(y_i(t) + \sum_{j=1}^{n} b_{ij} \int_{t-\tau_{ij}}^{t} F_j(y_j(s))ds)^2,$$

(2.3)

then

$$v_1' \leq \sum_{i=1}^{n}(-2)c_i(t)y_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}||c_i(t)| \int_{t-\tau_{ij}}^{t}(y_i^2(t) + F_j^2(y_j(s)))ds$$

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$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} (|a_{ij}(t)| + |b_{ij}|)[y_i^2(t) + F_j^2(y_j(t))]$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} (|a_{ij}(t)| + |b_{ij}|) \sum_{k=1}^{n} |b_{ik}| \int_{t-\tau_{ij}}^{t} (F_j^2(y_j(t)) + F_k^2(y_k(s))) ds$$

$$\leq \sum_{i=1}^{n} \{-2c_i(t) + \sum_{j=1}^{n} |b_{ij}||c_i(t)|\tau_{ij} + \sum_{j=1}^{n} (|a_{ij}(t)| + |b_{ij}|)\} y_i^2(t)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}||c_i| \int_{t-\tau_{ij}}^{t} F_j^2(y_j(s)) ds$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} (|a_{ij}(t)| + |b_{ij}|) F_j^2(y_j(t)) + \sum_{i=1}^{n} \sum_{k=1}^{n} |b_{ik}|\tau_{ik} \sum_{j=1}^{n} (|a_{ij}(t)|$$

$$+ |b_{ij}|) F_j^2(y_j(t)) + \sum_{i=1}^{n} \sum_{j=1}^{n} (|a_{ij}(t)|$$

$$+ |b_{ij}|) \sum_{k=1}^{n} |b_{ik}| \int_{t-\tau_{ik}}^{t} (F_k^2(y_k(s)) ds$$

$$\leq \sum_{i=1}^{n} (-2c_i(t) + P_i(t)) y_i^2(t) + \sum_{i=1}^{n} Q_i(t) y_i^2(t) + \sum_{i=1}^{n} [|c_i(t)|$$

$$+ \sum_{j=1}^{n} (|a_{ij}(t)| + |b_{ij}|)] \sum_{j=1}^{n} |b_{ij}| \int_{t-\tau_{ij}}^{t} F_j^2(y_j(s)) ds, \qquad (2.4)$$

Let

$$v_2(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| \int_{t-\tau_{ij}}^{t} [|c_i(u+\tau_{ij})| + \sum_{j=1}^{n} (|a_{ij}(u+\tau_{ij})| + |b_{ij}|)] \int_{u}^{t} F_j^2(y_j(s)) ds\, du$$

then

$$v_2'(t) \leq \sum_{i=1}^{n} R_i(t) y_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|[|c_i(t)|$$

$$+ \sum_{j=1}^{n} (|a_{ij}(t)| + |b_{ij}|)] \int_{t-\tau_{ij}}^{t} F_j^2(y_j(s)) ds, \qquad (2.5)$$

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By condition (2.2), there exists a constant $\alpha_i > 0$, for large t, we have

$$v'(t) \leq \sum_{i=1}^{n} -\alpha_i y_i^2(t), v = v_1 + v_2 \qquad (2.6)$$

Integrating both sides of (2.6) from 0 to t, we have $v(t) + \sum_{i=1}^{n} \alpha_i \int_0^t y_i^2(s) ds \leq v(0)$, so

$$\sum_{i=1}^{n} \alpha_i \int_0^{+\infty} y_i^2(s) ds \leq +\infty, \qquad (2.7)$$

Notices that $x_i(t)$ is bounded on R and by lemma ,$x_i'(t)$ is bounded on R^+.Therefore $y_i'(t)$ are bounded for $t \in [0, +\infty)$,so $y_i(t)$ is uniformly continuous on $t \in [0, +\infty)$, form (2.7),we have $\lim_{t \to +\infty} y_i(t) = 0$. Thus, $\lim_{t \to +\infty} x_i(t) = N_i$. The proof of the theorem is complete.

References

1. J.J.Hopfield, *Proc. Nat. Acad. Sci. U.S.A.* 81 3088 1984
2. C.M.Marcus and R.M.Westervelt, *Phys Rev. A* 39 347 1989
3. K. Gopalsamy and X. He, *Phys.D* 76 344 1994
4. K.Matsuoka, *Neural Networks* 5 495 1992
5. P.Van Den Driessche and Xingfu Zou, *SIAM J.APPL.MATH.* 58 1878 1998
6. Jinde Cao and Dongming Zhou, *Neural Networks* 11 1601 1998

GLOBAL ATTRACTIVITY OF A DELAY DIFFERENCE MODEL

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Consider the discrete Lasota-Wazewska model

$$N_{n+1} - N_n = -\mu N_n + pe^{-rN_{n-k}}, \quad n = 0, 1, 2, \cdots \qquad (*)$$

where$\mu \in (0,1)$, $r,p \in (0,\infty)$ and $k \in N$. We obtain a sufficient condition for all positive solutions of $(*)$ to be attracted to its equilibrium N^*. It improves correspondent result obtained in [11].

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1 Introduction

Consider the delay difference equation

$$N_{n+1} - N_n = -\mu N_n + pe^{-rN_{n-k}}, \quad n = 0, 1, 2, \cdots \tag{1}$$

where $\mu \in (0,1)$, $r, p \in (0\infty)$ and $k \in N$, where N denotes the set of non-negative integers. Equation (1) is a discrete analogue of the delay differential equation

$$\frac{dN(t)}{dt} = -\mu N(t) + pe^{-rN(t-\tau)}, \quad t \geq 0 \tag{2}$$

which was used first by Wazewska-Czyzewska and Lasota [1] as a model for the survival of red blood cells in an animal,see also Arino and Kimmel [2]. Here $N(t)$ denotes the number of red blood cells at time t, μ is the probability of death of a red blood cell, p and r are positive constants related to the production of red blood cells per unit time and τ is the time required to p reduce a red blood cell, see also [3,4] and [12,13].

Recently there have been a lot of activities concerning the oscillation and asymptotic behavior of delay difference equations, see for example, [5-10] and the references cited therein. The asymptotic behavior of (1) was studied by Chen and Yu [11], where the following result was obtained.

Theorem A. Assume that

$$M \stackrel{\Delta}{=} rN^*(1 - (1-\mu)^{k+1}) \leq 1. \tag{3}$$

Let $\{N_n\}$ be a positive solution of (1). Then

$$\lim_{n \to \infty} N_n = N^*.$$

By a solution of (1) we mean a sequence $\{N_n\}$ which is defined for $n \geq -k$ and which satisfies (1) for $n \geq 0$. Clearly if $a_{-k}, a_{-k+1}, \cdots, a_{-1}, a_0$ are $k+1$ given constants, then (1) has a unique solution satisfying the initial conditions

$$N_i = a_i, \quad \text{for} \quad i = -k, -k+1, \cdots, 0. \tag{4}$$

Furthermore,if the initial values are such that

$$a_i \geq 0, \quad \text{for} \quad i = -k, \cdots, -1, \quad \text{and} \quad a_0 > 0, \tag{5}$$

then the unique solution $\{N_n\}$ of the initial value problem (1) and (5) is positive for $n \geq 0$. The unique equilibrium N^* of (1) satisfies the equation

$$\mu N^* = pe^{-rN^*}. \tag{6}$$

Our aim in this paper is to obtain a sufficient condition under which all positive solutions of (1) are attracted to N^*. We will establish the following result.

Assume that

$$M \leq 1 + \frac{(1-\mu)^{k+1}}{rN^*}. \tag{7}$$

Let $\{N_n\}$ be a positive solution of (1), then

$$\lim_{n \to \infty} N_n = N^*.$$

Clearly, the condition (7) is much weaker than (3).

2 Some lemmas

The central problem in this section is to discuss the following system of inequalities

$$\begin{cases} y + (1-\mu)^{k+1}\ln(1 + \frac{y}{rN^*}) \leq M(e^{-x} - 1), \\ x + (1-\mu)^{k+1}\ln(1 + \frac{x}{rN^*} \geq M(e^{-y} - 1). \end{cases} \tag{8}$$

We will point out that (8) has a unique solution $x = y = 0$ on the proper condition in the region

$$D = \{(x,y) : -rN^* < x \leq 0 \leq y < \infty\}. \tag{9}$$

Lemma 1. Assume that $f(t)$ and $g(t)$ are continuously differentiable on $(t_0, t_1]$ and that $f'(t) < 0, g'(t) < 0$ and $g(t) < f(t)$ for $t \in (t_0, t_1)$, and that $g(t_0) = f(t_0)$ and $g'(t_0) \neq 0$. Then

$$\frac{f'(t_0)}{g'(t_0)} \leq 1. \tag{10}$$

Proof. Given any $t \in (t_0, t_1)$ by Cauchy mean value theorem, there exists a $\zeta_t \in (t_0, t)$ such that

$$\frac{f(t_0) - f(t)}{g(t_0) - g(t)} = \frac{f'(\zeta_t)}{g'(\zeta_t)} < 1.$$

Let $t \to t_0 + 0$, we complete the proof.

Lemma 2. If $M > 1$, then we find

$$\frac{(1-\mu)^{k+1}}{(rN^*)^2} < \frac{1}{4}. \tag{11}$$

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Proof. When $M > 1$, we have $\frac{1}{rN^*} < 1 - (1-\mu)^{k+1}$, which implies that $\frac{(1-\mu)^{k+1}}{rN^*} < (1-\mu)^{k+1}[1 - (1-\mu)^{k+1}] \leq \frac{1}{4}$, and so (11) holds.

Lemma 3. If $M \leq 1$, then (8) has a unique solution $x = y = 0$ in the region (9).

Proof. When $M \leq 1$, by (8) we get

$$
\begin{cases}
y \leq e^{-x} - 1, & \text{(a)} \\[2mm]
x \geq e^{-y} - 1. & \text{(b)}
\end{cases}
\tag{12}
$$

Combining (a) and (b) in (12), we obtain

$$
y \leq e^{1-e^{-y}} - 1, \qquad y \geq 0.
$$

Let $t = y$ and $\Phi(t) = e^{1-e^{-t}} - 1 - t$ for $t \geq 0$. Then $\Phi'(t) = e^{-t}e^{1-e^{-t}} - 1$ and $\Phi''(t) = e^{-t}e^{1-e^{-t}}(e^{-t} - 1) < 0$ for $t > 0$. It follows that $\Phi'(t) < \Phi'(0) = 0$ for $t > 0$. Thus $\Phi(t)$ is strictly decreasing when $t \geq 0$. And so $\Phi(t) < \Phi(0) = 0$ for $t > 0$. This implies that the inequality $\Phi(t) \geq 0$ has a unique solution $t = 0$ in $[0, \infty)$, i.e.(12) has a unique solution $x = y = 0$ in the region (9), then so does (8).

Lemma 4. The system of inequalities (8) has a unique solution $x = y = 0$ in the region (9) if and only if the system of equations

$$
\begin{cases}
y + (1-\mu)^{k+1}\ln(1 + \frac{y}{rN^*}) = M(e^{-x} - 1) & \text{(i)} \\[2mm]
x + (1-\mu)^{k+1}\ln(1 + \frac{x}{rN^*}) = M(e^{-y} - 1)) & \text{(ii)}
\end{cases}
\tag{13}
$$

has a unique solution $x = y = 0$ in the region (9).

Proof. In (13), the curve C_1 is denoted by (i):

$$
y_1 + (1-\mu)^{k+1}\ln(1 + \frac{y_1}{rN^*}) = M(e^{-x} - 1)
\tag{14}
$$

and the curve C_2 is denoted by (ii)

$$
x + (1-\mu)^{k+1}\ln(1 + \frac{x}{rN^*}) = M(e^{-y_2} - 1).
\tag{15}
$$

Assume that $f(t) = t + (1-\mu)^{k+1}\ln(1 + \frac{t}{rN^*})$ and $g(t) = M(e^{-t} - 1)$. Then (13) can be written as

$$
\begin{cases}
f(y_1) = g(x), \\[2mm]
f(x) = g(y_2).
\end{cases}
\tag{16}
$$

Since $f'(t) = 1 + \frac{(1-\mu)^{k+1}}{rN^*+t} > 0$ for $t \geq 0$, which implies that $f(t)$ is strictly increasing, thus its inverse function exists, i.e. we have

$$t = f^{-1}[g(\cdot)].$$

Then (8) can be written as

$$\begin{cases} y_1 \leq f^{-1}[g(x)], & \text{(i)}' \\ x \geq f^{-1}[g(y_2)]. & \text{(ii)}' \end{cases} \tag{17}$$

In (17), (i)$'$ denotes the region below the curve C_1 and (ii)$'$ denotes the region on the right of the curve C_2. Clearly, both (i)$'$ and (ii)$''$ have common points distinct from the origin in the region (9) if and only if the curve C_1 and C_2 have a cross over point distinct from the origin in the region (9).

Lemma 5. If (7) holds, the (13)has a unique solution $x = y = 0$ in the region (9)

Proof. In view of (14) and (15), we have

$$y_1' = \frac{-Me^{-x}}{(1 + \frac{(1-\mu)^{k+1}}{rN^*+y_1})} < 0, \tag{18}$$

$$y_2' = -\frac{(1 + \frac{(1-\mu)^{k+1}}{rN^*+x})}{Me^{-y_2}} < 0. \tag{19}$$

We consider the following two cases:

Case 1. $M < 1 + \frac{(1-\mu)^{k+1}}{rN^*}$. In this case by (18) and (19), we have $-1 < y_1'(0) < 0$ and $y_2'(0) < -1$, which implies that the curve C_2 is above the curve C_1 in the left-hand neighborhood sufficiently small at the origin. And

$$y_2''(x) = \frac{1}{Me^{-y_2}} \frac{(1-\mu)^{k+1}}{(rN^*+x)^2} + [y_2'(x)]^2 > 0 \quad \text{for} \quad -rN^* < x < 0. \tag{20}$$

Thus $y_2'(x) < y_2'(0) < -1$ for $-rN^* < x < 0$, i.e. the curve C_2 is above the right line $y = -x$. Assume, for the sake of contradiction, that the curve C_2 have a crossover point (x_0, y_0) distinct from the origin in the region (9), and that the curve C_1 and C_2 have no crossover point in $(x_0, 0)$. Clearly $y_0 > -x_0 > 0$.

In view of (18) and (19),it is not difficult to prove that $y_2(x)$ and $y_1(x)$ satisfy the conditions of Lemma 1. Thus we have

$$\frac{y_2'(x_0)}{y_1'(x_0)} \leq 1. \tag{21}$$

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Now set for $t \geq -x_0 > 0$

$$\phi(t) = (1 + \frac{(1-\mu)^{k+1}}{rN^* + x_0})(1 + \frac{(1-\mu)^{k+1}}{rN^* + t})e^{x_0 + t}.$$

We can find for $t \geq -x_0 > 0$

$$\phi'(t) = (1 + \frac{(1-\mu)^{k+1}}{rN^* + x_0})(1 + \frac{(1-\mu)^{k+1}}{rN^* + t} - \frac{(1-\mu)^{k+1}}{(rN^* + t)^2})e^{x_0 + t}.$$

When $M \leq 1$, by Lemma 3 the curve C_1 and C_2 have no crossover point distinct from the origin in the region (9).Thus we will only consider $M > 1$. By using Lemma 2, we have

$$\frac{(1-\mu)^{k+1}}{(rN^* + t)^2} < \frac{(1-\mu)^{k+1}}{(rN^*)^2} < \frac{1}{4} \quad \text{for} \ \ t \geq -x_0 > 0.$$

And so $\phi'(t) > 0$, i.e., $\phi(t)$ is strictly increasing for $t \geq -x_0 > 0$. Hence

$$\phi(y_0) > \phi(-x_0) = (1 + \frac{(1-\mu)^{k+1}}{rN^* + x_0})(1 + \frac{(1-\mu)^{k+1}}{rN^* - x_0})$$

$$= 1 + \frac{2rN^*(1-\mu)^{k+1}}{(rN^*)^2 - x_0^2} + \frac{(1-\mu)^{2k+2}}{(rN^*)^2 - x_0^2}$$

$$> (1 + \frac{(1-\mu)^{k+1}}{rN^*})^2 > M^2.$$

And so

$$\frac{y_2'(x_0)}{y_1'(x_0)} = \frac{\phi(y_0)}{M^2} > 1, \tag{22}$$

which contradicts (21). Thus when $M < 1 + \frac{(1-\mu)^{k+1}}{rN^*}$, the curve C_1 and C_2 have no crossover point distinct from the origin in the region (9)

Case 2. $M = 1 + \frac{(1-\mu)^{k+1}}{rN^*}$. Here it will suffice to prove that the curve C_2 is above the curve C_1 in the left-hand neighborhood sufficiently small at the origin. The proof of the rest is identical with one of Case 1. Finding derivatives with respect to x on (14) and (15) respectively, we get

$$y_1'(0) = y_2'(0) = -1, \quad y_1''(0) = y_2''(0) = 1 + \frac{1}{M} \cdot \frac{(1-\mu)^{k+1}}{(rN^*)^2}$$

and

$$y_2'''(0) - y_1'''(0) = \frac{3}{M^3} \cdot [\frac{(1-\mu)^{k+1}}{(rN^*)^2}]^2 - 1.$$

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In view of $M > 1$ and Lemma 2, we have

$$y_2'''(0) - y_1'''(0) < 3 \cdot (\frac{1}{4})^2 - 1 < 0.$$

Let $f(x) = y_2(x) - y_1(x)$. In the neighborhood at $x = 0$ we can write $f(x)'$s Taylor formula with Peano remainder term

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 + o(x^3).$$

Since $f(0) = f'(0) = f''(0) = 0$, we get

$$f(x) = \frac{1}{6}f'''(0)x^3 + o(x^3).$$

For a positive ε sufficiently small, when $x \in (-\varepsilon, 0)$ sign of $f(x)$ is identical with sign of $f'''(0)x^3$, while the latter is positive. Thus $f(x) > 0$, i.e., $y_2(x) > y_1(x)$ for $x \in (-\varepsilon, 0)$. Combining Case 1 and 2, the proof is completed.

3 Main results

3.1 Finding solution to global attractivity problem

Consider the delay difference model (1). The change of variables $N_n = N^* + \frac{1}{r}x_n$ reduces (1) to the delay difference equation

$$x_{n+1} - x_n + \mu x_n + r\mu N^*(1 - e^{-x_{n-k}}) = 0, \quad n = 0, 1, 2, \cdots \quad (23)$$

Let $\{x_n\}$ be an strictly oscillatory solution of (23). Then there must exist a sequence $\{n_i\}$ of positive integers such that

$$k < n_1 < n_2 < \cdots n_i < n_{i+1} < \cdots \quad \lim_{i \to \infty} n_i = \infty$$

$$x_{n_i} x_{n_{i+1}} < 0 \quad \text{for} \quad i = 1, 2, \cdots$$

and for each $i = 1, 2, \cdots$, the terms of the sequence x_j for $n_i < j < n_{i+1}$ assume both positive and negative values. Let M_i and m_i be integers in (n_i, n_{i+1}) such that for $i = 1, 2, \cdots$,

$$x_{M_i} = \max\{x_j : n_i < j < n_{i+1}\} \quad \text{and} \quad x_{m_i} = \min\{x_j : n_i < j < n_{n_{i+1}}\}.$$

And

$$M_i - 1, \ m_i - 1 \in (n_i, n_{i+1}).$$

Then for $i = 1, 2, \cdots$

$$x_{M_i} > 0 \quad \text{and} \quad x_{M_i} - x_{M_i-1} \geq 0 \tag{24}$$

while

$$x_{m_i} < 0 \quad \text{and} \quad x_{m_i} - x_{m_i-1} \leq 0. \tag{25}$$

Let

$$\beta \overset{\triangle}{=} \limsup_{i \to \infty} x_n \quad \text{and} \quad \alpha = \liminf_{i \to \infty} x_n. \tag{26}$$

Clearly, $\beta = \lim_{i \to \infty} x_{M_i}$ and $\alpha = \lim_{i \to \infty} x_{m_i}$. In a similar way to [12], we can prove that

$$-rN^* < \alpha \leq 0 \leq \beta < \infty. \tag{27}$$

Given any $\varepsilon > 0$, by (26) there exists a positive n_0 such that

$$\alpha - \varepsilon < x_n < \beta + \varepsilon \quad \text{for} \quad n \geq n_0 + k. \tag{28}$$

By (23) we have

$$x_{M_i} - x_{M_i-1} + \mu x_{M_i-1} + r\mu N^*(1 - e^{-x_{M_i-1-k}}) = 0. \tag{29}$$

By (24) we get

$$x_{M_i-1} + rN^*(1 - e^{-x_{M_i-1-k}}) \leq 0. \tag{30}$$

From (30) we find

$$x_{M_i-1} \leq -rn^*(1 - e^{-x_{M_i-1-k}}). \tag{31}$$

By(31), (29) can be written as

$$\begin{aligned} x_{M_i} &= (1-\mu)x_{M_i-1} - r\mu N^*(1 - e^{-x_{M_i-1-k}}) \\ &\leq (\mu-1)rN^*(1 - e^{-x_{M_i-1-k}}) - r\mu N^*(1 - e^{-x_{M_i-1-k}}) \\ &= -rN^*(1 - e^{-x_{M_i-1-k}}). \end{aligned} \tag{32}$$

From (32) we get

$$x_{M_i-1-k} \leq -\ln(1 + \frac{x_{M_i}}{rN^*}). \tag{33}$$

We can reduce (23) to

$$(1-\mu)^{-(n+1)}x_{n+1} - (1-\mu)^{-n}x_n + r\mu N^*(1-\mu)^{-(n+1)}[1 - e^{-x_{n-k}}] = 0. \tag{34}$$

By summing (34) form $n = M_i - 1 - k$ to $n = M_i - 1$, for i sufficiently large in view of (28) and (33), we obtain

$$(1 - \mu)^{-M_i} x_{M_i} = (1 - \mu)^{-(M_i - 1 - k)} x_{M_i - 1 - k}$$
$$- r\mu N^* \sum_{j = M_i - 1 - k}^{M_i - 1} (1 - \mu)^{-(j+1)} [1 - e^{-x_{j-k}}]$$
$$\leq (1 - \mu)^{-(M_i - 1 - k)} (-\ln(1 + \frac{x_{M_i}}{rN^*}))$$
$$+ rN^* (e^{-\alpha + \varepsilon} - 1)[(1 - \mu)^{-M_i} - (1 - \mu)^{-(M_i - 1 - k)}].$$

And so

$$x_{M_i} + (1 - \mu)^{k+1} \ln(1 + \frac{x_{M_i}}{rN^*}) \leq rN^* (e^{-\alpha + \varepsilon} - 1)(1 - \mu)^{k+1}).$$

Let $i \to \infty$ we have $\beta + (1 - \mu)^{k+1} \ln(1 + \frac{\beta}{rN^*}) \leq M(e^{-\alpha + \varepsilon} - 1)$. Since ε is arbitrary, this implies

$$\beta + (1 - \mu)^{k+1} \ln(1 + \frac{\beta}{rN^*}) \leq M(e^{-\alpha} - 1). \tag{35}$$

In the same way as above, we can prove

$$\alpha + (1 - \mu)^{k+1} \ln(1 + \frac{\alpha}{rN^*}) \geq M(e^{-\beta} - 1). \tag{36}$$

Connecting (32) with (33) to a system and letting $y = \beta$ and $x = \alpha$ we obtain (8).

Clearly every oscillatory solution of (23) tends to zero as $n \to \infty$ provided (8) has a unique solution $x = y = 0$ in the region (9).

3.2 Main results

Theorem 1. If $\{x_n\}$ is a non-oscillatory solution of (23), then we have

$$\lim_{n \to \infty} x_n = 0.$$

The proof is easy by noting its monotonic behavior.

By Lemma 5 and (A) in this section 3.1, we get

Theorem 2. Assume that (7) holds and that $\{x_n\}$ is an oscillatory solution of (23), then

$$\lim_{n \to \infty} x_n = 0.$$

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By Theorem 1 and 2, we obtain the following result.

Theorem 3. Assume that (7) holds, then N^* is a global attractor of Eq.(1).

Clearly,Theorem 3 improves Theorem A.

Example 1. Assume that $(1-\mu)^{k+1} = \frac{1}{2}$ and $rN^* = \sqrt{2}+1$, we calculate

$$M = 1 + \frac{(1-\mu)^{k+1}}{rN^*} = \frac{\sqrt{2}+1}{2} \approx 1.2071.$$

The conditions of Theorem 3 are satisfied.

Remark 1. Let us try to find out

$$M_0 \triangleq \max\{M : M \leq 1 + \frac{(1-\mu)^{k+1}}{rN^*}\}.$$

By Example 1, we know $M_0 \geq \frac{\sqrt{2}+1}{2}$. And when $M \geq \frac{\sqrt{2}+1}{2}$, we have

$$\frac{(1-\mu)^{k+1}}{rN^*} \leq \frac{2}{\sqrt{2}+1}[1-(1-\mu)^{k+1}](1-\mu)^{k+1} \leq \frac{2}{\sqrt{2}+1} \cdot \frac{1}{4} = \frac{\sqrt{2}-1}{2}.$$

And so

$$M \leq 1 + \frac{(1-\mu)^{k+1}}{rN^*} \leq \frac{\sqrt{2}+1}{2}.$$

which implies $M_0 \leq \frac{\sqrt{2}+1}{2}$. Thus $M_0 = \frac{\sqrt{2}+1}{2}$.

References

1. M.Wazewska-Czyzewska and A.Lasota, Mathematical problems of the dynamics of t he red blood cells Systems, Annals of the Polish Mathematical Society, Seires III, Applied Mathematics **6**, 23-40 (1976).
2. O.Arino and M.Kimmel, Stability analysis of models of cell production systems, Mathematical Modelling **17**, 1269-1300 (1986).
3. M.R.S.Kulenovic and G.Ladas, Linearized oscillation in population dynamies, Bulletin of Mathematical Bilolgy **49**, 515-527 (1987).
4. M.R.S.Kulenovic, Global attractivity in population dynamics, Computers Math.,Applic **18**, 925-928 (1989).

5. I.Gyore and G.Lades, Oscillation theory of delay differential equstions with applications,Clarendon Press.Oxford,1991.

6. V.LJ.Kocic and G.Ladas, Oscillation and global attrativity in a discret e model of Nichoson's blowflies, Applicable Analysis **38**, 1-31 (1990).

7. Joseph W.H.So and J.S.Yu,Global Attractivity and Uniform Persistence i n Nicho lson's Blowflies, Differential Equations and Dynamical Systems **2**, 11-18 (1994).

8. Li Jingwen, Global Attractivity in the Discrete Nicholson's Blowflies, Ann.of Diff.Eqs. **12**, 173-182 (1996).

9. V.L.J.Kocic and G.Ladas, Global attractivity in nonlinear delay difference equations, Proc.Amer.Math.Soc. **115**, 1083-1088 (1992).

10. CH.G.Philos,Oscillations in a nonautonmous delay Logistic difference equation, Proceediongs of the Edinburgh Mathematical Society **35**, 121-131 (1992).

11. Ming-Po Chen and J.S.Yu, Oscillation and Global Attractivity in the Discrete Lasota-Wazewaska Model. Soochow J.Math , to appear (1999).

12. Li Jingwen,Asymptotic Behavior of a Delay Differential Model, J.Biomath **9**, 91-95 (1994).

13. Li Jingwen, Global Attractivity of Delay Differential Model with Devia ting Argument, J.Biomath **10**, 122-129 (1995).

OSCILLATION AND GLOBAL ATTRACTIVITY OF GENERALIZED NICHOLSON'S BLOWFLY MODEL

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In this paper we investigate the oscillation and the global attractivity of generalized Nicholson's blowfly model. The results we obtain here extend some conclusions given in the relative literature.

1 Introduction

In [1], Kocic and Ladas investigated the delay difference equation

$$N_{n+1} - N_n = -\delta N_n + P N_{n-k} e^{-a N_{n-k}}, \quad n = 0, 1, \cdots, \tag{1}$$

where $k \in N$, and $a \in (0, \infty)$, $\delta \in (0, 1)$, $P \in (\delta, \infty)$. $\tag{2}$

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Eq.(1) is a discrete analogue of the delay differential equation

$$\dot{N}(t) = -\delta N(t) + PN(t-\tau)e^{-aN(t-\tau)}, \quad t \geq 0, \tag{3}$$

which was first used by Gurney, Blythe, and Nisbet[2] to model the dynamics of Nicholson's blowflies (see also[3,4,5]).

In this paper, we extend Eq.(1) to the following equation

$$N_{n+1} - N_n = -\delta N_n + \sum_{i=1}^{m} p_i N_{n-k_i} e^{-a_i N_{n-k_i}}, \quad n = 0, 1, \cdots, \tag{4}$$

where

$$k_i \in N, \text{ and } a_i \in (0, \infty), \ \delta \in (0, 1), \ \sum_{i=1}^{m} p_i > \delta, \ i = 1, 2, \cdots, m. \tag{5}$$

Let $a = \max\limits_{1 \leq i \leq m}\{a_i\}$, $k = \max\limits_{1 \leq i \leq m}\{k_i\}$. If $c_{-k}, \cdots, c_0$ are $(k+1)$ given constants such that, $c_n \geq 0$, for $n = -k, \cdots, -1$ and $c_0 > 0$, then Eq.(4) has a unique positive solution satisfying the initial conditions

$$N_n = c_n \text{ for } n = -k, \cdots, 0. \tag{6}$$

By a solution of (4)-(6) we mean a sequence $\{N_n\}$ which satisfies (4) and (5) for $n = 0, 1, 2, \cdots$ as well as the initial condition (6). Eq.(4) has a unique equilibrium N^*. Furthermore, N^* is a solution of the equation $\delta = \sum\limits_{i=1}^{m} p_i e^{-a_i N^*}$.

2   The Oscillation of N^*

Proposition 2.1 Let $\{N_n\}$ be a solution of (4)-(6). Then

$$\limsup_{n \to \infty} N_n \leq \sum_{i=1}^{m} \frac{p_i}{\delta e a_i}. \tag{7}$$

Proof Let $f(x) = xe^{-ax}$, then $\max\{f(x) : x \geq 0\} = \frac{1}{ae}$. By (4), we see immediately that $N_{n+1} \leq (1-\delta)N_n + \sum\limits_{i=1}^{m} p_i \frac{1}{a_i e}$. Define a sequence $\{w_n\}$ by

$$\begin{cases} w_{n+1} = (1-\delta)w_n + \sum\limits_{i=1}^{m} p_i \frac{1}{a_i e}, \\ w_0 \quad = N_0. \end{cases}$$

By Lemma 1.6.1 and 1.6.2[1], $N_n \leq w_n = (1-\delta)^n w_0 + \frac{1-(1-\delta)^{n+1}}{\delta} \sum\limits_{i=1}^{m} p_i \frac{1}{a_i e}$. Letting $n \to \infty$, we obtain the desired result.

Theorem 9 *Assume that*

$$a_i N^* > 1, \ i = 1, 2, \cdots, m, \tag{8}$$

and

$$\sum_{i=1}^{m} [p_i e^{-a_i N^*} (a_i N^* - 1)](1 - \delta)^{-k_i - 1} \frac{(k_i + 1)^{k_i + 1}}{k_i^{k_i}} > 1. \tag{9}$$

Then every positive solution of (4)-(6) oscillates about the positive equilibrium N^*.

Proof Let

$$N_n = N^* + \frac{1}{a} x_n, \ n = -k, -k + 1, \cdots. \tag{10}$$

Then $\{x_n\}$ is a solution of the delay difference equation

$$x_{n+1} - x_n + \delta x_n + a\delta N^* - \sum_{i=1}^{m} [aN^* + x_{n-k_i}] p_i e^{-a_i N^* - \frac{a_i}{a} x_{n-k_i}} = 0, n = 0, 1, \cdots. \tag{11}$$

Since $N_n > 0$ for n sufficiently large, it follows that $x_n > -aN^*$ for n sufficiently large. By Proposition 2.1, $\{x_n\}$ is bounded above. Now set

$$\mu = \limsup_{n \to \infty} x_n \text{ and } \lambda = \liminf_{n \to \infty} x_n. \tag{12}$$

Then $-aN^* \leq \lambda \leq \mu < \infty$. Assume for the sake of contradiction that Eq.(4) has a positive solution $\{N_n\}$ which does not oscillate about N^*. Then $\{x_n\}$ is a nonoscillatory solution of Eq.(11). Without loss of generality, we assume that $\{x_n\}$ is eventually positive. Eq.(11) can be rewritten in the form

$$x_{n+1} = x_n + F(x_n, x_{n-k_1}, \cdots, x_{n-k_m}) \tag{13}$$

where $F(x_n, x_{n-k_1}, \cdots, x_{n-k_m}) = \sum_{i=1}^{m} [aN^* + x_{n-k_i}] p_i e^{-a_i N^* - \frac{a_i}{a} x_{n-k_i}} - \delta x_n - a\delta N^*$, and $F \in C[(0, \infty)^{m+1}, (-\infty, \infty)]$. Since $\frac{\partial F}{\partial x_{n-k_i}} = p_i e^{-a_i N^* - \frac{a_i}{a} x_{n-k_i}} [1 - a_i N^* - \frac{a_i}{a} x_{n-k_i}], i = 1, 2, \cdots, m$, it follows from (8) that $\frac{\partial F}{\partial x_{n-k_i}} < 0, i = 1, 2, \cdots, m$, which, together with $\frac{\partial F}{\partial x_n} = -\delta < 0$, yields that $F(u_0, u_1, \cdots, u_m)$ is nonincreasing in each of its arguments. Thus

$$x_{n+1} = x_n + F(x_n, x_{n-k_1}, \cdots, x_{n-k_m}) \leq x_n + F(0, 0, \cdots, 0) = x_n.$$

Hence $\{x_n\}$ is monotonic for $n \geq n_0 + k$ and

$$\lim_{n \to \infty} x_n = 0. \tag{14}$$

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Eq.(11) can be rewritten in the form

$$x_{n+1} - x_n + \delta x_n + \sum_{i=1}^{m} P_i(n) x_{n-k_i} = 0,$$

where $P_i(n) = \dfrac{aN^* p_i e^{-a_i N^*} - aN^* p_i e^{-a_i N^* - \frac{a_i}{a} x_{n-k_i}} - x_{n-k_i} p_i e^{-a_i N^* - \frac{a_i}{a} x_{n-k_i}}}{x_{n-k_i}}$, and $\lim\limits_{n\to\infty} P_i(n) = p_i e^{-a_i N^*}(a_i N^* - 1)$. One can see that the hypotheses of Lemma 7.4.1[3] are satisfied and so the linear equation

$$y_{n+1} - y_n + \delta y_n + \sum_{i=1}^{m} [p_i e^{-a_i N^*}(a_i N^* - 1)] y_{n-k_i} = 0, n = 0, 1, \cdots, \quad (15)$$

has an eventually positive solution. Let $\{y_n\}$ be an eventually positive solution of Eq.(15). Then $z_n = (1-\delta)^{-n} y_n$ is an eventually positive solution of

$$z_{n+1} - z_n + \sum_{i=1}^{m} [p_i e^{-a_i N^*}(a_i N^* - 1)](1-\delta)^{-k_i-1} z_{n-k_i} = 0, \ n = 0, 1, \cdots. \quad (16)$$

According to Theorem 7.3.1[3], Eq.(16) has no nonoscillatory solutions and this contradiction complete the proof.

3   The Global Attractivity of N^*

Theorem 10 *Assume that (8) holds and*

$$aN^*[(1-\delta)^{-k-1} - 1] < 1. \quad (17)$$

Then the positive equilibrium N^* of (4)-(6) is a global attractor.

Proof   The proof will be accomplished by introducing the transformation (10) and by showing that (14) is satisfied. A slight modification of Theorem 1 shows that (14) is true when $x_n \geq 0$ or $x_n \leq 0$ eventually. Therefore it remains to establish (14) when $\{x_n\}$ is strictly oscillatory. To this end, let $\{x_{f_j+1}, x_{f_j+2}, \cdots, x_{h_j}\}$ be the ith positive semicycle of $\{x_n\}$ followed by the ith negative semicycle $\{x_{h_j+1}, x_{h_j+2}, \cdots, x_{f_{j+1}}\}$, and x_{M_j}, x_{m_j} be the extreme values in these two semicycles, respectively, with the smallest possible indices M_j and m_j. It is easy to see that

$$M_j - f_j \leq k+1, \quad m_j - h_j \leq k+1. \quad (18)$$

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Let

$$\left.\begin{array}{l}\lambda = \liminf_{n\to\infty} x_n = \liminf_{j\to\infty} x_{m_j} \leq 0 \\ \mu = \limsup_{n\to\infty} x_n = \limsup_{j\to\infty} x_{M_j} \geq 0\end{array}\right\}. \qquad (19)$$

Clearly $-aN^* \leq \lambda \leq \mu < \infty$. To proof that (14) holds it is sufficient to show that

$$\lambda = \mu = 0. \qquad (20)$$

From (19) it follows that for any $\epsilon > 0$, there exists $n_0 \in N$ such that

$$\lambda - \epsilon \leq x_n \leq \mu + \epsilon \quad \text{for} \quad n \geq n_0 + k, \qquad (21)$$

Furthermore we have

$$x_{n-k_i} e^{-x_{n-k_i}} \leq \mu + \epsilon \quad \text{for} \quad n \geq n_0 + k, i = 1, 2, \cdots, m. \qquad (22)$$

Eq.(11) can be written in the form

$$x_{n+1} - (1-\delta)x_n = -\delta a N^* + \sum_{i=1}^{m} (aN^* + x_{n-k_i}) p_i e^{-a_i N^* - \frac{a_i}{a} x_{n-k_i}}. \qquad (23)$$

By multiplying Eq.(23) by $(1-\delta)^{-n-1}$ and summing up from $n = f_j$ to $n = M_j - 1$, for j sufficiently large, we obtain

$$(1-\delta)^{-M_j} x_{M_j} - (1-\delta)^{-f_j} x_{f_j}$$
$$= [-\delta a N^* + \sum_{i=1}^{m} (aN^* + x_{n-k_i}) p_i e^{-a_i N^* - \frac{a_i}{a} x_{n-k_i}}] \sum_{b=f_j}^{M_j-1} (1-\delta)^{-b-1}.$$

Since $x_{f_j} < 0$ and $x_{n-k_i} e^{-\frac{a_i}{a} x_{n-k_i}} \leq \mu + \epsilon, i = 1, 2, \cdots, m$, it follows that

$$x_{M_j} \leq [-\delta a N^* + \sum_{i=1}^{m} [aN^* p_i e^{-a_i N^* - \frac{a_i}{a}(\lambda-\epsilon)} + (\mu+\epsilon) p_i e^{-a_i N^*}]] \frac{1 - (1-\delta)^{M_j-f_j}}{\delta}.$$

Noting that $M_j - f_j \leq k + 1$ and ϵ is arbitrary, we have

$$\mu \leq [-\delta a N^* + \sum_{i=1}^{m} [(aN^* p_i e^{-a_i N^* - \frac{a_i}{a}\lambda} + \mu p_i e^{-a_i N^*}]] \frac{1 - (1-\delta)^{k+1}}{\delta}.$$

Because $\lambda \leq 0$, it follows that $e^{-\frac{a_i}{a}\lambda} \leq e^{-\lambda}, i = 1, 2, \cdots, m$. As $\delta = \sum_{i=1}^{m} p_i e^{-a_i N^*}$, we find

$$\mu \leq [aN^*(e^{-\lambda} - 1) + \mu][1 - (1 - \delta)^{k+1}].$$

So it yields that

$$\mu \leq aN^*(e^{-\lambda} - 1)[(1 - \delta)^{-k-1} - 1].$$

Combing this with (17), we have

$$\mu \leq e^{-\lambda} - 1. \tag{24}$$

From (19), we obtain

$$x_{n-k_i} e^{-x_{n-k_i}} \geq \lambda - \epsilon, i = 1, 2, \cdots, m. \tag{25}$$

After multiplying Eq.(23) by $(1 - \delta)^{-n-1}$ and summing up from $n = h_j$ to $n = m_j - 1$, for j sufficiently large, we obtain

$$(1 - \delta)^{-m_j} x_{m_j} - (1 - \delta)^{-h_j} x_{n_j}$$
$$= [-\delta aN^* + \sum_{i=1}^{m} (aN^* + x_{b-k_i}) p_i e^{-a_i N^* - \frac{a_i}{a} x_{b-k_i}}] \sum_{b=h_j}^{m_j - 1} (1 - \delta)^{-b-1}.$$

Since $x_{h_j} > 0$, and ϵ is arbitrary, it holds that

$$\lambda \geq [-\delta aN^* + \sum_{i=1}^{m} [(aN^* p_i e^{-a_i N^*} e^{-\frac{a_i}{a}\mu} + \lambda p_i e^{-a_i N^*}]] \frac{1 - (1 - \delta)^{m_j - h_j}}{\delta}.$$

Since $\mu \geq 0$, we get that

$$\lambda \geq [-\delta aN^* + \sum_{i=1}^{m} [(aN^* p_i e^{-a_i N^*} e^{-\mu} + \lambda p_i e^{-a_i N^*}]] \frac{1 - (1 - \delta)^{m_j - h_j}}{\delta}.$$

Solving for λ, we have

$$-\lambda \leq aN^*[(1 - \delta)^{-k-1} - 1](1 - e^{-\mu}),$$

which, together with (17), yields

$$\lambda \geq e^{-\mu} - 1. \tag{26}$$

It is not difficult to prove that $\lambda = \mu = 0$ by (24) and (26) (see[4]). This complete the proof.

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References

1. Kocic and Ladas, Global Behavior of Nonlinear Difference Equation of Higher Order with Application, (Kluwer Academic Publishers, 1993).
2. W.S.Gurney,S.P.Blythe, and R.M.Nisbet, Nicholson's blowflies revisited, Nature, **287**,17-21(1980).
3. I.Györi and G.Ladas. Oscillation Theory of Delay Differential Equation with Application, (Oxford.Clarendon Press, 1991).
4. Joseph WHso and Yu Jianshe, *Differential Equation and Dynamical Systems*, **2**, 11 (1994).
5. Joseph WHso and Yu Jianshe, *J. Math. Anal. Appl.*, **193**, 233(1995).

HOPF BIFURCATION IN A THREE-UNIT NEURAL NETWORK WITH DELAY

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A system of the three-unit network with no self-connection is investigated, the general formula for bifurcation direction of Hopf bifurcation is calculated, and the estimate formula of period for periodic solution is given.

Dynamical characteristics of neural networks have become recently a subject of intense research activity. J.Bélair and S.Dufour[1] investigated a system of neural networks introduced by Hopfield[5].Especially, they studied the three-unit network system with no self-connection

$$\frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{3} T_{ij} f_j(x_j(t-\tau)), \quad i = 1, 2, 3. \tag{1}$$

where $f_j(0) = 0$, $j = 1, 2, 3$ and $T_{ii} = 0, i = 1, 2, 3$, gave the stability properties of the null solution. [2] discussed the linear stability regions of the system (1)

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with $f_j(x) = f(x) = \beta\ tanh(x(t)), j = 1, 2, 3$, and showed a supercritical Hopf bifurcation to occur.

In this paper, the method of Hassard[4] is used to study the system

$$\frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{3} a_{ij}\beta\ tanh(x_j(t-\tau)), \quad i = 1, 2, 3, \qquad (2)$$

where $a_{ii} = 0, i = 1, 2, 3$. The general formula for bifurcation direction of Hopf bifurcation is calculated and the estimate formula of period for bifurcation periodic solution is given.

Consider System (2). The linear parts of this system are

$$\frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{3} a_{ij}\beta x_j(t-\tau), \quad i = 1, 2, 3. \qquad (3)$$

The characteristic equation of Eq. (3) is

$$\det[\lambda I + I - \beta A e^{-\lambda\tau}] = (\lambda+1)^3 - P\beta^3 e^{-3\lambda\tau} - Q\beta^2(\lambda+1)e^{-2\lambda\tau} = 0, \qquad (4)$$

where $P = a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}, Q = a_{23}a_{32} + a_{13}a_{31} + a_{12}a_{21}$. As a general case, we consider $P \neq 0$ and $Q \neq 0$. [1] and [2] made $P\beta^3$ and $Q\beta^2$ as parameters to discuss the stability region of null solution. Here we use β to be parameter to give two lemmas for the existence of Hopf bifurcation.

Lemma 1. For any fixed positive value of τ. (i) If ω is not a root of the equation $\omega = -\tan(\omega\tau)$ and $1+\omega^2+2(1-\omega^2)\cos(2\omega\tau)-4\omega\sin(2\omega\tau) \neq 0$, then (4) has a pair of purely imaginary roots $\lambda = \pm i\omega$, when

$$\beta = \frac{2Q(1+\omega^2)(\omega\sin(\omega\tau) - \cos(\omega\tau))}{P(1+\omega^2+2(1-\omega^2)\cos(2\omega\tau) - 4\omega\sin(2\omega\tau))}. \qquad (5)$$

(ii) If ω is a root of the equation $\omega = -\tan(\omega\tau)$, then (4) has a pair of purely imaginary roots $\lambda = \pm i\omega$, when β is the root of the following equation

$$\beta^3 P\sin(3\omega\tau) + \beta^2 Q(\sin(2\omega\tau) - \omega\cos(2\omega\tau)) + 3\omega - \omega^3 = 0. \qquad (6)$$

We denote β_0 for β in (5) or satisfying (6). Letting λ be the function of β in (4), the derivative is obtained

$$\frac{dRe\lambda(\beta)}{d\beta}\bigg|_{\beta=\beta_0} = \frac{1}{M^2+N^2}\{M[3\beta_0^2 P\cos(3\omega\tau) + 2\beta_0 Q(\cos(2\omega\tau) + \omega\sin(2\omega\tau))]$$

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$$+N[-3\beta_0^2 P \sin(3\omega\tau) + 2\beta_0 Q(\omega \cos(2\omega\tau) - \sin(2\omega\tau))]\}, \tag{7}$$

where

$$M = 3(1 - \omega^2) + 3\tau\beta_0^3 P \cos(3\omega\tau) + \beta_0^2 Q((2\tau - 1)\cos(2\omega\tau) + 2\tau\omega \sin(2\omega\tau)),$$

$$N = 6\omega - 3\tau\beta_0^3 P \sin(3\omega\tau) + \beta_0^2 Q(2\omega\tau \cos(2\omega\tau) - (2\tau - 1)\sin(2\omega\tau)).$$

We denote

$$\alpha'(\beta_0) = \left.\frac{dRe(\lambda(\beta))}{d\beta}\right|_{\beta=\beta_0},$$

and obtain following result.

Lemma 2. For any fixed positive value of τ, suppose that $\beta = \beta_0$, all roots in (4) have negative real parts except a pair of pure imaginary roots, and $\alpha'(\beta_0) \neq 0$, then there exists a periodic solution of Eq. (2) which bifurcate from $\beta = \beta_0$.

Next, we discuss the direction of Hopf bifurcation and the stability of bifurcation periodic solution. We write delay differential equation (2) to a form of ordinary differential equation

$$\dot{X}_t = A(\mu)X_t + RX_t \tag{8}$$

using Riesz expressing theorem, and the method of Hassard [5], we give the formulas of Hopf bifurcation periodic solution as follows.

Theorem. If $g_{21} \neq 0$, the bifurcation periodic solution of equation (2) at origin can be described by following formulas

(i) The bifurcation direction of periodic solution is determined by the sign of

$$\mu(\varepsilon) = \mu_2\varepsilon^2 + \cdots,$$

where $\mu_2 = -Reg_{21}/2\alpha'(\beta_0)$. Here $\alpha'(\beta_0)$ is given in (7) . If $\alpha'(\beta_0) > 0$, then when $\mu_2 > 0$ ($\mu_2 < 0$), equation (2) has supercritical (subcritical) bifurcation.

(ii) The period of the bifurcation periodic solution can be estimated by

$$T(\varepsilon) = \frac{2\pi}{\omega}(1 + t_2\varepsilon^2 + \cdots),$$

where

$$t_2 = -\frac{1}{\omega}\left(\frac{1}{2}Img_{21} + \mu_2 \left.\frac{dIm\lambda(\beta)}{d\beta}\right|_{\beta=\beta_0}\right).$$

(iii) The stability of bifurcation periodic solution is determined by the sign of

$$B(\varepsilon) = b_2\varepsilon^2 + \cdots.$$

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where $b_2 = Re g_{21}$. when $b_2 > 0$ ($b_2 < 0$), the bifurcation solution is unstable (stable), where

$$g_{21} = -2\beta_0 e^{-i\omega\tau}[\overline{q_1^* D}(a_{12}q_2^2\overline{q_2}+a_{13})+\overline{q_2^* D}(a_{21}q_1^2\overline{q_1}+a_{23})+\overline{D}(a_{31}q_1^2\overline{q_1}+a_{32}q_2^2\overline{q_2})],$$

and

$$q_1 = -\frac{a_{32}}{a_{31}}(q_2 - \frac{(i\omega + 1)e^{i\omega\tau}}{a_{32}\beta_o}),$$

$$q_2 = \frac{a_{23}a_{31}\beta_0 + a_{21}(i\omega + 1)e^{i\omega\tau}}{a_{21}a_{32}\beta_0 + a_{31}(i\omega + 1)e^{i\omega\tau}}.$$

$$q_1^* = -\frac{1}{a_{12}\beta_0}((i\omega - 1)e^{-i\omega\tau}q_2^* + a_{32}\beta_0),$$

$$q_2^* = \frac{a_{13}a_{32}\beta_0 - a_{12}(i\omega - 1)e^{-i\omega\tau}}{a_{12}a_{23}\beta_0 - a_{13}(i\omega - 1)e^{-i\omega\tau}}.$$

$$D = [\overline{q_1^*}(q_1 + a_{12}\tau e^{-i\omega\tau}\beta_0 q_2 + a_{13}\tau e^{-i\omega\tau}\beta_0) + \overline{q_2^*}(a_{21}\tau e^{-i\omega\tau}\beta_0 q_1 + q_2$$

$$+a_{23}\tau e^{-i\omega\tau}\beta_0) + a_{31}\tau e^{-i\omega\tau}\beta_0 q_1 + a_{32}\tau e^{-i\omega\tau}\beta_0 q_2 + 1]^{-1},$$

here $\overline{q_i}$ ($\overline{q_i^*}$) is the conjugate of $q_i(q_i^*)$.

For a example, let matrix A in (2) be (see [2])

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1.25 & 0 & 1 \\ 1.25 & 1 & 0 \end{pmatrix}$$

$\tau = 4.265$ and $\omega = 0.3899$. From (5), $\beta_0 = 0.96$ and from (7) ,$\alpha'(\beta_0) = 0.202$,which is not equal to zero . We calculate that $Re\ g_{21} = -0.07$,and $Im g_{21} = 0.3433$. Then $\mu_2 = 0.1732$.

Since $\alpha'(\beta_0) > 0$ and $\mu_2 > 0$, Equation (2) has supercritical bifurcation .The bifurcation periodic solution is stable. The period of the bifurcation solution is

$$T(\varepsilon) = \frac{2\pi}{\omega}(1 - 0.446\varepsilon^2 + \cdots).$$

Acknowledgments

References

1. J.Bélair and S. Dufour, Canad. Appl. Math. Quart. **4**, 135 (1996).
2. J. Bélair, S. A. Campbell and P. V. D. Driessche, SIAM J. Appl. Math. **56**, 245 (1996).
3. J.K.Hale and S.Verduyn Lunel, Introduction to Functional Differential Equations, (Springer-Verlag, New York, 1993).
4. B. D. Hassard, N. D. Kazarinoff and Y.-H. Wan, Theory and Applications of Hopf Bifurcation, (Cambridge University Press, London, 1981).
5. J. J. Hopfield, Proc. Natl. Acad. Sci. **81**, 3088 (1984).

ASYMPTOTIC BEHAVIOR FOR A CLASS OF DELAY 2-D DISCRETE DYNAMIC SYSTEM

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In this paper, we discuss the asymptotic behaviour of solutions for a class of delay 2-dimension discrete dynamic system

$$A_{m-1,n} + A_{m,n-1} - A_{mn} + p_{mn}A_{m+\sigma,n+\tau} = 0,$$

where,$m, n = 1, 2, ...$, and we also obtained sufficient conditions for the oscillation of all solutions of the above Eq.

1 Introduction

Oscillation criteria for a class of delay 2-dimension discrete dynamic system of the form

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{m-\sigma,n-\tau} = 0, m, n = 1, 2, \cdots. \quad (1)$$

have been derived in [1].This paper is concerned with a class of advanced type delay 2-dimension discrete dynamic system of the form

$$A_{m-1,n} + A_{m,n-1} - A_{m,n} + p_{m,n}A_{m+\sigma,n+\tau} = 0, m, n = 1, 2, \cdots \quad (2)$$

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where σ, τ are non-negative integers and $p_{m,n} > 0$ for $m, n > 0$. This equation can be regarded as a discrete analog of the elliptic partial differential equation

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - \frac{\partial A}{\partial x} - \frac{\partial A}{\partial y} + A(x, y) + p(x, y)A(x + \sigma, y + \tau) = 0.$$

We first settle the question of existence and uniqueness of solutions of our equations (2). For the sake of convenience, let us denote the set $\{a, a+1, \cdots, b\}$ of consecutive integers by $[a, b]$, the ray $\{a, a + 1, \cdots, \}$ ofr integers by $[a, \infty)$, and the cross product $[0, \infty) \times [0, \infty)$ by Z. In case $\sigma = \tau = 0$, we may write (2) in the form

$$A_{mn} = \frac{A_{m-1,n} + A_{m,n-1}}{1 - p_{mn}},$$

provided that $p_{m,n} \neq 1$ for $m, n \geq 0$. Furthermore, if the values of $A_{m,0}$ and $A_{0,n}$ are given for $m \geq 1$ and $n \geq 1$ respectively, then we may successively calculate

$$A_{11}, A_{21}, A_{12}, A_{31}, A_{22}, A_{13}, \cdots$$

in a unique manner. The following result is now clear.

Theorem 1.1 *Assume that $\sigma = \tau = 0$ and that $p_{m,n} \neq 1$ for $m, n \geq 0$. Then given any two sequences $\{\phi_m\}_{m=1}^{\infty}$ and $\{\psi_m\}_{m=1}^{\infty}$, there is a unique solution $\{A_{mn}\}$ of (2) which is defined for $(m, n) \in Z \setminus \{(0, 0)\}$ and satisfies the conditions*

$$A_{m,0} = \phi_m, \quad m = 1, 2, \cdots$$

and

$$A_{0,n} = \psi_n, \quad n = 1, 2, \cdots$$

In case one of the non-negative integers σ or τ is positive, the by writing equation (2) in the form

$$A_{m+\sigma, n+\tau} = \frac{1}{p_{mn}}(A_{mn} - A_{m-1,n} - A_{m,n-1}),$$

it is also easy to see that the following result holds.

Theorem 1.2 *Assume that at least one of the non-negative integers σ and τ is positive. Then given any functions $\{\psi_{mn}\}$ defined for*

$$(m, n) \in \Omega = Z \setminus \{(0, 0)\} \setminus [\sigma + 1, \infty) \times [\tau + 1, \infty),$$

there is a unique solution $\{A_{mn}\}$ of (2) which is defined for $(m, n) \in Z \{(0, 0)\}$ and satisfies the conditions

$$A_{mn} = \psi_{mn}, \quad (m, n) \in \Omega.$$

In view of these Theorems, it will be understood that a solution $\{A_{mn}\}$ is always defined for $(m,n) \in Z \setminus \{(0,0)\}$. Furthermore, a solution $\{A_{mn}\}$ will be said to be eventually positive or eventually negative if $A_{mn} > 0$, or respectively $A_{mn} < 0$, for all large m and n. It will be said to be oscillatory if it is neither eventually positive nor eventually negative.

An elementary result will be used repeatedly in the sequel.

Lemma 1.1 *For any integers a, b, c and d wcich satisfy $a \leq b$ and $c \leq d$, the following formal identity holds for any double sequence $\{x_{mn}\}$ defined for $(m,n) \in [a-1,b] \times [c-1,d]$, we have*

$$\sum_{m=a}^{b} \sum_{n=c}^{d} (x_{m-1,n} + x_{m,n-1} - x_{mn})$$

$$= -x_{bd} + \sum_{m=a}^{b-1} \sum_{n=c}^{d-1} x_{mn} + \sum_{m=a}^{b} x_{m,c-1} + \sum_{n=c}^{d} x_{a-1,n}.$$

We omit the proof here.

2 Preparatory Lemmas

We will be interested in obtaining oscillation criteria for solutions of (2). There are several preparatory results which will be useful for achieving our goals. First of all, it is easy to see that eventually positive solution of (2) is increasing in m and in n.

Lemma 2.1 *Let $\{A_{mn}\}$ be an eventually positive solution of (2). Then $\{A_{mn}\}$ is strictly increasing in m and in n for all large m, n.*

Lemma 2.2 *Assume that there is a positive number δ such that for all large m and n, we have $\sum_{m=i+1}^{i+\sigma} \sum_{n=j+1}^{j+\tau} p_{mn} \geq \delta$.*

Let $\{A_{mn}\}$ be an eventually positive solution of (2). Then for all large s and t,

$$\frac{A_{s+\sigma,t+\tau}}{A_{s,t}} \leq \frac{16}{\delta^4}.$$

Lemma 2.3 *Suppose $\tau = 0$. Assume that there is a positive number δ such that for all large i and j, we have*

$$\sum_{m=i+1}^{i+\sigma} p_{mj} \geq \delta.$$

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Let $\{A_{mn}\}$ be an eventually positive solution of (2). Then for all large s and t,

$$\frac{A_{s+\sigma}}{A_{s+1,t}} \leq \frac{4}{\delta^2}.$$

3 Oscillation Theorems

In case both σ and τ are zero, an oscillation theorem is easily obtained. The idea [1,Theorem 3.1] of the proof is well known and is thus omitted.

Theorem 3.1 *Assume that $\sigma = \tau = 0$, and that $1 - p_{mn}$ is not eventually negative, then every solution of (2) is oscillatory.*

Theorem 3.2 *Assume that*

$$\lim_{m,n\to\infty} \sup \sum_{m=i}^{i+\sigma} \sum_{n=j}^{j+\tau} p_{mn} > 1,$$

then every solution of (2) is oscillatory.

Theorem 3.3 *Assume that $\sigma, \tau > 0$, and that*

$$\lim_{m,n\to\infty} \inf \sum_{m=i+1}^{i+\sigma} \sum_{n=j+1}^{j=\tau} p_{mn} > \frac{1}{2}\frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}},$$

where $\lambda = \frac{2\sigma\tau}{\sigma+\tau}$. Then every solution of (2) is oscillatory.

By means of Lemma 2.3 and the same reasoning used in the proof of Theorem 3.3, we may also obtain the following oscillation criterion.

Theorem 3.4 *Assume that $\sigma > 0$ and $\tau = 0$, and that*

$$\lim_{m,n\to\infty} \inf \frac{1}{\sigma} \sum_{m=i+1}^{i+\sigma} p_{mj} > \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}.$$

Then every solution of (2) is oscillatory.

4 Acknowledgements

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References

1. B.G.Zhang, S.T.Liu and S.S.Cheng , *J.Difference Eq. Appl.*, **1**, 215 (1995).
2. S.T.Liu,S.S.Cheng, *Far east Journal Mathematical Science*, **5**, 387 (1997).
3. I.Gyori and J.Ladas, *Oscillation Theory of Delay Differential Equations*, (Oxford Mathematical Monographs, Claredon Press, Oxford, 1991).

VISCOSITY SOLUTIONS OF FULLY NONLINEAR SYSTEMS PARABOLIC EQUATIONS

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In this note, we are concerned with the following problem

$$\begin{cases} F_i(t,x;u;\partial_t u_i, D_x u_i, D_x^2 u_i) = 0, & (t,x) \in \mathcal{Q}, \\ u_i(t,x) = 0, & (t,x) \in \Gamma, \\ u_i(0,x) = \phi(x), & x \in \Omega, \\ \quad i = 1,2,\cdots,m. \end{cases} \tag{1}$$

Here $\mathcal{Q} = (0,T) \times \Omega$, $\Gamma = (0,T) \times \partial\Omega$ in which $(0,T)$ is an interval in R and Ω is a bounded open subset of R^n, $u = (u_1,\cdots.u_m) : \mathcal{Q} \to R^m$ is the unknown function, $F = (F_1,\cdots,F_m) : \mathcal{Q} \times R^m \times R \times R^n \times \mathbf{S}(n) \to R^m$ is a given function which is locally bounded, $\mathbf{S}(n)$ denotes the set of real symmetric $n \times n$ matrices equipped its usual order. $D_x u_i$ and $D_x^2 u_i$ denote the gradient and Hessian matrix of the function u_i with respect to the argument x.

A scalar equation $F_i = 0$ in (1) is said to be *degenerated parabolic* if

$$F_i(t,x;u;a,p,X) - F_i(t,x;u;a,p,Y) \geq 0 \tag{2}$$

for all $(t,x;u;a,p) \in \mathcal{Q} \times R^m \times R \times R^n$ and $X \leq Y \in \mathbf{S}(n)$, and there is $\delta_i > 0$ such that

$$F_i(t,x;u;b,p,X) - F_i(t,x;u;a,p,X) \geq \delta_i(b-a) \tag{3}$$

for all $(t, x; u; p, X) \in \mathcal{Q} \times R^m \times R^n \times \mathbf{S}(n)$ and $a \leq b \in R$. A system (1) is said to be degenerated parabolic if all components $F_i = 0, i = 1, \cdots, m$ are degenerated parabolic.

The theory of viscosity solution depends on the comparison. Unfortunately, the comparison does not hold true in general for systems. In the previous works ([1],[4],[5]), only quasi-decreasing system, for which comparison holds true, was discussed. By using of Perron's method and combining with the technique of coupled solution, we extend the technique of viscosity solution to the general quasi-monotone system and the non-quasi-monotone systems.

For a function $u : \mathcal{Q} \to R^m$, $u^*(t, x) = (u_1^*, \cdots, u_m^*)$ and $u_*(t, x) = (u_{1*}, \cdots, u_{m*})$ denote its upper and lower envelopes, we know that u^* and u_* are upper and lower semi-continuous functions (USC and LSC in short), respectively, on $\bar{\mathcal{Q}}$ with value in $R^m \cup \{\pm\infty\}^m$ and $u_* \leq u \leq u^*$ on $\bar{\mathcal{Q}}$. And we know that if F is quasi-monotone, so are F^* and F_*, and converse is not necessary.

To give the definition of sub and super-solution, we first put that $A = \{1, \cdots, m\}$ and $A_i \subset A \backslash \{i\}$ is called the decreasing index set of F_i, i.e., F_i is decreasing with respect to the coupled argument u_j as $j \in A_i$ and increasing with respect to u_k as $k \notin A_i$. Then, we define

$$W^i(u, v; A_i) : R^m \times R^m \times 2^A \to R^{m-1}$$

as $W^i(u, v; A_i) = \{W_1^i, \cdots, W_{i-1}^i, W_{i+1}^i, \cdots, W_m^i\}$ in which

$$W_j^i(u, v; A_i) = u_j, \quad \text{as } j \in A_i, \quad W_j^i(u, v; A_i) = v_j, \quad \text{as } j \notin A_i.$$

The set $\mathcal{P}_\Omega^{2,+} u(t, x)$ (or $\mathcal{P}_\Omega^{2,-} u(t, x)$) and $\overline{\mathcal{P}}_\Omega^{2,+} u(t, x)$ (or $\overline{\mathcal{P}}_\Omega^{2,-} u(t, x)$) denote the parabolic superjet (or subjet) of the second order for the function u at (t, x) and its closure defined as that in [2].

Definition 1 Suppose that F is degenerated parabolic, locally bounded and quasi-monotone. $U(t, x) = (U_1, \cdots, U_m)$ and $V(t, x) = (V_1, \cdots, V_m)$ are called a pair of viscosity coupled sub-solution and super-solution of (1) if they are locally bounded in $\mathcal{Q}$ such that, for $i = 1, \cdots, m$,

$$F_{i*}(t, x; W^i(U^*, V_*; A_i), U_i^*; a_i, p_i, X_i) \leq 0$$

$$\leq F_i^*(t, x; W^i(V_*, U^*; A_i), V_{i*}; b_i, q_i, Y_i),$$

$$\forall (t, x) \in \mathcal{Q},$$

$$(a_i, p_i, X_i) \in \mathcal{P}_\Omega^{2,+} U_i^*(t, x), \quad (b_i, q_i, Y_i) \in \mathcal{P}_\Omega^{2,-} V_{i*}(t, x),$$

$$\quad (4)$$

$$U_i^*(t, x) \leq 0 \leq V_{i*}(t, x), \quad \text{for } (t, x) \in \Gamma,$$

$$U_i^*(0, x) \leq \phi(x) \leq V_{i*}(0, x), \quad \text{for } x \in \Omega,$$

$$\quad (5)$$

Definition 2 $\{U, V\}$ is called a viscosity coupled solution of (1) if $\{U, V\}$ and $\{V, U\}$ are both couples of viscosity sub and super-solutions of (1).

Definition 3 $u(t, x)$ is called a viscosity solution of (1) if $\{u, u\}$ is a viscosity coupled solution of (1).

It could be called the technique of coupled solution for such a way to define the sub and super-solutions. In fact, the set of sub and super-solutions defined like above is a totally ordered subset of that in the classic sense. It is a useful tool to deal with systems (cf. [3],[7],[8]).

Then, by the Perron's method, we prove the following theorem.

Theorem 4 Let the system F be degenerated parabolic, locally bounded, quasi-monotone. Suppose also that the comparison for (1) holds. If U and V are a pair of viscosity coupled sub and super-solution of (1), then the problem (1) has unique viscosity solution $u(t, x) \in C(\mathcal{Q})$ satisfying $U^* \leq u \leq V_*$ on $\bar{\mathcal{Q}}$.

To prove the comparison, we need the following three conditions.

(H1) There are modular functions $\omega_i : (0, \infty) \to (0, \infty)$ that satisfy $\omega_i(0^+) = 0$ such that, for each fixed $t \in R$,

$$F_i^*(t, y; W_i, r_i; a_i, \alpha(x - y), Y) - F_{i*}(t, x; W_i, r_i; a_i, \alpha(x - y), X)$$
$$\leq \omega_i(\alpha|x - y|^2 + |x - y|), \tag{6}$$

whenever $x, y \in \Omega$, $r_i, \alpha \in R$, $W_i \in R^{m-1}$ and $X, Y \in \mathbf{S}(n)$ satisfying

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{7}$$

(H2) There are non-negative constants β_{ij} ($\beta_{ii} = 0$), $i, j = 1, \cdots, m$, such that

$$|F_{i*}(t, x; v^i, r; a, p, X) - F_i^*(t, x; u^i, r; a, p, X)| \leq \sum_{j \neq i} \beta_{ij}|v_j - u_j|, \tag{8}$$

for $u, v \in R^m$, $(t, x; r; a, p, X) \in \bar{\mathcal{Q}} \times R \times R \times R^n \times \mathbf{S}(n)$.

(H3) there exist real numbers $\gamma_i > 0$ such that

$$F_{i*}(t, x; W_i, r_i; a, p, X) - F_i^*(t, x; W_i, s_i; a, p, X)$$
$$\geq -\gamma_i(r_i - s_i), \tag{9}$$

for $r_i \geq s_i, (t, x; W_i; a, p, X) \in \bar{\mathcal{Q}} \times R^{m-1} \times R \times R^n \times \mathbf{S}(n), i = 1, \cdots, m$.

Theorem 5 Let the system F be degenerated parabolic, locally bounded, quasi-monotone. Suppose that the conditions **(H1)-(H3)** hold. If U and V are a couple of viscosity sub and super-solution of (1), then $U^* \leq V_*$, in $\bar{\mathcal{Q}}$.

For the non-quasi-monotone system, we give the following concept like that in [1].

Definition 6 Suppose that F is degenerated parabolic, locally bounded. U and V are called a pair of viscosity sub-solution and super-solution in strong sense of (1) if they satisfy the condition

(C)   there is a $h(t,x) \in C(\mathcal{Q})$ between U and V,

such that, for every $w(t,x) \in C(\mathcal{Q})$ between U and V(i.e., $U \leq w \leq V$ as $U \leq V$ or $V \leq w \leq U$ as $V \leq U$),

$$F_{i*}(t,x;w^i,U_i^*;a_i,p_i,X_i) \leq 0$$

$$\leq F_i^*(t,x;w^i,V_{i*};b_i,q_i,Y_i), \tag{10}$$

$$\forall (t,x) \in \mathcal{Q},$$
$$(a_i,p_i,X_i) \in \mathcal{P}_\Omega^{2,+}U_i^*(t,x), \quad (b_i,q_i,Y_i) \in \mathcal{P}_\Omega^{2,-}V_{i*}(t,x),$$

where $w^i = (w_1,\cdots,w_{i-1},w_{i+1},\cdots,w_m)$, and (5) holds.

We prove the following results by the comparison for the scalar parabolic equations and fixed-point theorem.

Theorem 7 Let the system F be degenerated parabolic, locally bounded and satisfy **(H1)**, **(H2)** and **(H3)**. Suppose that U and V are a couple of viscosity sub and super-solution in the strong sense of (1). Then,

i) $U_i^* \leq V_{i*}$, $i = 1,\cdots,m$, on $\bar{\mathcal{Q}}$;

ii) there is a unique viscosity solution $u(t,x) \in C(\mathcal{Q})$ satisfying

$$U^* \leq u \leq V_*, \text{ on } \bar{\mathcal{Q}}.$$

Concerning to viscosity sub and super-solutions in the strong sense of (1), we have

Theorem 8 Suppose that the system F is degenerated parabolic, locally bounded and satisfies **(H2)** and **(H3)**. If U and V are a couple of viscosity sub and super-solution in the classic sense of (1). Then, there is a couple of viscosity sub and super-solutions in the strong sense for (1).

In fact, a couple of viscosity sub and super-solutions in the strong sence of (1) are

$$\hat{U}(t,x) = U(t,x) - h(t), \quad \hat{V}(t,x) = V(t,x) + h(t)$$

where $h(t)$ is a non-negative solution of linear system of ordinary differential equations

$$\begin{cases} h' - (\mathbf{\Gamma} + \mathbf{B})h = \mathbf{B}d, \\ h(0) = c \geq 0, \end{cases} \tag{11}$$

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where $\Gamma = \mathrm{diag}(\hat{\gamma}_i)$, $\mathbf{B} = (\hat{\beta}_{ij})$ are $m \times n$ matrices in which

$$\hat{\gamma}_i = \frac{\gamma_i}{\delta_i}, \qquad \hat{\beta}_{ij} = \frac{\beta_{ij}}{\delta_i}$$

$\gamma_i, \beta_{ij}, \delta_i, i, j = 1, \cdots, m$ are constants in conditions (H3), (H2) and (3), while

$$d = (d_1, \cdots, d_m), \quad d_i = \sup_{(t,x) \in \mathcal{Q}} \frac{V_i - U_i}{\delta_i} \geq 0.$$

Thus, we have

Theorem 9 Let the system F be degenerated parabolic, locally bounded and satisfy **(H1)**, **(H2)** and **(H3)**. Suppose that U and V are a couple of viscosity sub and super-solution in the strong sense of (1), which satisfy the condition (C). Then, there is a unique viscosity solution $u(t, x) \in C(\mathcal{Q})$ satisfying

$$U^* - h \leq u \leq V_* + h, \quad \text{on } \bar{\mathcal{Q}},$$

where $h(t)$ is a function mentioned above.

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References

1. H.Ishii and S.Koike, *Comm.PDE* **16**,1095 (1991).
2. M.G.Crandall, H.Ishii and P.L.Lions, *Bull. AMS(new series)* **27**, 1-67(1992).
3. Qixiao Ye and Zhengyuan Li, *Introduction of reaction-diffusion equations*, (Sinic Sci. Press, Beijing,1994) (in Chinese).
4. H.Engler and S.M.Lenhart, *Proc. London Math. Soc.*, **63**, 212-240(1991).
5. S.M.Lenhart, *J. Math. Anal. Appl.* **131**, 180-193(1988).
6. M.G. Crandall and H. Ishii, *Diff. Int. Eq.* **3**, 1001-1014(1990).
7. G. Lu and B.D. Sleeman, *Proc. Roy. Soc. Edin.* **123A**, 857-885(1993).
8. G. Lu and B.D. Sleeman, *Math. Met. Appl. Sci.* **17**, 1005-1016(1994).

INVARIANT MANIFOLDS FOR A RAPIDLY OSCILLATORY PERTURBATION PROBLEM

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1 Introduction

This work was motivated by the papers of Lorenz[6] and Kopell[5]. Lorenz[6] studied a homogeneous incompressible fluid. By truncating the governing partial differential equations, he obtained a 9-dimensional ordinary differential equation model. A further reduction to a 3-dimensional *quasi-geostrophic* model was then made formally. Kopell[5] developed a result on the continuation of normally hyperbolic invariant manifolds of problems involving singularities and applied the result to justify the reduction. Unfortunately, there is a gap in the proof of the abstract result[1]. In this paper, we give a similar version of the result and apply it to the Lorenz model.

The situation encountered here can be explained as follows. Roughly speaking, the problem with parameter ϵ considered is either singular at $\epsilon = 0$ for certain time scales or the manifold interested is not normally hyperbolic at $\epsilon = 0$ for other time scales. The standard persistence results of normally hyperbolic invariant manifold (see the works of Fenichel[2] and Hirsch, etc.[4]) do not apply directly. Following the idea of Kopell, we introduce an auxiliary system with two parameters, which reduces to the original one when the two parameters agree. Normally hyperbolic invariant manifold theory can be applied when one of the parameters is fixed away from the singular value and the other is set to the singular value. To continue the singular parameter to the fixed one, the method developed by Chicone and Liu[1] is employed. This method, combining the normally hyperbolic theory and a continuation argument, seems very useful in treating problems with singularities caused by the parameters.

The paper is organized as follows. In Section 2, we recall the definition and the results of normally hyperbolic invariant manifolds. Our results are stated in Section 3 followed by a proof. We then apply the results to the 9-dimensional atmospheric model in Section 4.

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discussions.

2 Normal hyperbolicity

Let us recall the definition of normal hyperbolicity[2]. Consider a smooth differential equation

$$\dot{x} = F(x), \qquad x \in \mathcal{R}^n$$

with flow ϕ^t that has an overflowing invariant manifold $\bar{M} = M \cup \partial M$. Let TM denote the tangent bundle of M and let N denote the bundle orthogonal to TM. Thus, $T_M \mathcal{R}^n = TM \oplus N$. Let $\Pi : T_M \mathcal{R}^n \to N$ be the orthogonal projection. For each $m \in M$ and $t \geq 0$, define two operators

$$A_t(m) := D\phi^{-t}|_{T_m M} : T_m M \to T_{\phi^{-t}(m)} M,$$

$$B_t(m) := \Pi_m D\phi^t(\phi^{-t}(m))|_{N_{\phi^{-t}(m)}M} : N_{\phi^{-t}(m)} M \to N_m M, \qquad (1)$$

and define the Lyapunov type numbers

$$\nu(m) := \limsup_{t \to \infty} \|B_t(m)\|^{1/t}, \quad \sigma(m) := \limsup_{t \to \infty} \frac{\ln \|A_t(m)\|}{-\ln \|B_t(m)\|}. \qquad (2)$$

The invariant manifold M is called k-*normally hyperbolic* if $\nu(m) < 1$ and $\sigma(m) < \frac{1}{k}$ for $m \in M$.

Theorem 2.1 *(Fenichel[2], Hirsch, etc.[4]) If M is a k-normally hyperbolic invariant manifold, then it is C^k and persists under C^1 perturbations.*

3 The main results

Consider the system

$$x' = f_1(x, y; \epsilon), \quad y' = \epsilon h_1(x, y; \epsilon) \qquad (3)$$

where $x \in \mathcal{R}^n$, $y \in M$ and M is a m-dimensional smooth compact manifold without boundary or a compact ball in $\mathcal{R}^m$; f_1, h_1 are C^r ($r \geq 2$) in x, y, and $\epsilon > 0$, except that f_1 and h_1 might be singular at $\epsilon = 0$.

We assume that, for $\epsilon \neq 0$, there exist functions $x = \bar{x}(y; \epsilon)$, with $\bar{x}_y$ bounded uniformly in ϵ such that $f_1(\bar{x}(y; \epsilon), y; \epsilon) = 0$. Making a change of variables

$$u = x - \bar{x}(y; \epsilon), \qquad (4)$$

the transformed system becomes,

$$u' = f(u, y; \epsilon) - \epsilon \bar{x}_y(y; \epsilon) h(u, y; \epsilon), \quad y' = \epsilon h(u, y; \epsilon), \qquad (5)$$

where $f(u, y; \epsilon) = f_1(u + \bar{x}(y; \epsilon), y; \epsilon)$, $h(u, y; \epsilon) = h_1(u + \bar{x}(y; \epsilon), y; \epsilon)$ and $f(0, y; \epsilon) = 0$ for $\epsilon \neq 0$.

Theorem 3.1 *Consider system (5). Assume that, for $y \in M$, $f(0, y; \epsilon) = 0$ for $\epsilon \neq 0$; and $D_x f(0, y; \epsilon) = diag\{A_1(y; \epsilon), \cdots, A_k(y; \epsilon)\}$, where*

$$A_i(y; \epsilon) := \begin{pmatrix} -\alpha_i(y; \epsilon) & \beta_i(y; \epsilon) \\ -\beta_i(y; \epsilon) & -\alpha_i(y; \epsilon) \end{pmatrix}, \text{ or } A_i(y; \epsilon) = -\alpha_i(y; \epsilon)$$

and $\alpha_i(y; \epsilon) \geq \alpha_0 > 0$ is smooth in ϵ and y, $\beta_i(y; \epsilon) > 0$ is smooth in y but might be singular at $\epsilon = 0$, $|\partial_y \alpha_i(y; \epsilon)| \leq C$, and $|\partial_y \beta_i(y; \epsilon)| \leq C$. Then, for $\epsilon \neq 0$ small, (5) has a normally hyperbolic invariant manifold $M(\epsilon)$ parametrized by M, and $M(\epsilon) \to M$ as $\epsilon \to 0$.

Proof. It is a corollary of Theorem 3.2 to follow. $\square$

Consider a slightly general system

$$x' = f(x, y; \epsilon) + R(x, y; \epsilon), \quad y' = H(x, y; \epsilon) \tag{6}$$

where f and H might be singular at $\epsilon = 0$. Following Kopell[5], we now introduce the auxiliary family of systems as

$$x' = f(x, y; \delta) + R(x, y; \epsilon), \quad y' = H(x, y; \delta). \tag{7}$$

Theorem 3.2 *Consider system (7). Assume that, there exist constants $\delta_0 > 0$, $\alpha_0 > 0$, $L > 0$ and $C_0 > 0$ such that*

$$f(0, y; \delta) = 0, \quad R(x, y; \epsilon) \leq L(|x|^2 + \epsilon), \quad \langle x, D_x f(x, y; \delta) x \rangle \leq -\alpha_0 |x|^2 \tag{8}$$

for $0 \leq \epsilon \leq \delta$, $0 < \delta < \delta_0$, $|x| \leq C_0$, and all $y \in M$. Moreover, assume, for some $k \geq 2$, $M(\delta, 0) = \{(x, y) : x = F^0(y) = 0\}$ is a k-normally hyperbolic and stable invariant manifold of the system (7) corresponding to $\epsilon = 0$, and there exist $0 < \lambda < 1$, $r > k$ and $T > 0$, such that for $\delta \in (0, \delta_0]$ and for $(0, y) \in M(\delta, 0)$ the operators in (1) satisfy

$$\|B_T^0(0, y)\| < \lambda, \quad \frac{\ln \|A_T^0(0, y)\|}{-\ln \|B_T^0(0, y)\|} < \frac{1}{r}. \tag{9}$$

Then, there exists $\delta_1 > 0$ such that for $\delta \in (0, \delta_1]$ and $0 \leq \epsilon \leq \delta$, the corresponding system (7) has a k-normally hyperbolic invariant manifold $M(\delta, \epsilon)$ parametrized by M, and $M(\delta, \epsilon) \to M$ as $\delta \to 0$. In particular, system (6) has a k-normally hyperbolic invariant manifold for $0 < \epsilon \leq \delta_1$ small.

Proof. We follow the idea of Kopell[5] and the procedure used by Chicone and Liu[1] to give a proof.

Fix $\delta > 0$ small and let $\mathcal{A}^\delta$ denote the set of $\epsilon' \in [0, \delta]$ such that, for $\epsilon \in [0, \epsilon']$, the corresponding equation (7) has a k-normally hyperbolic invariant manifold $M(\delta, \epsilon)$ given by the graph of a function $x = F^\epsilon(y)$. We will prove that $\mathcal{A}^\delta$ is nonempty, open and closed if δ is small enough. The fact that $\mathcal{A}^\delta$ is nonempty follows from the assumption that $M(\delta, 0)$ is a k-normally hyperbolic invariant manifold of the system (7) for $\epsilon = 0$. That $\mathcal{A}^\delta$ is open is a consequence of Theorem 2.1. Proposition 3.3 below claims that $\mathcal{A}^\delta$ is also closed. Assuming Proposition 3.3, the proof is then completed. $\square$

Proposition 3.3 *There exists $\delta_1 > 0$ such that the set $\mathcal{A}^\delta$ is closed in $[0, \delta]$ for $0 < \delta \leq \delta_1$.*

Let $\epsilon_* = \sup \mathcal{A}^\delta$. We must show that $\epsilon_* \in \mathcal{A}^\delta$. To accomplish this, first, we show that there is a C^1 invariant manifold $M(\delta, \epsilon_*)$ continued from $M(\delta, \epsilon)$ for $0 \leq \epsilon < \epsilon_*$. Secondly, we check directly that $M(\delta, \epsilon_*)$ is k-normally hyperbolic.

Suppose, for $0 \leq \epsilon < \epsilon_*$, the normally hyperbolic invariant manifold of the corresponding system (7) is given by $M(\delta, \epsilon) = \{(x, y) : x = F^\epsilon(y)\}$. The invariance of $M(\delta; \epsilon)$ is equivalent to

$$F_y^\epsilon(y) H(F^\epsilon(y), y; \delta) = f(F^\epsilon(y), y; \delta) + R(F^\epsilon(y), y; \epsilon). \tag{10}$$

For $\kappa > 0$ small, consider the cylinder $\mathcal{C}_\kappa := \{(x, y) : |x^2| \leq \kappa^2\}$.

Lemma 3.4 *Assume the conditions (8) in Theorem 3.2. If*

$$\kappa_\epsilon = \frac{\alpha_0}{2L} - \sqrt{\frac{\alpha_0^2}{4L^2} - \epsilon}$$

(which is of order of ϵ as $\epsilon \to 0$), then the invariant manifold $M(\delta, \epsilon) \subset \mathcal{C}_{\kappa_\epsilon}$.

Proof. This is because $\mathcal{C}_\kappa$ is positively invariant for $\kappa > \kappa_\epsilon$. $\square$

Let $(F^\epsilon(y^\epsilon(s, q)), y^\epsilon(s, q))$ be the solution of (7) with the initial condition $(F^\epsilon(q), q)$. The linear variational equation of (7) along this solution has the form

$$W'(s) = \begin{pmatrix} f_x + R_x & f_y + R_y \\ H_x & H_y \end{pmatrix} W(s) \tag{11}$$

where the functions f_x, f_y, H_x and H_y in the coefficient matrix are evaluated at $(F^\epsilon(y^\epsilon(s, q)), y^\epsilon(s, q); \delta)$, R_x and R_y are evaluated at $(F^\epsilon(y^\epsilon(s, q)), y^\epsilon(s, q); \epsilon)$.

Lemma 3.5 *The variational equation (11) has the following set of independent solutions:*

$$X_j^\epsilon(s, q) := \begin{pmatrix} F_y^\epsilon(y^\epsilon(s, q)) \, Y_j(s, q; \epsilon) \\ Y_j(s, q; \epsilon) \end{pmatrix} \tag{12}$$

where $Y(s, q; \epsilon) = (Y_1(s, q; \epsilon), Y_2(s, q; \epsilon), \cdots, Y_m(s, q; \epsilon))$ is the principal matrix solution of

$$Z'(s) = \left(H_x F_y^\epsilon(F^\epsilon(y^\epsilon(s, q))) + H_y\right) Z(s), \tag{13}$$

and the argument of H_x and H_y is $(F^\epsilon(y^\epsilon(s, q)), y^\epsilon(s, q); \delta)$.

Moreover this set of solutions spans the tangent space of $M(\delta, \epsilon)$ at each point of the solution $(F^\epsilon(y^\epsilon(s, q)), y^\epsilon(s, q))$.

Proof. One can check directly by differentiating (10) with respect to y. $\square$

Lemma 3.6 *There is a constant $C > 0$ such that for $\epsilon < \epsilon_*$,*

$$|F_y^\epsilon| \le C\epsilon, \quad |F_{yy}^\epsilon| \le C.$$

Hence, $\lim_{\epsilon \to \epsilon_*} F^\epsilon(y)$ exists in C^1 norm and the graph of the limiting function is an invariant manifold of system (7) for ϵ_*.

Proof. To prove it, we will use the graph transform to derive an operator for which the function $q \mapsto F_y^\epsilon(q)$ is a fixed point and then use the resulting fixed point equation to obtain the desired estimates.

Let $\Psi^\epsilon(s, q)$ denote the principal matrix solution of (11) at $s = 0$ and let us write $\Psi^\epsilon(s, q)$ in the block form with respect to the x and y coordinates

$$\Psi^\epsilon(s, q) := \begin{pmatrix} \Psi_1^\epsilon(s, q) & \Psi_2^\epsilon(s, q) \\ \Psi_3^\epsilon(s, q) & \Psi_4^\epsilon(s, q) \end{pmatrix}.$$

By Lemma 3.5, we have,

$$\begin{pmatrix} F_y^\epsilon(y^\epsilon(s, q))Y(s, q; \epsilon) \\ Y(s, q; \epsilon) \end{pmatrix} = \Psi^\epsilon(s, q) \begin{pmatrix} F_y^\epsilon(q) \\ I \end{pmatrix}$$

where I is the identity matrix of size l. From which,

$$F_y^\epsilon(y^\epsilon(s, q)) = \left(\Psi_1^\epsilon(s, q)F_y^\epsilon(q) + \Psi_2^\epsilon(s, q)\right) Y^{-1}(s, q; \epsilon). \tag{14}$$

Consider the space $\Gamma := \{\eta : M \to \mathcal{R}^{n \times n}, \eta \in C^1\}$ and define a graph transform $\Lambda^\epsilon : \Gamma \to \Gamma$ by

$$(\Lambda^\epsilon \eta)(p) = \left(\Psi_1^\epsilon(T, q^\epsilon)\eta(q^\epsilon) + \Psi_2^\epsilon(T, q^\epsilon)\right) Y^{-1}(T, q^\epsilon; \epsilon) \tag{15}$$

where T is as in Theorem 3.2, $0 \le \epsilon < \epsilon_*$ and q^ϵ is related to p by $p = y^\epsilon(T, q^\epsilon)$. By Lemma 3.4 and an application of Gronwall's inequality, there exists $C_1(T)$ such that, for $s \in [0, T]$,

$$|(F^\epsilon(y^\epsilon(s, q^\epsilon)), y^\epsilon(s, q^\epsilon)) - (0, y^0(s, q^0))| \le C_1(T)\epsilon,$$

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and

$$\|\Psi^\epsilon(s, q^\epsilon) - \Psi^0(s, q^0)\| \le C_1(T)\epsilon. \tag{16}$$

In view of the definition of $Y(s, q; \epsilon)$ in Lemma 3.5 and the fact that $\|Y(T, q; 0)\|$ and its inverse are bounded away from zero uniformly in q, there exist $K_0 > 0$ and $\rho > 0$ such that, as long as $|F_y^\epsilon| \le \rho$, $\|Y^{-1}(T, q^\epsilon; \epsilon)\| \le K_0$.

We will show that if $\delta > 0$ is small, then $|F_y^\epsilon| \le \rho$ for $0 \le \epsilon < \epsilon_*$. In doing so, we also obtain that $|F_y^\epsilon| \le C\epsilon$ for $0 \le \epsilon < \epsilon_*$. Let $\epsilon_0 = \sup\{\epsilon' : |F_y^\epsilon| \le \rho$ for $\epsilon \in [0, \epsilon']\}$. Suppose, on the contrary, that $\epsilon_0 < \epsilon_*$. The relation (14) says that the function F_y^ϵ is a fixed point of the operator Λ^ϵ; that is,

$$F_y^\epsilon(p) = \left(\Psi_1^\epsilon(T, q^\epsilon)F_y^\epsilon(q^\epsilon) + \Psi_2^\epsilon(T, q^\epsilon)\right) Y^{-1}(T, q^\epsilon; \epsilon). \tag{17}$$

By the estimate (16) and also noticing that

$$B_T^0(0, q^0) = \Psi_1^0(T, q^0), \quad A_T^0(0, q^0) = Y^{-1}(T, q^0; 0),$$

we obtain that

$$
\begin{aligned}
|F_y^\epsilon(p)| &\le |\left(\Psi_1^0(T, q^0)F_y^\epsilon(q^\epsilon) + \Psi_2^0(T, q^0)\right) Y^{-1}(T, q^0; 0)| + (1 + \rho)K_0 C_1(T)\epsilon \\
&\le \|B_T^0(0, p)\| \, \|A_T^0(0, p)\| \, |F_y^\epsilon(q^\epsilon)| + (1 + \rho)K_0 C_1(T)\epsilon \\
&\le \lambda |F_y^\epsilon| + (1 + \rho)K_0 C_1(T)\epsilon,
\end{aligned}
$$

which implies that $|F_y^\epsilon| \le \frac{(1+\rho)K_0 C_1(T)}{1-\lambda}\epsilon$. Taking the limit as $\epsilon \to \epsilon_0$, due to the continuity of normally hyperbolic invariant manifold in C^k norm, we have $|F_y^{\epsilon_0}| \le \frac{(1+\rho)K_0 C_1(T)}{1-\lambda}\epsilon_0$. If $\delta \le \delta_1 < \min\{\frac{(1-\lambda)\rho}{(1+\rho)K_0 C_1(T)}, \delta_0\}$, then, for $\epsilon \in [\epsilon_0, \epsilon_*)$ and close to ϵ_0, $|F_y^\epsilon| \le \rho$. This contradicts to the maximality of ϵ_0 and, in turn, we have $\epsilon_0 = \epsilon_*$. Since now $|F_y^\epsilon| \le \rho$ for $0 \le \epsilon < \epsilon_*$, the above argument also yields $|F_y^\epsilon| \le \frac{(1+\rho)K_0 C_1(T)}{1-\lambda}\epsilon$ for all $0 \le \epsilon < \epsilon_*$.

We now verify that $\|F_{yy}^\epsilon\| \le C(T)$ for some $C(T) > 0$. Differentiating (17) with respect to p, one obtains

$$|F_{yy}^\epsilon(p)| \le \|\Psi_1^\epsilon(T, q^\epsilon)\| \cdot |\frac{dq^\epsilon}{dp}| \cdot \|Y^{-1}(T, q^\epsilon; \epsilon)\| \cdot |F_{yy}^\epsilon(q^\epsilon)| + \mathcal{R}(T),$$

where $\mathcal{R}(T)$ is uniformly bounded in $0 \le \epsilon < \epsilon_*$. From $p = y^\epsilon(T, q^\epsilon)$, we have $|\frac{dq^\epsilon}{dp}| = \|Y^{-1}(T, q^\epsilon; \epsilon)\|$. Thus there exists $C_2(T) > 0$ such that

$$
\begin{aligned}
|F_{yy}^\epsilon(p)| &\le \left(\|\Psi_1^0(T, q^0)\| \cdot \|Y^{-1}(T, q^0; 0)\|^2 + C_3(T)\epsilon\right) |F_{yy}^\epsilon(q^\epsilon)| + \mathcal{R}(T) \\
&= \left(\|B_T^0(0, p)\| \cdot \|A_T^0(0, p)\|^2 + C_3(T)\epsilon\right) |F_{yy}^\epsilon(q^\epsilon)| + \mathcal{R}(T) \\
&\le (\lambda + C_3(T)\epsilon)|F_{yy}^\epsilon| + \mathcal{R}(T).
\end{aligned}
$$

If we require further that $\delta_1 < \frac{1-\lambda}{C_3(T)}$, then

$$|F_{yy}^\epsilon| \leq \frac{\mathcal{R}(T)}{1 - \lambda - C_3(T)\epsilon} \leq \frac{\mathcal{R}(T)}{1 - \lambda - C_3(T)\delta_1}.$$

The family of functions $\{F^\epsilon : 0 \leq \epsilon < \epsilon_*\}$ is then an equicontinuous family in C^1 norm. Hence there exists a C^1 limit F^{ϵ_*} by the Ascoli-Arzela Theorem and its graph $M(\delta, \epsilon_*)$ is an invariant manifold of the system (7) at ϵ_*. $\square$

Next, we show that $M(\delta, \epsilon_*)$ is k-normally hyperbolic.

We take N to be the orthogonal complement of $TM(\delta, 0)$ in $T_{M(\delta,0)}\mathcal{R}^{n+l}$. Let $\nu(M(\delta, \epsilon_*); m)$ and $\sigma(M(\delta, \epsilon_*); m)$ denote the Lyapunov type numbers defined as in (2) for $m \in M(\delta, \epsilon_0)$.

Lemma 3.7 *There exists δ_1 such that if $\delta \leq \delta_1$ (hence $\epsilon_* \leq \delta_1$), then, for $m \in M(\delta, \epsilon_*)$, one has $\nu(M(\delta, \epsilon_*); m) < 1$ and $\sigma(M(\delta, \epsilon_*); m) < \frac{1}{k}$.*

Proof. Let $m^{\epsilon_*} = (F^{\epsilon_*}(p), p)$ be any point on $M(\delta, \epsilon_*)$ and let $m^0 = (0, p)$ be the corresponding point on $M(\delta, 0)$ for $\epsilon = 0$. Since $|F_y^{\epsilon_*}| \leq C\epsilon_*$, we have $\|\Pi_{m^{\epsilon_*}} - \pi_{m^{\epsilon_*}}\| \leq C\epsilon_*$ where $\pi_{m^{\epsilon_*}}$ is the orthogonal projection to $N_{m^{\epsilon_*}} M(\delta, \epsilon_*)$.

If T is as in Theorem 3.2, then, from Lemma 3.4 and using the Gronwall's inequality,

$$\|\Psi^{\epsilon_*}(T, q^{\epsilon_*})\| \leq (1 + C_1(T)\epsilon_* T)\|\Psi^0(T, q^0)\|$$

for some $C_1(T) > 0$, and hence, there exists $C_2(T) > 0$ such that

$$\begin{aligned}
\|B_T^{\epsilon_*}(m^{\epsilon_*})\| &\leq \|(\Pi_{m^{\epsilon_*}} - \pi_{m^{\epsilon_*}})\Psi^{\epsilon_*}(T, \phi^{\epsilon_*}(-T, m^{\epsilon_*})\| \\
&\quad + \|\pi_{m^{\epsilon_*}}\Psi^{\epsilon_*}(T, \phi^{\epsilon_*}(-T, m^{\epsilon_*})\| \\
&\leq C_2(T)\epsilon_* + \|B_T^0(m^0)\|.
\end{aligned} \tag{18}$$

Similarly, there exists $C_3(T) > 0$ so that $\|A_T^{\epsilon_*}(m^{\epsilon_*})\| \leq C(T)\epsilon_* + \|A_T^0(m^0)\|$.

For s large, write $s = dT + r$ for some integer d and $0 \leq r < T$, and denote $m_j^{\epsilon_*} := (F^{\epsilon_*}(p_j), p_j) := \phi^{\epsilon_*}(-jT, m^{\epsilon_*})$ and $m_j^0 = (F^0(p_j), p_j) = (0, p_j)$ for $j = 0, 1, \cdots, d$. Then

$$B_s^{\epsilon_*}(m^{\epsilon_*}) = B_r^{\epsilon_*}(m_d^{\epsilon_*}) \cdot B_T^{\epsilon_*}(m_{d-1}^{\epsilon_*}) \cdot B_T^{\epsilon_*}(m_{d-2}^{\epsilon_*}) \cdots B_T^{\epsilon_*}(m^{\epsilon_*}),$$

and hence, $\|B_s^{\epsilon_*}(m^{\epsilon_*})\| \leq \|B_r^{\epsilon_*}(m_d^{\epsilon_*})\| \Pi_{j=0}^{d-1} \|B_T^{\epsilon_*}(m_j^{\epsilon_*})\|$.

Similarly, $\|A_s^{\epsilon_*}(m^{\epsilon_*})\| \leq \|A_r^{\epsilon_*}(m_d^{\epsilon_*})\| \Pi_{j=0}^{d-1} \|A_T^{\epsilon_*}(m_j^{\epsilon_*})\|$.

Therefore, from (18), if δ_1, and hence ϵ_*, are small,

$$\begin{aligned}
\nu(M(\delta, \epsilon_*); m^{\epsilon_*}) &= \limsup_{s \to \infty} \|B_s^{\epsilon_*}(m^{\epsilon_*})\|^{1/s} \\
&\leq \limsup_{d \to \infty} \left(\|B_r^{\epsilon_*}(m_d^{\epsilon_*})\| \Pi_{j=0}^{d-1} \|B_T^{\epsilon_*}(m_j^{\epsilon_*})\|\right)^{\frac{1}{dT+r}} \\
&\leq (C_2(T)\epsilon_* + \lambda)^{\frac{1}{T}} < 1.
\end{aligned}$$

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To estimate $\sigma(M(\delta, \epsilon_*); m^{\epsilon_*})$, let us note that for each $j = 0, 1, \cdots, d-1$,

$$\frac{\ln\|A_T^\epsilon(m_j^{\epsilon_*})\|}{-\ln\|B_T^\epsilon(m_j^{\epsilon_*})\|} \leq \frac{\ln(C_3(T)\epsilon_* + \|A_T^0(m_j^0)\|)}{-\ln(C_2(T)\epsilon_* + \|B_T^0(m_j^0)\|)} < \frac{1}{r}. \tag{19}$$

Thus,

$$\begin{aligned}
\sigma(M(\delta, \epsilon_*); m^{\epsilon_*}) &= \limsup_{s \to \infty} \frac{\ln\|A_s^\epsilon(m^{\epsilon_*})\|}{-\ln\|B_s^\epsilon(m^{\epsilon_*})\|} \\
&\leq \limsup_{d \to \infty} \frac{\sum_{j=0}^{d-1}\ln\|A_T^{\epsilon_*}(m_j^{\epsilon_*})\| + \ln\|A_r^\epsilon(m_d^{\epsilon_*})\|}{-\sum_{j=0}^{d-1}\ln\|B_T^\epsilon(m_j^{\epsilon_*})\| - \ln\|B_r^\epsilon(m_d^{\epsilon_*})\|} \\
&= \limsup_{d \to \infty} \frac{\sum_{j=0}^{d-1}\ln\|A_T^{\epsilon_*}(m_j^{\epsilon_*})\|}{-\sum_{j=0}^{d-1}\ln\|B_T^\epsilon(m_j^{\epsilon_*})\|} \leq \frac{1}{r},
\end{aligned}$$

where the last step uses the estimates (19). $\square$

Corollary 3.8 *The invariant manifold $M(\delta, \epsilon_*)$ is k-normally hyperbolic and C^k. Hence $\epsilon_* \in \mathcal{A}^\delta$.*

4   A quasi-geostrophic invariant manifold for an atmospheric model

In this last section, we apply our result to justify the reduction of a 9-dimensional atmospheric model made by Lorenz[6].

Lorenz[6] considered a homogeneous incompressible fluid of average depth H, moving over a surface of variable topographic height h, under the force F, as well as some physical conditions assumed. The 9-dimensional atmospheric model, derived by truncating the governing partial differential equations of the fluid, is

$$\begin{aligned}
x_i' &= b_i x_j x_k - \frac{c}{a_i}(a_i - a_k)x_j y_k + \frac{c}{a_i}(a_i - a_j)x_k y_j \\
&\quad - \frac{2c^2}{a_i}y_j y_k - \nu_0 a_i x_i + y_i - z_i, \\
y_i' &= \frac{a_k b_k}{a_i}x_j y_k - \frac{a_j b_j}{a_i}x_k y_j + \frac{c}{a_i}(a_k - a_j)y_j y_k - x_i - \nu_0 a_i y_i, \\
z_i' &= -b_k x_j(z_k - h_k) - b_j(z_j - h_j)x_k + cy_j(z_k - h_k) - c(z_j - h_j)y_k \\
&\quad + g_0 a_i x_i - \kappa_0 a_i z_i - F_i, \tag{20}
\end{aligned}$$

and the other six equations are obtained from above by cyclic permutations (i, j, k) of the indices $(1, 2, 3)$. In the system, a_i, b_i and c are positive di-

mensionless constants; ν_0, κ_0, and g_0 are the kinematic viscosity, diffusivity coefficient, and the acceleration of gravity, respectively.

To further reduce the model to a 3-dimensional so called *quasi-geostrophic* one, we first rescale the parameters used by Kopell[5].

Let ϵ be the Rossby number and scale the space variables as follows

$$x_i = \epsilon^2 X_i, \ y_i = \epsilon Y_i, \ z_i = \epsilon Z_i.$$

The scaling of the others is made under the assumption, in this case, that ν_0, F_i are small and g_0 is large. So they are scaled as

$$\nu_0 = \epsilon N, \ F_i = \epsilon \bar{F}_i, \ G = \epsilon g_0.$$

Using these scaled variables, the system is transformed into

$$X_i' = \frac{1}{\epsilon}(Y_i - Z_i) - \frac{2c^2}{a_i} Y_j Y_k + \epsilon \mathcal{R}_i(X, Y, Z, \epsilon),$$
$$Y_i' = \epsilon \mathcal{S}_i(X, Y, Z, \epsilon),$$
$$Z_i' = -Ga_i X_i - \kappa_0 a_i Z_i - cY_j h_k + ch_j Y_k + \bar{F}_i + \epsilon \mathcal{Q}_i(X, Y, Z, \epsilon).$$

One observes that, to the lowest order of ϵ, $Y_i = Z_i$. Introducing the new variables $W_i = Y_i - Z_i$, the system becomes

$$X_i' = \frac{1}{\epsilon}W_i - \frac{2c^2}{a_i} Y_j Y_k + \epsilon R_i(X, W, Y, \epsilon),$$
$$W_i' = -Ga_i X_i - \kappa_0 a_i W_i + (-ch_k Y_j + ch_j Y_k) + \bar{F}_i + \epsilon S_i(X, W, Y, \epsilon),$$
$$Y_i' = \epsilon Q_i(X, W, Y, \epsilon). \tag{21}$$

To apply Theorem 3.1, we first make a change of variables as in (4) to transform (21) into

$$X_i' = \frac{1}{\epsilon}W_i + \epsilon \bar{R}_i(X, W, Y, \epsilon),$$
$$W_i' = -Ga_i X_i - \kappa_0 a_i W_i + \epsilon \bar{S}_i(X, W, Y, \epsilon),$$
$$Y_i' = \epsilon \bar{Q}_i(X, W, Y, \epsilon). \tag{22}$$

Secondly, we apply the linear change of variables

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = P \begin{pmatrix} X_i \\ W_i \end{pmatrix},$$

where

$$P = \begin{pmatrix} -\kappa_0 a_i \sqrt{\epsilon} + Ga_i & -\frac{2}{\sqrt{\epsilon}} + \frac{\kappa_0 a_i}{2} \\ \sqrt{4Ga_i - \kappa_0^2 a_i^2 \epsilon} \cdot \sqrt{\frac{4Ga_i - \kappa_0^2 a_i^2 \epsilon}{4\epsilon}} \end{pmatrix},$$

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which transfers the linear part to its Jordan Form. It is important to note that, although the transform P is singular at $\epsilon = 0$, the inverse P^{-1} is regular. Thus, the system becomes

$$u_i' = -\frac{\kappa_0 a_i}{2} u_i + \frac{\sqrt{4Ga_i - \kappa_0^2 a_i^2 \epsilon}}{4\epsilon} v_i + \sqrt{\epsilon} \hat{R}_i(u, v, Y, \epsilon),$$

$$v_i' = -\frac{\sqrt{4Ga_i - \kappa_0^2 a_i^2 \epsilon}}{4\epsilon} u_i - \frac{\kappa_0 a_i}{2} v_i + \sqrt{\epsilon} \hat{S}_i(u, v, Y, \epsilon),$$

$$Y_i' = \epsilon \hat{Q}_i(u, v, Y, \epsilon), \tag{23}$$

with $\hat{R}_i(u, v, Y, \epsilon)$ and $\hat{S}_i(u, v, Y, \epsilon)$ being regular with respect to ϵ. If we replace ϵ by μ^2 in (23), it will be in the form described in Theorem 3.1. Hence, a 3-dimensional invariant manifold parametrized by Y_i exists, which justifies the reduction of the 9-dimensional model to a 3-dimensional system. The flow on this invariant manifold is now a regular perturbation problem. Furthermore, the 3-dimensional invariant manifold is normally hyperbolic if it is overflowing (otherwise, it is locally attracting and persistent). Thus nearby solutions approaches the invariant manifold exponentially and the rapid oscillations are damped out.

References

1. C. Chicone, and W. Liu, Invariant tori for periodically perturbed oscillators. *SIAM J. Math. Anal.*, to appear.
2. N. Fenichel, Persistence and smoothness of invariant manifolds for flows. *Indiana U. Math. J.* **21**(1971), 193–226.
3. N. Fenichel, Geometric singular perturbation theory for ordinary differential equation. *J. Differential. Equations* **31**(1979), 53–98.
4. M. Hirsch, C. Pugh, and M. Shub, *Invariant Manifolds.* Lecture Notes in Math. **583**, Springer-Verlag, New York, 1977.
5. N. Kopell, Invariant manifolds and the initialization problem for some atmospheric equations. *Physica* **14D** (1985), 203–215.
6. E.N. Lorenz, Attractor sets and quasi-geostrophic equilibrium. *J. Atmos. Sci.* **37** (1980), 1685-1699.

GLOBAL STABILITY FOR A POPULATION MODEL WITH TIME DELAY

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In this paper , we give a sufficient condition that guarantees every positive solution of the populution model with time delay

$$N'(t) = r(t)N(t)\frac{1 - N(t - \tau)}{1 - \lambda N(t - \tau)}, t \geq 0$$

convergence to the equilibrium $N^* = 1$ as $t \to \infty$. The results of Joseph and Yu are improved.

1 Introduction

Consider the following population Model with delay (Hutchinson in [1])

$$N'(t) = rN(t)(1 - \frac{N(t - \tau)}{K}, t \geq 0 \qquad (1.1)$$

where $N(t)$ is the population of a single species at time t. r is the growth rate of the population, $K > 0$ is the carrying of the habitat and $\tau > 0$ is the delay. Many models improved from (1.1) were proposed by biomathematicians. One of the well-known model is

$$N'(t) = rN(t)\frac{1 - N(t - \tau)}{1 - \lambda N(t - \tau)}, t \geq 0. \qquad (1.2)$$

When the growth rate r in (1.2) depends on time t, we have

$$N'(t) = r(t)N(t)\frac{1 - N(t - \tau)}{1 - \lambda N(t - \tau)}, t \geq 0, \qquad (1.3)$$

where $r(t) \in C([0, +\infty], (0, +\infty)), \lambda \in (0, 1)$. In (1.2), when $\lambda = 0$, (1.2) becomes (1.1) with $K = 1$. The global stability of (1.1) or (1.2) is considered by Gopalsamy, Kulenovic and Ladas [2], Kuang, Zhang and Zhao [3], Yu[4], Joseph and Yu [6]. The initial value problem of (1.3) takes the form

$$N(\theta) = \phi(\theta), \theta \in [-\tau, 0], \qquad (1.5)$$

$$\phi \in C([-\tau, 0), (0, \frac{1}{\lambda}) \ with \ \phi(0) > 0. \qquad (1.6)$$

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We now state our main result.

Theorem 1.1 If

$$\int_{t-r}^{t} r(s)ds \le 1 - \lambda, t \ge 0, \tag{1.7}$$

$$\int_{0}^{+\infty} r(s)ds = \infty, \tag{1.8}$$

then every solution of initial value problem (1.3) with (1.4) tends to 1 as $t \to \infty$.

In this case, when $r(t) \equiv r > 0$, Th1.1 improves the results in [4].

2 Basic lemmas

Lemma 2.1 Assume that (1.4),(1.6),(1.7) hold, $\lambda \in (0,1)$, then the solution $N(t; 0, \phi)$ of IVP(1.3) with (1.5) exists on $[0, +\infty)$ and satisfies

$$0 < N(t; 0, \phi) < \frac{1}{\lambda}.$$

Lemma 2.2 Assume that (1.4),(1.6),(1.8) hold, let $N(t)$ be the solution of IVP(1.3) and (1.5), if $N(t)$ is eventually greater (reps. Less) than 1, then $\lim(N(t) = 1$.

Lemma 2.3 Assume that (1.4),(1.6),(1.7) hold and $\lambda \in (0,1)$.$N(t)$ is a solution of IVP(1.3) with (1.5) and is oscillatory about 1, then $N(t)$ is bounded above away from $1/\lambda$ and is bounded below a way from 0.

Lemma 2.4 Assume $\lambda \in (0,1)$ then the system of inequalities

$$u \le (1 - \lambda)\frac{1 - e^v}{1 - \lambda e^v}, \tag{2.1}$$

$$v \ge (1 - \alpha)\frac{1 - e^u}{1 - \lambda e^u}, \tag{2.2}$$

$$-\infty < v \le 0 \le u < ln\frac{1}{\lambda}, \tag{2.3}$$

has a unique solution $(u, v) = (0, 0)$

Proof from (2.2), we have

$$u \ge ln\frac{v - 1 + \lambda}{\lambda v - 1 + \lambda},$$

then

$$(1 - \lambda)\frac{1 - e^v}{1 - \lambda e^v} \ge ln\frac{v - 1 + \lambda}{\lambda v - 1 + \lambda}. \tag{2.4}$$

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If $v \neq 0$ then $v < 0$, set

$$f(v) = (1 - \lambda)\frac{1 - e^v}{1 - \lambda e^v} - ln\frac{v - 1 + \lambda}{\lambda v - 1 + \lambda}.$$

Clearly $f(0) = 0$, and

$$f'(v) = \frac{(1 - \lambda)^2(1 - \lambda e^v)^2 - (1 - \lambda)^2(v - 1 + \lambda)(\lambda v - 1 + \lambda)e^v}{(1 - \lambda e^v)^2(v - 1 + \lambda)(\lambda v - 1 + \lambda)}$$

Set $g(v) = (1 - \lambda e^v)^2 - (v - 1 + \lambda)(\lambda v - 1 + \lambda)e^v$, clearly $g(0) = 0$, and

$$g'(v) = -2\lambda e^v(1 - \lambda e^v) - (\lambda v - 1 + \lambda)e^v - \lambda(v - 1 + \lambda)e^v - (v - 1 + \lambda)(\lambda v - 1 + \lambda)e^v.$$

Set $h(v) = -2\lambda(1 - \lambda e^v) - (\lambda v - 1 + \lambda) - \lambda(v - 1 + \lambda) - (v - 1 + \lambda)(\lambda v - 1 + \lambda)$, clearly $h(0) = 0, h'(v) = 2\lambda^2 e^v - 2\lambda - 2\lambda v + 1 - \lambda^2; h'(0) = (a - 1)^2 > 0, h''(v) = 2\lambda^2 e^v - 2\lambda = 2\lambda(a\lambda^v - 1) < 0$, hence due to $v < 0$, we have $h'(v) > h'(0) > 0$, then $h(v) < h(0) = 0$, then $g'(v) < 0$, and then $g(v) > g(0) = 0$, thus $f'(v) > 0$, then $f(v) < f(0) = 0$. Hence

$$(1 - \lambda) = frac1 - e^v 1 - \lambda e^v < ln\frac{v - 1 + \lambda}{\lambda v - 1 + \lambda},$$

contradicting(2.4), thus $v = 0$. Similarly we can prove $u = 0$. The proof is complete.

3   Proof of Theorem 1.1

To complete the proof of Theorem 1.1, all we need is to show

Lemma 3.1 If (1.4),(1.6),(1.7) hold and $\lambda \in (0, 1)$, then every oscillatory solution of IVP(1.3) with (1.5) about 1 tends to 1 as $t \to \infty$.

Proof Let $N(t)$ be an oscillatory solution of IVP(1.3) with (1.5). By Lemma 2.3 $N(t)$ is bounded above away from $1/\lambda$ and is bounded below way from 0. Set

$$lnN(t) = x(t). \tag{3.1}$$

Then (1.3) becomes

$$x'(t) = r(t)\frac{1 - e^{x(t-\tau)}}{1 - \lambda e^{x(t-\tau)}}, t \geq 0. \tag{3.2}$$

Let lim sup $x(t) = u$, lim inf $x(t) = v$. Then we have

$$-\infty < v \leq 0 < u < ln(1/\lambda). \tag{3.3}$$

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Let t_n^* be an increasing infinite sequence of real numbers such that

$$x'(t_n^*) = 0 \ and \ \lim x(t_n^*) = u.$$

We may assume that t_n^* is the left local maximum point of $x(t)$, it is easy to show there exists $\xi_n \in [t_n^* - \tau, t_n^*)$ such that $x(\xi_n) = 0$ and $x(t) > 0$ for $t \in (\xi_n, t_n^*)$. In the other hand, for any $0 < \epsilon < ln(1/\lambda) - u$, by (3.3) there exists $T > 0$, such that $v_1 = v - \epsilon < x(t) < u + \epsilon = u_1$ for $t \geq T$. Then by (3.2) we have

$$x'(t) \leq r(t)\frac{1 - e^{v_1}}{1 - \lambda e^{v_1}}, t \geq T, \tag{3.4}$$

$$x'(t) \geq r(t)\frac{1 - e^{u_1}}{1 - \lambda e^{u_1}}, t \geq T, \tag{3.5}$$

Integrating (3.4) from ξ_n to t_n^*, and let $n \to \infty, \epsilon \to 0$, we have

$$u \leq (1 - \lambda)\frac{1 - e^v}{1 - \lambda e^v}. \tag{3.6}$$

Again set s_n^* be a sequence of real numbers such that $x'(s_n^*) = 0, x(s_n^*) < 0$ and

$$\lim_{n \to \infty} x(s_n^*) = v.$$

We may assume that s_n^* is the left local minimum point of $x(t)$, it is easy to see that there exists $\eta_n \in [s_n^* - \tau, s_n^*)$, such that $x(\eta_n) = 0$ and $x(t) < 0$ for $t \in (\eta_n, s_n^*)$. Integrating (3.5) from η_n to s_n^*, and let $n \to \infty, \epsilon \to 0$, we have

$$v \geq (1 - \lambda)\frac{1 - e^u}{1 - \lambda e^u}. \tag{3.7}$$

Combining (3.3),(3.6),(3.7) and from lemma 2.4, we have $u = v = 0$, thus

$$\lim_{t \to \infty} x(t) = 0, \ then \ \lim_{t \to \infty} N(t) = 1.$$

Remark 3.1 In [4], J.S. Yu proved that if

$$\int_{t-\tau}^t r(s)ds \leq \delta \leq \frac{3}{2}(1 - \lambda e^\delta) \ and \ \int_0^{+\infty} r(s)ds = \infty, \tag{3.8}$$

then every positive solution of (1.3) tends to 1 as $t \to \infty$. Our result improves condition (3.8). In fact, when $3\lambda e^\delta - 2\lambda > 1$, we have $(3/2)(1 - \lambda e^\delta) < 1 - \lambda$.

Remark 3.2 It is easy to see that when $1/3 < \lambda < 1$, we have $3\lambda^2 - 4\lambda + 1 < 0$, then $3\lambda e^{1-\lambda} > 3\lambda(2 - \lambda) > 1 + 2\lambda$, then $1 - \lambda > 3/2(1 - \lambda e^{1-\lambda})$. By this view, (1.7) improves (3.8).

Example 3.1 Consider following equation

$$N'(t) = \frac{1}{3}N(t)\frac{1 - N(t-1)}{1 - \frac{2}{3}N(t-1)}, t \geq 0, \tag{3.9}$$

$$\lambda = \frac{2}{3}, r \equiv \frac{1}{3}, \int_{t-\tau}^{t} r(s)ds = \frac{1}{3} = 1 - \lambda.$$

From theorem 1.1, every positive solution of (3.9) tends to 1 as $t \to \infty$. But (3.8) is not satisfied.

Acknowledgments

References

1. G. E Hutchinson, Circular caused systems in ecology, Ann. New York Acad. Sci. **50**, 221-246 (1948)
2. K. Gapalsamy, M.R.S. Kulenovic and G. Ladas, Time Delays in a "Food-Limited" populations model, Applicable Analysis **31**, 225 (1988)
3. Y.Kuang, B.G. Zhang and T. Zhao, Qualitative analysis of a nonlinear delay differential equation; Toho Ku Math.J. **43**, 509 (1991)
4. J.S. Yu, Global attractivity of the zero solution of a class of functional differential equations and its applications, Sci. In china (Seri. A) **39**, 225-237 (1996)
5. J.K. Hale, Theory of Functional Differential Equations, (New York: Springer-Verlag, 1977).
6. W.-H. Joseph and J.S. Yu, Global attractivity for a population model with time delay, Proc. of the Amer. Math. Soc. **123**, 2687 -2694 (1995)

DISTRIBUTION OF THE ZEROES OF THE SOLUTIONS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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In this paper, we first prove two comparison theorems of Sturm-type by which we obtain an estimate for the distribution of the zeros of the equation

$$x(t) + px(t - \tau) - g(t)x'(x - t) + a(t)x(t) + b(t)x(x - t) = 0 \qquad (1)$$

A sufficient condition for the oscillation of the solutions of equation (1) is also obtained.

1 Introduction

The oscillation theory for the solutions of neutral differential equations with delay has been developed intensively. However, the most oscillation results are just qualitative ones. From these results we only knew that the solutions of a certain differential equation have zeroes sequence tending to infinity. But no information about distribution of the zeroes of the solutions is gained. The study of the distribution of the zeroes is belong to quantitative analysis and hence more difficult. It is well known that Sturm comparison theorem is for the zeroes distribution of ordinary differential equations [1]. Li Bingtuan [2] studied the distribution of the zeroes of the solution s of first order differential equations with delay of the zeroes of the solutions of the form

$$d/dt[x(t) + p(t)(t - \tau)] + Q(t)x(t - \sigma) = 0$$

In this paper, we will establish some theorems estimating the distance between the zeros of the solutions of second order neutral differential equation with delay as follows

$$1[x] = x''(t) + px''(t-\tau) - g(t)x'(t-\tau) + a(t)x(t) + b(t)x(t-\tau) = 0 \ t \in [t_1, t_2] \quad (1)$$

where $\tau \geq 0, p > 0$ are constant, $a(t), b(t) \in C^1[t_1, t_2]$, $g(t) \in C^1[t_1, t_2 + \tau]$

Consider the neutral differential inequality

$$L[y] = y''(t) + py''(t + \tau) + (g(t + \tau)y(t + \tau))' + A(t)y(t) \geq 0 \qquad (2)$$

where $\tau \geq 0, p > 0$ are constant, $A(t) \in C^1[t_1, t_2], g(t) \in C^1[t_1, t_2 + \tau]$

Definition 1 The function $x(t)$ is said to be a solution of the equation (1) if

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$x(t) \in C^2[t_1 - \tau, t_2]$ and $x(t)$ satisfies (1) for $t \in [t_1, t_2]$.

Definition 2 The function $y(t)$ is said to be a solution of the differential inequality (2) if $y(t) \in C^2[t_1 - \tau, t_2]$ and $y(t)$ satisfies (2) for $t \in [t_1, t_2]$.

We shall give a new Picone identity and prove two comparison theorem of Sturmian type. By using them, the results for distribution of the zeros of equation (1) are obtained.

A new identity of Piconian type

Theorem 1 Let differential operators l and L be defined by (1) and (2) respectively. We have the identity of Piconian type

$$
xL[y] + px(t-\tau)y'' - pxy''(t+\tau) + x(t-\tau)(g(t)y(t))' - x(g(t+\tau)y(t+\tau))'
$$
$$
+[a(t) - A(t)]xy + b(t)x(t-\tau)y - yl[x]
$$
$$
= d/dt\{y'[x + px(t-\tau)] - y[x' + px'(t-\tau) - g(t)x(t-\tau)]\}. \tag{3}
$$

Proof In fact, the left-hand side of (3) is

$$
x[y'' + py''(t+\tau) + (g(t+\tau)y(t+\tau))' + A(t)] + px(t-\tau)y'' - pxy''(t+\tau)
$$
$$
+x(t-\tau)(g(t)y(t))' - x(g(t+\tau)y(t+\tau))' + [a(t) - A(t)]xy + b(t)x(t-\tau)y
$$
$$
-y[x'' + px''(t-\tau) - g(t)x'(t-\tau) + a(t)x + b(t)x(t-\tau)]
$$
$$
= xy'' + px(t-\tau)y'' + x(t-\tau)(g(t)y)' - x''y - px''(t-\tau)y + g(t)x'(t-\tau)y
$$

The right-hand side of (3) is

$$
y''[x + px(t-\tau)] + y'[x' + px'(t-\tau)] - y'[x' + px'(t-\tau) - g(t)x(t-\tau)]
$$
$$
-y[x'' + px''(t-\tau) - g'(t)x(t-\tau) - g(t)x'(t-\tau)]
$$
$$
= xy'' + px(t-\tau)y'' - x''y - px''(t-\tau)y + g(t)x'(t-\tau)y + g(t)x(t-\tau)y'
$$
$$
+g'(t)x(t-\tau)y
$$

From the equality $g(t)x(t-\tau)y' + g'(t)x(t-\tau)y = x(t-\tau)(g(t)y)'$ we know that the left-hand side of (3) is equal to the right-hand side of (3) and therefor (3) is true.

Corollary 2 If the solution of the equation $l[x] = 0$ is positive in the interval $[t_1 - \tau, t_2]$, we have for any nonnegative function $y(t) \in C^2[t_1, t_2 + \tau]$

$$
\int_{t_1}^{t_2} xL[y]dt + \int_{t_1}^{t_1+\tau} px(t-\tau)y''dt
$$

$$
-\int_{t_2}^{t_2+\tau} px(t-\tau)y''dt + \int_{t_1}^{t_2} x(t-\tau)(g(t)y)'dt
$$

$$- \int_{t_2}^{t_2+\tau} x(t-\tau)(g(t)y)'dt + \int_{t_1}^{t_2} [a(t) - A(t)]xydt + \int_{t_1}^{t_2} b(t)x(t-\tau)ydt$$

$$\geq y'(t_2)[x(t_2) + px(t_2 - \tau)] - y'(t_1)[x(t_1) + px(t_1 - \tau)]$$

$$+ y(t_1)[x'(t_1) + p'x(t_1 - \tau) - g(t_1)x(t_1 - \tau)]$$

$$- y(t_2)[x'(t_2) + px'(t_2 - \tau) - g(t_2)x(t_2 - \tau)]. \tag{4}$$

Proof It is immediately seen that

$$\int_{t_1}^{t_2} px(t-\tau)y''dt - \int_{t_1}^{t_2} pxy''(t+\tau)dt$$

$$= \int_{t_1}^{t_2} px(t-\tau)y''dt - \int_{t_1+\tau}^{t_2+\tau} px(t-\tau)y''dt$$

$$= (\int_{t_1}^{t_2} px(t-\tau)y''dt + \int_{t_2}^{t_1+\tau} px(t-\tau)y''dt)$$

$$- (\int_{t_2}^{t_1+\tau} px(t-\tau)y''dt + \int_{t_1+\tau}^{t_2+\tau} px(t-\tau)y''dt)$$

$$= \int_{t_1}^{t_1+\tau} px(t-\tau)y''dt - \int_{t_2}^{t_1+\tau} px(t-\tau)y''dt \tag{5}$$

Similar to (5) we have

$$\int_{t_1}^{t_2} x(t-\tau)(g(t)y)'dt - \int_{t_1}^{t_2} x(g(t+\tau)y(t+\tau))'dt$$

$$= \int_{t_1}^{t_1+\tau} x(t-\tau)(g(t)y)'dt - \int_{t_2}^{t_1+\tau} x(t-\tau)(g(t)y)'dt \tag{6}$$

Integrate both sides of inequality (3) with respect to t over the interval [t1, t2] and take into account that (5), (6) and $yl[x] = 0$ we know that (4) is true. This completes the proof of the Corollary 2.

2 Comparison Theorem

We shall say that condition (p) are met if the following conditions hold:
$(p_1)y(t_1) = y(t_2) = 0, y(t) > 0$, for $t \in (t_1, t_2)$
  $y(t) \leq 0$ for $t \in [t_2, t_2 + \tau]$ $(p_2)y'' \geq 0$ for $t \in [t_1, t_1 + \tau]$
  $y'' \leq 0$ for $t \in [t_2, t_2 + \tau]$ $(p_3)(g(t)y)' \geq 0$ for $t \in [t_1, t_1 + \tau]$
  $(g(t)y)' \leq 0$ for $t \in [t_2, t_2 + \tau]$
Theorem 3 If there exists a solution of differential inequality $L[y] \geq 0$ which meets condition (p) and

(1) $a(t) \geq A(t)$ for $t \in [t_1, t_2]$

(2) $b(t) \geq 0$ for $t \in [t_1, t_2]$ and at least one of (1) (2) is fulfilled strictly at least at one point of the respective interval. Then equation $l[x] = 0$ has no solution $x(t)$ which is positive in the interval $[t_1 - \tau, t_2]$.

Theorem 4 Suppose that the conditions of Theorem 3 hold. Then the equation $l[x] = o$ has no solution $x(t)$ which is negative in the interval $[t_1 - \tau, t_2]$.

3 Distribution of the zeroes and quantitative conditions for oscillation

Theorem 5 Suppose that there exists a solution of differential inequality $L[y] \geq 0$ which meets condition (P) and

(1) $a(t) \geq A(t)$, for $t \in [t_1, t_2]$

(2) $b(t) \geq 0$, for $t \in [t_1, t_2]$

(3) At least one of inequalities (1), (2) is fulfilled strictly at least at one point $t_0 \in [t_1, t_2]$. Then each solution of equation (1) has at least one zero in the interval $[t_1 - \tau, t_2]$. **Theorem 6** Suppose that there exist two increase sequence $\{t'_n\}$ and $\{t"_n\}$ such that (1) $t'_n \to \infty, t"_n \to \infty$ as $n \to \infty$ (2) $t'_n - \tau < t"_n$, for $n = 1, 2,$

Furthermore suppose that there exist a solution y_n of differential inequality $L[y] \geq 0$ which satisfies the conditions of Theorem 5 in the interval $[t'_n - \tau, t"_n + \tau]$, for $n = 1, 2,$then all solutions of equation (1) oscillate and they have at least one zero in the interval $[t'_n - \tau, t'_n]$ for $n = 1, 2,$

Remark In Theorem 5 let $p = 0, g(t) \equiv 0, b(t) \equiv 0$, we get well known Sturm Theorem about oscillation for second order ordinary differential equations.

References

1. C Sturm, J. Math. Pures. Appl **1**, 106-186 (1836)
2. Li Bingtuan, Acta, Math. Appl. Sinica **13**, 467-472 (1990)
3. Lin Shizhong, Acta, Math. Appl. Sinica **17**, 3 (458-461) 1994

MONOTONE SOLUTIONS OF SECOND ORDER QUASILINEAR DIFFERENTIAL EQUATIONS

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A necessary and sufficient condition is obtained for existence of monotone solutions
of a quasilinear differential equation. Relations between this equation and an
advanced type nonlinear differential equation are also discussed.

1 Introduction

This paper is concerned with a class of quasilinear differential equation of the
form

$$[r(t)(y'(t))^\alpha]' + q(t)(y(t))^\alpha = 0, \quad t \geq t_0, \tag{1}$$

where α is positive number, $q : [t_0, \infty) \to [0, \infty)$ is a continuous function
such that $q(t) \not\equiv 0$, and $r : [t_0, \infty) \to (0, \infty)$ is a continuous function. The
oscillatory behavior of (1) has been studied by several authors, the reader is
referred to the papers [1], [4] for oscillation results and [1], [3] for nonoscillation
results. A more general equation than (1) has been investigated by Li [2], Wong
and Agarwal [4]. In these papers, some sufficient conditions are obtained for
oscillation of all solutions as well as for existence of positive monotone solutions.

Recently, Li [3], Kusano and Yoshida [1] considered (1) under the condition

$$\int_{t_0}^\infty \frac{ds}{r(s)^{1/\alpha}} = \infty, \tag{2}$$

and obtained some results for nonoscillation of all solutions of (1). Here we
are interested in the existence of a **S-monotone** solution which satisfies

$$y(t) > 0, r(t)(y'(t))^\alpha \geq 0 \quad \text{and} \quad (r(t)(y'(t))^\alpha)' \leq 0.$$

without requiring the condition (2). Similar questions related to equation of
the form

$$[r(t)(y'(t))^\alpha]' + q(t)(y(t+\tau))^\alpha = 0, \quad t \geq t_0, \tag{3}$$

are also considered in Section 2. These studies show that there are some subtle
relations and differences between these two equations, as we will see below.

We base our investigation on a Riccati-type transformation. Monotone
methods are then used to derive a necessary and sufficient condition for the
existence of a S-monotone solution of (1). As applications, an existence cri-
terion and a comparison theorems are derived. Finally, we will consider the
relations of the two equations.

2 Main results

Definition A n-times differentiable function $y(t)$ defined on $[t_0, \infty)$ is called
a **S-monotone** solution if it satisfies

$$y(t) > 0, (-1)^i y^{(i)}(t) \leq 0 \quad \text{for} \quad 1 \leq i \leq n, \quad t \geq t_0.$$

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A slightly more general type of function can be introduced by requiring

$$y(t) > 0, r(t)(y'(t))^\alpha \ge 0 \quad \text{and} \quad (r(t)(y'(t))^\alpha)' \le 0.$$

Let $y(t)$ be a S-monotone solution of (1), and let the function $w(t)$ be defined by

$$w(t) = r(t)(y't))^\alpha / y(t)^\alpha, \quad t \ge t_0, \tag{4}$$

then $y\prime(t)/y(t) = (w(t)/r(t))^{1/\alpha}$, and

$$w'(t) + w(t)\alpha(w(t)/r(t))^{1/\alpha} + q(t) = 0, \quad t \ge t_0, \tag{5}$$

For the sake of convenience, we will write $F(x, y, z) = z(\frac{x}{y})^{1/z}$, which has been introduced in [3] and which has the following properties: $F(x, y, z) > 0$, $F_x(x, y, z) > 0$ and $F_y(x, y, z) < 0$ for $x, y, z > 0$. Then we can also write (5) in the simpler form

$$w'(t) + w(t)F(w(t), r(t), \alpha) + q(t) = 0, \quad t \ge t_0. \tag{6}$$

We have thus shown that if $y(t)$ is a S-monotone solution of (1), then $y(t)$ will satisfy (6) and hence the following inequality

$$w'(t) + w(t)F(w(t), r(t), \alpha) + q(t) \le 0, \quad t \ge t_0. \tag{7}$$

Note that $w(t)$ is nonnegative. Therefore, if we now integrate the inequality (7) from t to ∞, we will obtain

$$w(t) \ge \int_t^\infty w(s)F(w(s), r(s), \alpha)ds + \int_t^\infty q(s)ds, \quad t \ge t_0. \tag{8}$$

Theorem 1 Equation (1) has a S-monotone solution $y(t)$ for $t \ge t_0$ if and only if there is a nonnegative and continuous function $w(t)$ for $t \ge t_0$ which satisfies the integral inequality (8).

We remark that the proof is similar to that of Theorem 1 [3] and we omit it here.

As a direct application, we deduce a comparison theorem for the existence of a S-monotone solution of (1). Consider, together with (1), the following equation

$$[R(t)(y'(t))^\alpha]' + Q(t)(y(t))^\alpha = 0, \quad t \ge t_0, \tag{9}$$

where $R(t)$ and $Q(t)$ satisfy conditions similar to those imposed on $r(t)$ and $q(t)$. The following is now clear from Theorem 1.

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Theorem 2 In addition to the conditions imposed on the equations (1) and (9), suppose further that $r(t) \leq R(t)$ for $t \geq t_0$ and

$$\int_t^\infty Q(s)ds \leq \int_t^\infty q(s)ds, \quad t \geq t_0. \tag{10}$$

If equation (1) has a S-monotone solution, then so does equation (9).

As another application, we derive an explicit existence criterion based on Theorem 1.

Theorem 3 Suppose that $\int_{t_0}^\infty q(s)ds < \infty$ and let $\phi(t) = 2\int_t^\infty q(s)ds < \infty$, $t \geq t_0$. Suppose further that

$$\int_{t_0}^\infty \left(\frac{\phi(s)}{r(s)}\right)^{1/\alpha} ds \leq \frac{1}{2\alpha}. \tag{11}$$

Then equation (1) has a nonnegative solution.

Next, we will consider the following class of advanced type differential equation of the form

$$[r(t)(y'(t))^\alpha]' + q(t)(y(t+\tau))^\alpha = 0, \quad t \geq t_0, \tag{12}$$

where $\tau \geq 0$. By means of the same Riccati transformation (4), we may proceed in a similar manner as in Theorem 1 and obtain the following extension of Theorem 1.

Theorem 4 Equation (12) has a S-monotone solution $y(t)$ if and only if there is a nonnegative and continuous function $w(t)$ which satisfies the following inequality

$$w(t) \geq \int_t^\infty w(s)F(w(s), r(s), \alpha)ds + \int_t^\infty q(s)[\exp(\int_s^{s+\tau} (\frac{w(v)}{r(v)})^{\frac{1}{\alpha}}dv)]^\alpha ds,$$

for $t \geq t_0$.

Since

$$\int_t^\infty q(s)ds \leq \int_t^\infty q(s)[\exp(\int_s^{s+\tau} (\frac{w(v)}{r(v)})^{\frac{1}{\alpha}}dv)]^\alpha ds,$$

for $u > 0$. We immediately obtain from Theorems 1 and 4 the following corollary.

Corollary If equation (12) has a S-monotone solution, then so does equation (1).

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References

1. T. Kusano and N. Yoshida, J. Math. Anal. Appl. **189**, 115-127(1995).
2. W. T. Li, J. Math. Anal. Appl. **217**, 1-14(1998).
3. W. T. Li, J. Math. Anal. Appl. **221**, 326-337(1998).
4. P. J. Y. Wong and R. P. Agarwal, J. Math. Anal. Appl. **198**, 337-354(1996).

STABLE SET OF NONLINEAR NEUTRAL-TYPE LARGE-SCALE DYNAMICAL SYSTEMS

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This paper studies asymptotically stable set of the equilibrium 0 of nonlinear neutral-type large-scale dynamical systems. Sufficient conditions are given for stable set of the systems.

1 Introduction

In nonlinear systems, there usually exists more than one equilibrium. Multiple equilibria rule out global stability and stability (if present) is restricted to a finite region. How far can initial conditions be allowed to vary without disrupting the stability properties established in the immediate vicinity of equilibrium states? In the last a few years, many investigators, such as Hale *et al*[1], Siljak[3], Xu *et al*[4] and Zhang *et al*[6], respectively studied stable set of the equilibrium 0 of some nonlinear systems.

In this paper, we will discuss the stable set of the equilibrium 0 of nonlinear neutral-type large-scale dynamical systems and obtain some sufficient conditions for determining stable set.

2 Main Results

Throughout the paper, $R = (-\infty, +\infty)$, $R_+ = [0, +\infty)$, and $C^1(Y, Z)$ be the class of continuously differential mappings from a topological space Y to a topological space Z. We assume that $C_m^1 = C^1([-\alpha, 0], R^m)$ where α could be constant or $+\infty$ and m is some positive integer. For $w \in C_m^1$, we define

$$\|w\|_\alpha = \sup_{-\alpha \le \theta \le 0} |w(\theta)|, \|\dot{w}\|_\alpha = \sup_{-\alpha \le \theta \le 0} |\dot{w}(\theta)|, \qquad (*)$$

where $|\cdot|$ is some vector norm in R^m.

For matrices (or vectors) A and B, $A \le B$ means that each pair of corresponding elements of A and B satisfies this inequality "$\le$". For $x, y \in R^m$, $x < y$ means if $x_i < y_i$ for all $i = 1, \cdots, m$. For any $m \times m$-matrix $A = (a_{ij})_{m \times m}$, $\|A\|_\infty \triangleq \max_{1 \le i \le m} \sum_{i=1}^m |a_{ij}|$.

The symbol $\rho(A)$ denotes the spectral radius of a square matrix A. From Horn and Johnson[2], it is easy that if A is a nonnegative square matrix (*i.e.*, $A \ge 0$), then $\rho(A)$ is an eigenvalue of A and there is a nonnegative vector $x \ge 0$ with $x \ne 0$ such that $Ax = \rho(A)x$. The notation $W_\rho(A)$ is used to denote the characteristic space associated with $\rho(A)$ (the collection of all $x \in W_\rho(A)$ such that $Ax = \rho(A)x$).

Consider the following large-scale systems of nonlinear neutral type

$$\dot{x}_i(t) = A_i(t)x_i(t) + f_i(t, x(t), x(t - \tau(t)), \dot{x}(t - \tau(t))), \quad t \ge t_0, \qquad (1)$$

with the initial value

$$x_i(t_0 + s) = \phi_i(s), \quad \dot{x}_i(t_0 + s) = \dot{\phi}_i(s), \qquad -\alpha \le s \le 0, \qquad (2)$$

where $i = 1, \cdots, r$, $x = (x_1^T, \cdots, x_r^T)^T \in R^n$, $x_i \in R^{n_i}$, $n_1 + \cdots + n_r = n$; $A_i(t)$ are $n_i \times n_i$ continuous function matrices, $a_i = \sup_{t \ge t_0} \|A_i(t)\|_\infty < +\infty$, $\phi_i : [-\alpha, 0] \to R^{n_i}$ are n_i-dimensional continuously differentiable functions. $f = (f_1, \cdots, f_r)^T$, $f_i : R \times R^n \times R^n \times R^n \to R^{n_i}$ are sufficiently smooth with $f_i(t, 0, 0, 0) = 0$ so that for any initial function $\phi = [\phi_1^T, \cdots, \phi_r^T]^T$ at $t = t_0$, system (2.10) has unique continuous solution $x(t)$. $\tau(t) : [t_0, +\infty) \to R^+$ is continuous function with $0 \le \tau(t) \le \alpha$ and $\lim_{t \to +\infty}(t - \tau(t)) = +\infty$.

Definition The set $D \subset C_n^1$ $(D \backslash \{0\}$ is nonempty) is said to be an asymptotically stable set of the equilibrium 0 of (2.10) if for any $t_0 \ge 0$ and for any $\phi \in D$, there is a continuously nonnegative vector function $\nu : R^r \times R^r \to R^r \times R^r$ with $\nu(0) = 0$ such that

$$[x(t), \dot{x}(t)]^+ \le \nu([\phi, \dot{\phi}]_\alpha^+), \qquad t \ge t_0, \qquad (3)$$

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and

$$\lim_{t \to +\infty} [x(t), \dot{x}(t)]^+ = 0, \tag{4}$$

where

$$[x(t), \dot{x}(t)]^+ = [|x_1(t)|, \cdots, |x_r(t)|, |\dot{x}_1(t)|, \cdots, |\dot{x}_r(t)|]^T$$

and

$$[\phi, \dot{\phi}]_\alpha^+ = [\|\phi_1\|_\alpha, \cdots, \|\phi_r\|_\alpha, \|\dot{\phi}_1\|_\alpha, \cdots, \|\dot{\phi}_r\|_\alpha]^T. \tag{5}$$

We make assumptions

(A_1) The transition matrices $Y_i(t, s)$ of the linear equations $x_i(t) = A_i(t)x_i(t)$ satisfy

$$|Y_i(t, s)| \le M_i e^{-\omega_i(t-s)} \quad \text{for} \quad t_0 \le s \le t$$

where $M_i \ge 1$ and $\omega_i > 0$ are constants.

(A_2) Functions f_i satisfy

$$|f_i(t, x(t), x(t - \tau(t)), \dot{x}(t - \tau(t)))| \le \sum_{j=1}^{r} a_{ij}([x(t), \dot{x}(t)]_\alpha^+) \|x_j(t)\|_\alpha$$

$$+ \sum_{j=1}^{r} b_{ij}([x(t), \dot{x}(t)]_\alpha^+) \|\dot{x}_j(t)\|_\alpha, \quad \text{for} \quad t \ge t_0, i = 1, \cdots, r,$$

where $a_{ij}, b_{ij} : R_+^{2r} \to R_+$ are monotonically nondecreasing continuous functions, $i, j = 1, \cdots, r$. and $[x(t), \dot{x}(t)]_\alpha^+$ is defined by (*) and (5).

For convenience, we set that $H(K) = [h_{ij}(K)]_{2r \times 2r}$ is a $2r \times 2r$-matrix, where

$$h_{ij}(K) = \frac{a_{ij}(MK)M_j}{\omega_i}, \quad \text{for} \quad i = 1, \cdots, r; \; j = 1, \cdots, r,$$

$$h_{ij}(K) = \frac{b_{i,j-r}(MK)}{\omega_i}, \quad \text{for} \quad i = 1, \cdots, r; \; j = r + 1, \cdots, 2r,$$

$$h_{ij}(K) = \begin{cases} a_{i-r}M_{i-r} + a_{i-r,j}(MK)M_j, & \text{for } i - r = j, \\ a_{i-r,j}(MK)M_j, & \text{for } i - r \ne j; \\ i = r + 1, \cdots, 2r; \; j = 1, \cdots, r, \end{cases}$$

$$h_{ij}(K) = b_{i-r,j-r}(MK), \quad \text{for} \quad i = r + 1, \cdots, 2r; j = r + 1, \cdots, 2r,$$

$M = \text{diag}(M_1, \cdots, M_r, 1, \cdots, 1)$ is $2r$ − dimensional diagonal matrix.

Theorem 1 *Assume that (A_1) and (A_2) hold, and the set $D_1 \backslash \{0\}$ is nonempty, where*

$$D_1 = \{y \in C_m^1 : [y, \dot{y}]_\alpha^+ < K \in R_+^{2r}, [y, \dot{y}]_\alpha^+ \in W_\rho(H(K)), \rho(H(K)) < 1\}, \tag{6}$$

then D_1 is an asymptotically stable set of the equilibrium 0 of (2.10).

Sketch of the Proof By the variation of parameter, we have, from (2.10), for $t \geq t_0$,

$$x_i(t) = Y_i(t, t_0)\phi_i(0) + \int_{t_0}^t Y_i(t, s)f_i(s, x(s), x(s - \tau(s)), \dot{x}(s - \tau(s)))ds. \quad (7)$$

In view of the assumptions (A_1) and (A_2), we obtain, from (7),

$$|x_i(t)| \leq M_i e^{-\omega_i(t - t_0)}|\phi_i(0)| + \int_{t_0}^t M_i e^{-\omega_i(t - s)}[\sum_{j=1}^r a_{ij}([x(s), \dot{x}(s)]_\alpha^+)$$

$$\times \|x_j(s)\|_\alpha + \sum_{j=1}^r b_{ij}([x(s), \dot{x}(s)]_\alpha^+)\|\dot{x}_j(s)\|_\alpha]ds, \quad (8)$$

and, from (2.10),

$$|\dot{x}_i(t)| \leq a_i|x_i(t)| + \sum_{j=1}^r a_{ij}([x(t), \dot{x}(t)]_\alpha^+)\|x_j(t)\|_\alpha$$

$$+ \sum_{j=1}^r b_{ij}([x(t), \dot{x}(t)]_\alpha^+)\|\dot{x}_j(t)\|_\alpha, \quad t \geq t_0. \quad (9)$$

First, by making a contradiction, we can prove that, for $\phi \in D_1$, there exist a positive vector K and a sufficient small positive constant ε such that

$$[x(t), \dot{x}(t)]^+ < M([\phi, \dot{\phi}]_\alpha^+ + \vec{\beta}\varepsilon) < MK, \quad \text{for } t \geq t_0, \quad (10)$$

where $\vec{\beta}$ is a constant vector.
Letting $\varepsilon \to 0$, we get

$$[x(t), \dot{x}(t)]^+ \leq M[\phi, \dot{\phi}]_\alpha^+ < MK, \quad t \geq t_0. \quad (11)$$

Secondly, we prove that

$$\lim_{t \to +\infty} [x(t), \dot{x}(t)]^+ = 0. \quad (12)$$

From (11), there is a nonnegative constant vector $\sigma = [\sigma_1, \cdots, \sigma_{2r}]^T$ such that

$$\lim_{t \to +\infty} \sup |x_i(t)| = M_i\sigma_i < M_iK_i, \quad \text{and} \quad \lim_{t \to +\infty} \sup |\dot{x}_i(t)| = \sigma_{r+i} < K_{r+i}, \quad (13)$$

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where $i = 1, \cdots, r$ and K_i is i-th component of vector K.

According to definition of limsup, $\lim_{t \to +\infty}(t - \tau(t)) = +\infty$ and

$$\int_0^{+\infty} e^{-\omega_i s} ds = 1/\omega,$$

we can obtain, by estimating $|x_i(t)|$ and $|\dot{x}_i(t)|$ and letting $t \to +\infty$, that

$$\sigma_i \le \frac{1}{\omega_i}[\sum_{j=1}^r a_{ij}(MK)M_j\sigma_j + \sum_{j=1}^r b_{ij}(MK)\sigma_{r+j}], \qquad (14)$$

and

$$\sigma_{r+i} \le a_i M_i \sigma_i + \sum_{j=1}^r a_{ij}(MK)M_j\sigma_j + \sum_{j=1}^r b_{ij}(MK)\sigma_{r+j}. \qquad (15)$$

Combining with (14) and (15), if $\sigma \ge 0$ and $\sigma \ne 0$, then, by Theorem 8.3.2 of Horn and Johnson[2],

$$\rho(H(K)) \ge 1.$$

This contradicts $\rho(H(K)) < 1$. Hence (12) holds and the proof is complete.

Ttheorem 2 *Assume that (A_1) and (A_2) hold, and the set $D_2 \backslash \{0\}$ is nonempty, where*

$$D_2 = \{y \in C_m^1 : [y, \dot{y}]_\alpha^+ < kE_{2r}, \|H(kE_{2r})\|_\infty < 1\}, \qquad (16)$$

where $E_{2r} = [1, \cdots, 1]^T$ is a $2r-$dimensional vector, then D_2 is an asymptotically stable set of the equilibrium 0 of (2.10).

Corollary *In addition to (A_1) and (A_2), if there is a positive constant vector K such that $\rho(H(K)) < 1$, then the equilibrium 0 of (2.10) is asymptotically stable.*

Acknowledgements

References

1. J. K. Hale and S. M. Verduyn Lunel, Introduction to Functional Differential Equations, Spring-Verlag, New York, 1993.
2. R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, London, 1990.

3. D. D. Siljak, Large-scale Dynamic Systeme, Stability and Structure, New York, Elsevier North-Holland, inc., 1978.
4. D. Y. Xu, *Proc. of 12th World Congress of IMACS*, **1**(1988), 213-215, Paris, France.
5. D. Y. Xu and A. Xu, *Chinese science Bulletin*, **44**(1999), 121-123.
6. Zhang Yi and S. P. Banks, *IMA J. Math. Cont. Inf.,* **9**(1992), 89-99.

SOME RECENT CONTRIBUTIONS TO IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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This paper considers general impulsive functional differential equations. Some recent results on existence and uniqueness of solutions are given. Criteria on stability and boundedness of solutions are also established through the use of Lyapunov functions.

1 Introduction

The theory of impulsive systems of differential equations has become an important area of investigation in recent years. One of the most important features of such systems is that they are capable of modeling both continuous evolutions and discrete events occurring in a physical system. Impulsive systems arise in many applications such as orbital transfer of satellites, vibration suppression of flexible structures, inspection and quality control. For a detailed discussion of the basic theory and concepts, we refer the reader to [7]. There is a relatively large body of literature on impulsive systems involving ordinary differential equations. However, the corresponding theory involving functional differential equations has not yet been fully developed.

There are a number of difficulties one must face in developing the corresponding theory of impulsive delay differential equations. For example, in the classical theory of delay differential equations, the fact that the continuity of a function $x(t)$ in $\Re^n$ implies the continuity of the functional x_t in C^n plays a key role in establishing the existence of solutions of delay differential equations [5]. However, if a function $x(t)$ is piecewise continuous, which is typical for solutions of impulsive differential equations, then the functional x_t need not be

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piecewise continuous. In fact it can be discontinuous everywhere. Thus even if $f(t, \psi)$ is continuous in its two variables, we cannot, in general, say anything about the composition function $f(t, x_t)$ when $x(t)$ is piecewise continuous. To avoid this difficulty, Krishna and Anokhin[6] recently considered a special case, namely, an equation of the form $x'(t) = f(t, x(t - h(t)))$ with impulses, and established some interesting existence and uniqueness results where the foregoing difficulty does not arise.

In this paper, we shall consider general impulsive functional differential equations and discuss some recent contributions to this area. They include local and global existence of solutions, boundedness and stability properties.

2 Preliminaries

For $a, b \in \Re$ with $a < b$ and for $S \subset \Re^n$, let us define the following class of functions.
$PC([a, b], S) = \{\psi : [a, b] \to S | \ \psi(t^+) = \psi(t), \ \forall t \in [a, b), \ \psi(t^-)$ exists in $S, \ \forall t \in (a, b]$ and
$\psi(t^-) = \psi(t)$ for all but at most a finite number of points $t \in (a, b]\}$, $PC([a, b), S) = \{\psi : [a, b) \to S| \ \psi(t^+) = \psi(t), \ \forall t \in [a, b), \ \psi(t^-)$ exists in $S, \ \forall t \in (a, b)$ and
$\psi(t^-) = \psi(t)$ for all but at most a finite number of points $t \in (a, b)\}$, and

$$PC([a, \infty), S) = \{\psi : [a, \infty) \to S \ \big| \ \forall c > a, \ \psi|_{[a,c]} \in PC([a, c], S).$$

Given $r > 0$, we equip the linear space $PC([-r, 0], \Re^n)$ with the norm $\| \cdot \|_r$ defined by $\|\psi\|_r = \sup_{-r \leq s \leq 0} \|\psi(s)\|$. If $x \in PC([t_0 - r, \infty), \Re^n)$, where $t_0 \in \Re_+$, then for each $t \geq t_0$, we define $x_t \in PC([-r, 0], \Re^n)$ by $x_t(s) = x(t+s)$ for $-r \leq s \leq 0$.

Let $J \subset \Re_+$ be an interval of the form $[a, b)$ where $0 \leq a < b \leq \infty$ and let $D \subset \Re^n$ be an open set. Then we consider the system of impulsive functional equations $x'(t) = f(t, x_t), \ t \neq \tau_k(x(t^-))$,
$\Delta x(t) = I(t, x_{t-}), \ t = \tau_k(x(t^-))$. The initial condition for system (2) is given by

$$x_{t_0} = \phi, \tag{1}$$

where $t_0 \in \Re_+$ and $\phi \in PC([-r, 0], \Re^n)$. The functions τ_k are assumed to satisfy $0 = \tau_0(x) < \tau_1(x) < \tau_2(x) < \ldots$ and $\lim_{k \to \infty} \tau_k(x) = \infty$ for each $x \in \Re^n$.

Definition 2.1 *A function $x \in PC([t_0 - r, t_0 + \alpha], D)$, where $\alpha > 0$, $D \subset \Re^n$ and $[t_0, t_0 + \alpha] \subset J$, is said to be a solution of (2) if*

() the set $T = \{t \in (t_0, t_0 + \alpha] \big| \ t = \tau_k(x(t^-))$ for some $k\}$ of impulse times is finite (possibly empty);

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() x is continuous at each $t \in (t_0, t_0 + \alpha]\backslash T$;

() the derivative of x exists and is continuous at all but at most a finite number of points t in $(t_0, t_0 + \alpha)$;

() the right-hand derivative of x exists and satisfies the delay differential equation (2) for all $t \in [t_0, t_0 + \alpha)\backslash T$;

() x satisfies the delay difference equation (2) for all $t \in T$.

If in addition x satisfies the initial condition (1), then it is said to be a solution *of (the initial value problem) (2)-(1) and we write $x = x(t, t_0, \phi)$.*

Definition 2.2 *A function $x \in PC([t_0 - r, t_0 + \beta), D)$, where $0 < \beta \le \infty$ and $[t_0, t_0 + \beta) \subset J$, is said to be a* solution *of (2) (solution of (2)-(1)) if for each $0 < \alpha < \beta$ the restriction of x to $[t_0 - r, t_0 + \alpha]$ is a solution of (2) (solution of (2)-(1)) and if $\beta < \infty$ then the derivative of x exists and is continuous at all but at most a finite number of points t in $(t_0, t_0 + \beta)$ and the set $T = \{t \in (t_0, t_0 + \beta)| \ t = \tau_k(x(t^-)) \ for \ some \ k\}$ is finite.*

Note that the points where a solution fails to have a continuous derivative will generally include but may not be limited to impulse times. We require that there be at most a finite number of such exceptional points on any finite interval of time. In our definition, we do not explicitly require that solutions have a right-hand derivative satisfying (2) at the impulse times in T although in practice they usually will.

Definition 2.3 *A functional $f : J \times PC([-r, 0], D) \rightarrow \Re^n$ is said to be* composite-PC *if for each $t_0 \in J$ and $0 < \alpha \le \infty$, where $[t_0, t_0 + \alpha) \subset J$, if $x \in PC([t_0 - r, t_0 + \alpha), D)$ then the composite function g defined by $g(t) = f(t, x_t)$ is an element of the function class $PC([t_0, t_0 + \alpha), \Re^n)$.*

Definition 2.4 *A functional $f : J \times PC([-r, 0], D) \rightarrow \Re^n$ is said to be* quasi-bounded *if for each $t_0 \in J$ and $\alpha > 0$, where $[t_0, t_0 + \alpha] \subset J$, and for each compact set $F \subset D$ there exists some $M > 0$ such that $\|f(t, \psi)\| \le M$ for all $(t, \psi) \in [t_0, t_0 + \alpha] \times PC([-r, 0], F)$.*

Definition 2.5 *A functional $f : J \times PC([-r, 0], D) \rightarrow \Re^n$ is said to be* locally Lipschitz *in ψ if for each $t_0 \in J$ and $\alpha > 0$, where $[t_0, t_0 + \alpha] \subset J$, and for each compact set $F \subset D$ there exists some $L > 0$ such that $\|f(t, \psi_1) - f(t, \psi_2)\| \le L\|\psi_1 - \psi_2\|_r$ for all $t \in [t_0, t_0 + \alpha]$ and $\psi_1, \psi_2 \in PC([-r, 0], F)$.*

If f is locally Lipschitz in ψ then clearly it is also continuous in ψ. If in addition f is composite-PC then it is also quasi-bounded since $\|f(t, \psi)\| \le L\|\psi\|_r + \|f(t, 0)\|$ for $t \in [t_0, t_0 + \alpha]$ where $\|\psi\|_r \le \sup\{\|z\| \mid z \in F\}$ and where $\|f(t, 0)\|$ is bounded above by some constant since $f(t, 0)$ is a piecewise continuous (and hence bounded) function of t.

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3 Existence and Uniqueness

One of the fundamental differences between continuous delay differential equations and impulsive delay differential equations follows from the observation that if $x \in C([t_0 - r, t_0 + \alpha], D)$ (i.e. x is a continuous function mapping $[t_0 - r, t_0 + \alpha]$ into D) then x_t is a continuous function of t (with respect to $\| \cdot \|_r$) for $t \in [t_0, t_0 + \alpha]$ while if $x \in PC([t_0 - r, t_0 + \alpha], D)$ then x_t may be discontinuous at some (or all) $t \in [t_0, t_0 + \alpha]$. If x_t is a continuous function of t and if $f(t, \psi)$ is assumed to be continuous in its two variables, then the composition of functions $f(t, x_t)$ is also continuous. If x happens to be a solution of (2) then this would imply that x is in fact continuously differentiable for $t \in (t_0, t_0 + \alpha]$. Since impulses cause discontinuities in a solution x and since x_t need not be even piecewise continuous as a function of t then even if we were to impose a condition on f that $f(t, \psi)$ be continuous in its two variables then we could not, in general, say anything about the composition $f(t, x_t)$. In many practical examples it is clear that $f(t, x_t)$ is, say, piecewise continuous. For these reasons our theorems will usually impose assumptions on f that require checking properties of the composition $f(t, x_t)$ for each choice of piecewise continuous function x rather than checking the smoothness of f itself.

The proof of the following lemma can be found in many texts on functional differential equations including [5] and [1].

Lemma 3.1 *Assume $x \in C([t_0 - r, t_0 + \alpha], D)$. Then x_t is a continuous function of t (with respect to $\| \cdot \|_r$) for $t \in [t_0, t_0 + \alpha]$.*

To see how a single discontinuity can destroy the conclusion of Lemma 3.1, consider the function

$$x(t) = \begin{cases} 0, & t \in [-1, 0], \\ 1, & t \in (0, 1], \end{cases} \tag{2}$$

where $t_0 = 0$, $r = 1$ and $\alpha = 1$. Suppose $t_1, t_2 \in [0, 1]$ and $\delta > 0$ with $0 < t_1 - t_2 < \delta$. Then for $s = -t_2 \in [-r, 0]$, $|x(t_1 + s) - x(t_2 + s)| = |x(t_1 - t_2) - x(0)| = 1$ which implies $\|x_{t_1} - x_{t_2}\|_r = 1$. So clearly x_t is discontinuous at each $t \in [0, 1]$.

Theorem 3.2 *Assume f is composite-PC, quasi-bounded and continuous in ψ and that $\tau_k \in C^1(D, \Re_+)$ for some $(t^*, x^*) \in J \times D$ and some k then there exists a $\delta > 0$, where $[t^*, t^* + \delta] \subset J$, such that*

$$\nabla \tau_k(x(t)) \cdot f(t, x_t) \neq 1, \tag{3}$$

for all $t \in (t^*, t^* + \delta]$ and for all functions $x \in PC([t^* - r, t^* + \delta], D)$ which are continuous on $(t^*, t^* + \delta]$ and satisfy $x(t^*) = x^*$ and $\|x(s) - x^*\| < \delta$ for $s \in [t^*, t^* + \delta]$. Then for each $(t_0, \phi) \in J \times PC([-r, 0], D)$ there exists a solution $x = x(t_0, \phi)$ of (2)-(1) on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$.

In the next theorem, we give conditions which will ensure that as a solution of (2) evolves it will intersect each impulse hypersurface at most once.

Theorem 3.3 *Assume f is composite-PC, quasi-bounded and continuous in ψ and that $\tau_k \in C^1(D, \Re_+)$ for $k = 1, 2, \ldots$ and the limit $\lim_{k \to \infty} \tau_k(x) = \infty$ is uniform in x. Furthermore, assume that*

$$\nabla \tau_k(\psi(0)) \cdot f(t, \psi) < 1, \tag{4}$$

for all $(t, \psi) \in J \times PC([-r, 0], D)$ and $k = 1, 2, \ldots$. Finally, assume that $\psi(0) + I(\tau_k(\psi(0)), \psi) \in D$ and

$$\tau_k(\psi(0) + I(\tau_k(\psi(0)), \psi)) \leq \tau_k(\psi(0)), \tag{5}$$

for all $\psi \in PC([-r, 0], D)$ for which $\psi(0^-) = \psi(0)$ and for all $k = 1, 2, \ldots$. Then, for every continuable solution x of (2) there exists a continuation y of x which is noncontinuous. Moreover, any solution x of (2) can intersect each impulse hypersurface (in the sense that $t = \tau_k(x(t^-))$) at most once.

In Theorem 3.3 we assumed that $\psi(0) + I(\tau_k(\psi(0)), \psi) \in D$ for all k and for all $\psi \in PC([-r, 0], D)$ for which $\psi(0^-) = \psi(0)$. This will of course be necessary if we want solutions to be defined and continuable beyond impulse times. It ensures that solutions remain in the domain of the functional f following the impulsive action. In most applications, $D = \Re^n$ and so this condition is trivially satisfied.

Theorem 3.4 *Assume that all of the conditions of Theorem 3.3 are satisfied and let x be any solution of (2). If x is defined on a closed interval of the form $[t_0 - r, t_0 + \alpha]$, where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$, then x is continuable. If x is defined on an interval of the form $[t_0 - r, t_0 + \beta)$, where $0 < \beta < \infty$ and $[t_0, t_0 + \beta] \subset J$, and if x is noncontinuable then for every compact set $F \subset D$, there exists a sequence of numbers $\{s_k\}_{k=1}^\infty$ with $t_0 < s_1 < s_2 < \ldots < s_k < \ldots < t_0 + \beta$ and $\lim_{k \to \infty} s_k = t_0 + \beta$ such that $x(s_k) \notin F$.*

The first part of Theorem 3.4 states that a maximal interval of existence of a solution of (2) is open on the right. The second part says that a noncontinuous solution x is either defined for all $t \in J$ or it is defined on a bounded proper sub-interval of J and in this latter case the solution either becomes unbounded as $t \to t_0 + \beta$ (i.e. $\limsup_{t \to t_0 + \beta} \|x(t)\| = \infty$) or it takes on values arbitrarily close to the boundary of D as $t \to t_0 + \beta$. In the special case where $J = \Re_+$ and $D = \Re^n$, Theorem 3.4 essentially says that bounded solutions are continuable to $t = \infty$. The next result gives existence of solutions on $[t_0 - \gamma, \infty)$.

Theorem 3.5 *Assume $J = \Re_+$, $D = \Re^n$, and the conditions of Theorem 3.3 are satisfied. Suppose that there exist functions $h_1, h_2 \in PC(\Re_+, \Re_+)$ such that $\|f(t, \psi)\| \leq h_1(t) + h_2(t)\|\psi\|_r$ for all $(t, \psi) \in \Re_+ \times PC([-r, 0], \Re^n)$. Then for*

each $(t_0, \phi) \in \Re_+ \times PC([-r, 0], \Re^n)$ there exists a (local) solution $x = x(t_0, \phi)$ of (2)-(1) and any such solution can be continued to $[t_0 - r, \infty)$.

As with ordinary differential equations, additional smoothness assumptions on f are required if we want to expect uniqueness of solutions. In the following result, we show that the addition of a local Lipschitz condition on f is sufficient to guarantee uniqueness of solutions of (2)-(1).

Theorem 3.6 *Assume f is composite-PC and locally Lipschitz in ψ. Then there exists at most one solution of (2)-(1) on $[t_0 - r, t_0 + \beta)$ where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$.*

4 Stability and Boundedness

In what follows, we assume that $f(t, 0) \equiv 0$, and $I(t, 0) \equiv 0$ so that (1) admits the zero solution. Furthermore, we assume that $\tau_k(x) \equiv \tau_k$ for all $k = 1, 2, 3 \ldots$

Definition 4.1 *The zero solution of (1) is said to be*

(S_2) *stable, if for any $\varepsilon > 0$ and $t_0 \in J$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\| \phi \|_r < \delta$ implies $|x(t)| < \varepsilon$ for $t > t_0$, where $x(t) = x(t, t_0, \phi)$ is any solution of (1);*

(S_2) *uniformly stable, if the δ in (S_1) is independent of t_0;*

(S_3) *asymptotically stable, if it is stable and there is a $\delta_0 = \delta(t_0) > 0$ such that $\| \phi \|_r < \delta_0$ implies that $\lim_{t \to \infty} x(t) = 0$;*

(S_4) *uniformly asymptotically stable, if it is uniformly stable and there is a $\delta_0 > 0$, such that for any $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$, $\| \phi \|_r < \delta_0, t_0 > 0$ imply $|x(t)| < \varepsilon$ for $t > t_0 + T$;*

(S_5) *unstable if (S_2) fails.*

Definition 4.2 *The solutions of (1) are said to be*

(i) *uniformly bounded (UB) if $B_1 > 0$, $B_2 > 0$ such that*

$$\forall \quad t_0 \in J \ \| \phi \|_r < B_1 \quad implies \quad \| x(t) \| < B_2, \quad t \geq t_0,$$

where $x(t) = x(t, t_0, \phi)$ is any solution of (1);

(ii) *uniformly ultimately bounded (UUB) with the bound B if for any $B_3 > 0$, there exists $T > 0$ such that for any $t_0 \in J$, $\| \phi \|_r < B_3$ implies*

$$\| x(t) \| < B, \quad t \geq t_0 + T.$$

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Let $V \in C[R_+ \times R^n, R_+]$ and $\phi \in PC([-\gamma, 0], R^n)$, we define

$$D^+V(t, \phi(0)) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, \phi(0) + hf(t,\phi)) - V(t, \phi(0)] \quad (6)$$

and

$$\Delta V(\tau_k, x) = V(\tau_k, x + I(\tau_k, x)) - V(\tau_k, x). \quad (7)$$

We define the following classes of functions for later use.

$K_0 = \{\phi \in C[J, R_+], \phi(s) > 0, s > 0\}.$

$K_1 = \{\phi \in K_0, \phi(s) > 0, s > 0\}.$

$K_2 = \{\phi \in K_0, \phi(s) < s, s > 0\}.$

$K_3 = \{\phi \in K_0, \phi(s) \text{ is nondecreasing }\}.$

$K = \{\phi \in K_0, \phi(s) \text{ is strictly increasing and } \phi(0) = 0\}.$

$\mathcal{V}_l = \{\mathcal{V} : \mathcal{J} \times \mathcal{R}^\backslash \to \mathcal{R}_+, \text{ continuous on } (-r, \tau_0) \times R^n \text{ and } [\tau_{k-1}, \tau_k) \times R^n$
and $\lim_{(t,y) \to (\tau_k^-, x)} V(t, y) = V(\tau_k^-, x) \text{ exists for } k = 1, 2, 3, \ldots\}.$

$$RK = \{\phi \in K, \lim_{s \to \infty} \phi(s) = \infty\}.$$

Let us first give a theorem which shows that impulses may make an unstable system asymptotically stable.

Theorem 4.3 *Assume that*

(i) $V \in \mathcal{V}_l$, $a, b, \in K$ *such that*

$$b(|x|) \leq V(t, x) \leq a(|x|), \quad (t, x) \in J \times s(\rho);$$

(ii) *there exists* $\lambda \in PC[J, R_+]$, $C \in K$, $p \in K_2$ *such that*

$$D^+V(t, \phi(0)) \leq \lambda(t)c(|\phi(0)|) \quad (8)$$

whenever $V(t, \phi(0)) \geq p(V(t+s, \phi(s)))$, $s \in [-\gamma, 0]$, *where* $t \neq \tau_k$, $\phi \in PC[-\gamma, 0]$, $(t, \phi(0)) \in J \times s(\rho);$

(iii) *there exists* $\psi_k \in K_3$, $\psi_k \circ b^{-1}(u) + u \leq p(u)$, $u \in (0, \rho)$, $k = 1, 2, \ldots$, *such that*

$$\Delta V(\tau_k, \phi(0)) \leq \psi_k(|\phi(0)|) \quad (9)$$

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(iv) $\sup\limits_{k \in N} \{\tau_k - \tau_{k-1}\} = \tau < \infty$ *and*

$$\sup_{t \geq 0} \int_t^{t+\tau} \lambda(s)ds < \inf_{q>0} \int_{p(q)}^q \frac{du}{c \circ b^{-1}(a)} < \infty. \qquad (10)$$

Then the trivial solution of (1) is uniformly asymptotically stable.

Our next result gives sufficient conditions under which stability properties of a system are preserved under impulsive perturbations.

Theorem 4.4 *Assume that*

(i) $V \in \mathcal{V}_I, \dashv, \lfloor \in \mathcal{K}$ *such that*

$$b(|x|) \leq V(t,x) \leq a(|x|), \quad (t,x) \in J \times s(\rho);$$

(ii) *there exists* $\lambda \in PC[J, R_+]$, $c \in K$, $p \in K$, *such that*

$$D^+ V(t, \phi(0)) \leq -\lambda(t)C(|\phi(0)|) \qquad (11)$$

whenever $p(V(t,\phi(0))) \geq V(t+s, \phi(s))$, $s \in [-\sigma, 0]$, $\sigma \in (0, \gamma]$, *where* $t \neq \tau_k$, $\phi \in PC[-\gamma, 0]$, $(t, \phi(0)) \in J \times s(\rho)$;

(iii) *there exists* $\psi_k \in K_3$, $\psi_k \circ b^{-1}(u) + u \leq p(u)$, $u \in (0, \rho)$, $k = 1, 2, \ldots$, *such that*

$$\Delta V(\tau_k, \phi(0)) \leq \psi_k(|\phi(0)|); \qquad (12)$$

(iv) $\inf\limits_{k \in N} \{\tau_k - \tau_{k-1}\} = \sigma + 2\tau$ *and*

$$\sup_{q>0} \int_q^{p(q)} \frac{du}{c \circ b^{-1}(u)} < \inf_{t \geq 0} \int_t^{t+\tau} \lambda(s)ds < \infty. \qquad (13)$$

Then the trivial solution of (1) is uniformly asymptotically stable.

The final result gives a criterion on boundedness of solutions.

Theorem 4.5 *Assume that*

(i) $V \in C[R_+ \times R^n, R]$, $V(t,x)$ *is locally Lipschitz in* x *and*

$$b(|x|) \leq V(t,x) \leq a(|x|),$$

where $a \in K$, $b \in RK$;

(ii) there exists $d_k \in K$ such that

$$\Delta V(\tau_k, \phi(0)) \leq d_k(|\phi(0)|), \quad k = 1, 2, \ldots; \qquad (14)$$

(iii) $\forall \alpha > 0$, $\sum_{k=1}^{\infty} \dfrac{\psi_k(\alpha)}{\alpha} < \infty$ and $\limsup_{k \to \infty} \dfrac{\tilde{\psi}_k(\alpha)}{\alpha} \leq M$, where $M > 0$,

$\psi_k = d_k \circ b^{-1}$ and $\tilde{\psi}_k = (I + \psi_k) \circ \cdots \circ (I + \psi_1);$

(iv) there exists $A, H > 0$, $c \in K$ and

$$D^+ V(t, \phi(0)) \leq A - c(|\phi(0)|), \quad t \geq t_0, \quad |x| \geq H, \qquad (15)$$

$$p(V(t, \phi(0)) > V(t + s, \phi(s)), \quad s \in [-\gamma, 0],$$

where $p \in C[R_+, R_+]$, $p(s) > Ms$ for $s > 0$.

Then the solutions of (1) are UB and UUB.

To conclude this section, we discuss some examples.

Example 4.1: Consider the nonlinear equation

$$x'(t) = -ax(t-1)[1 + x(t)] \qquad (16)$$

where $a > 0$. This equation has applications to population dynamics and it was shown in [10] that if $a > \frac{\pi}{2}$, then there exists $\varepsilon > 0$ such that $\lim_{t \to \infty} |x(t_0, \phi)(t)| \geq \varepsilon$, $\forall \phi \in C[[-1, 0], R]$, $\phi(s) \neq 0$ for $s \in (-1, 0)$.

This implies that the zero solution of (17) is unstable. Now let $V(x) = |x|$. Then for $\phi \in PC[[-1, 0], R]$, we have

$$D^+ V(\phi(0)) \leq a|\phi(-1)|(1 + |\phi(0)|) = aV(\phi(-1))(1 + V(\phi(0))).$$

Let $\alpha \in (0, 1)$ and $p(s) = \alpha s$, $s \in R_+$. Then whenever $V(\phi(0)) \geq p(V(\phi(s)))$, $s \in [-1, 0]$, we have

$$D^+ V(\phi(0)) \leq \frac{a}{\lambda} |\phi(0)|(1 + |\phi(0)|)$$

since $V(\phi(-1)) \leq \frac{1}{\lambda} V(\phi(0))$.

Let $\rho > 0$ and for $\| \phi \|_r \leq \rho$, we have

$$D^+ V(\phi(0)) \leq \frac{a(1 + \rho)}{\lambda} |\phi(0)| = \lambda(t) c(|\phi(0)|)$$

where $\lambda(t) = \dfrac{a(1 + \rho)}{\lambda}$ and $c(s) = s$.

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Choose a sequence $\{\tau_k\}$ so that $\tau_k - \tau_{k-1} = \tau > 0$ for all k, where

$$\tau < \frac{1}{a(1+\rho)}[-\alpha \ln \alpha].$$

Define $I(\tau_k, \phi)$ so that

$$|\phi(0) + I(t_k, \phi)| \leq \alpha|\phi(0)|.$$

Then direct computation shows that all conditions of Theorem 4.1 are met and thus the zero solution of (17) is stabilizable through the use of impulses. A simple choice of I which will work is

$$I(t, \phi) = (\alpha - 1)\phi(0).$$

An optimal choice of α is

$$\alpha = \frac{1}{e}$$

at which τ is the largest. $\qquad\square$

Example 4.2: Consider the scalar equation

$$\{ x'(t) = A(t)x(t) + \int_{t-\gamma}^{t} C(t-s)x(s)ds + f(t), \Delta x = h_k x(t), \qquad t = \tau k, \quad (17)$$

where A, C and f are continuous functions,

$$|f(t)| \leq L, \quad L > 0, \quad h_k \geq 0 \quad and \quad \sum_{k=1}^{\infty} h_k < \infty.$$

Assume that $A(t) < 0$ and

$$A(t) + M \int_{0}^{\gamma} |C(u)|du \leq -\alpha,$$

where $\alpha > 0$ and $M = \prod_{k=1}^{\infty}(1 + h_k)$.

Let $V(t, x) = |x|$ and $q > 1$ such that

$$A(t) + Mq \int_{0}^{\gamma} |C(u)|du \leq -\frac{\alpha}{2},$$

and let $p(s) = Mqs$. Then, for any solution $x(t)$, if

$$p(V(t, x(t))) > V(t+s, x(t+s)), \quad s \in [-\gamma, 0], \quad t \geq t_0,$$

we have $\quad D^+V(t, x(t)) \leq A(t)|x(t)| + \int_{t-\gamma}^t |C(t-s)| \, |x(s)| ds + |f(t)|$

$\leq L + A(tOx(t) + \int_0^\gamma |C(u)| \, |x(t-u)| du$

$\leq L + \left(A(t) + Mq \int_0^\gamma |C(u)| du \right) |x(t)|$

$\leq L - \frac{\alpha}{2}|x(t)|$

$\leq A - \frac{\alpha}{r}|x(t)|, \quad |x(t)| \geq H, \quad \text{where } A = \frac{L}{2}, \ H = \frac{2L}{\alpha}.$

$$\Delta V(\tau_k, x) \leq h_k |x|, \quad k = 1, 2, \ldots .$$

Thus all the conditions of Theorem 4.3 are met and the solutions of (18) are UB and UUB. $\qquad\qquad\square$

1. R.D. Driver, *Ordinary and Delay Differential Equations*, Springer-Verlag, New York, 1977.
2. George Ballinger and Xinzhi Liu, Existence and uniqueness results for impulsive delay differential equations", *DCDIS*, **5**, (1999), 579-591.
3. George Ballinger and Xinzhi Liu, Existence, uniqueness and boundedness results for impulsive delay differential equations, *Applicable Anal.*, to appear.
4. K. Gopalsamy and B.G. Zhang, On Delay Differential Equations with Impulses, *Journal of Mathematical Analysis and Applications*, **139**, 1989, 110-122.
5. J.K. Hale and S.M.V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.
6. S.V. Krishna and A.V. Anokhin, Delay Differential Systems with Discontinuous Initial Data and Existence and Uniqueness Theorems for Systems with Impulse and Delay, *Journal of Applied Mathematics and Stochastic Analysis*, **7**, No. 1, 1994, 49-67.
7. V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
8. X.Z. Liu, Further Extensions of the Direct Method and Stability of Impulsive Systems, *Nonlinear World*, **1**, 1994, 341-354.
9. Xinzhi Liu and Jianhua Shen, Razumikhin-type theorems on boundedness for impulsive functional differential equations, *Dynamic systems Appl.*, to appear.
10. E.M. Wright, A nonlinear difference-differential equation, *J. für die Reine und Angewandte Math.*, **194**, 1955, 66-87.

STABILITY OF SOLUTION FOR A CLASS OF THE FOURTH ORDER NONLINEAR AUTONOMOUS SYSTEMS

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In this paper, we have discussed stability of solution for a class of the fourth order nonlinear autonomous systems by means of the integral method of Liapunov function generation for nonlinear autonomous systems(see [1]).

Consider the equation

$$\frac{d^4x}{dt^4} + A\frac{d^3x}{dt^3} + B\frac{d^2x}{dt^2} + C\frac{dx}{dt} + D(\frac{d^2x}{dt^2} + \frac{dx}{dt} + x)^m = 0, \qquad (1)$$

where A, B, C, D are real constants, $m(> 1)$ is a odd number.

Equation (1) is equivalent to the following systems of equations

$$\begin{cases} \frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = x_3, \frac{dx_3}{dt} = x_4, \\ \frac{dx_4}{dt} = -Ax_4 - Bx_3 - Cx_2 - D(x_3 + x_2 + x_1)^m = -g(x_1, x_2, x_3, x_4). \end{cases} \qquad (2)$$

By means of the integral method of Liapunov function generation for nonlinear autonomous systems [1]m, we construct a Liapunov function using the following steps.

step 1. $\quad \frac{\partial H}{\partial x_1} = h_1(x_1, x_2, x_3, x_4) = \int \frac{\partial}{\partial x_1} g(x_1, x_2, x_3, x_4) dx_3$

$\qquad\qquad = \int Dm(x_3 + x_2 + x_1)^{m-1} dx_3 = D(x_3 + x_2 + x_1)^m,$

$\quad \frac{\partial H}{\partial x_2} = h_2(x_1, x_2, x_3, x_4) = \int \frac{\partial}{\partial x_2} g(x_1, x_2, x_3, x_4) dx_3$

$\qquad\qquad = \int [C + Dm(x_3 + x_2 + x_1)^{m-1}] dx_3$

$\qquad\qquad = Cx_3 + D(x_3 + x_2 + x_1)^m,$

$\quad \frac{\partial H}{\partial x_3} = g(x_1, x_2, x_3, x_4) = Ax_4 + Bx_3 + Cx_2 + D(x_3 + x_2 + x_1)^m,$

$\quad \frac{\partial H}{\partial x_4} = x_4 + h_4(x_1, x_2, x_3, x_4)$

$\qquad\qquad = x_4 + \int \frac{\partial}{\partial x_4} g(x_1, x_2, x_3, x_4) dx_3 = x_4 + \int A dx_3 = x_4 + Ax_3,$

where $H, h_i (i = 1, 2, 3, 4)$ can be defined as [1].

step 2.
$$\frac{\partial V}{\partial x_1} = \frac{\partial H}{\partial x_1} + f_1 = D(x_3 + x_2 + x_1)^m + f_1,$$

$$\frac{\partial V}{\partial x_2} = \frac{\partial H}{\partial x_2} + f_2 = Cx_3 + D(x_3 + x_2 + x_1)^m + f_2,$$

$$\frac{\partial V}{\partial x_3} = \frac{\partial H}{\partial x_3} + f_3 = Ax_4 + Bx_3 + Cx_2 + D(x_3 + x_2 + x_1)^m + f_3,$$

$$\frac{\partial V}{\partial x_4} = \frac{\partial H}{\partial x_4} + f_4 = x_4 + Ax_3 + f_4,$$

where f_i are undetermined functions and satisfies the conditions $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ $(i, j = 1, 2, 3, 4)$, V can be defined as [1].

step 3.
$$\frac{dV}{dt}\Big|_{(2)} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial V}{\partial x_3}\frac{dx_3}{dt} + \frac{\partial V}{\partial x_4}\frac{dx_4}{dt}$$

$$= [D(x_3 + x_2 + x_1)^m + f_1]x_2$$

$$+ [Cx_3 + D(x_3 + x_2 + x_1)^m + f_2]x_3$$

$$+ [Ax_4 + Bx_3 + Cx_2 + D(x_3 + x_2 + x_1)^m + f_3]x_4$$

$$+ [x_4 + Ax_3 + f_4][-Ax_4 - Bx_3 - Cx_2 - D(x_3 + x_2 + x_1)^m].$$

Taking

$$f_1 = 0, \qquad\qquad f_2 = (B + C)x_2 + Ax_3 + x_4,$$

$$f_3 = Ax_2 + (A - 1)x_3 - (A - 1)x_4, \qquad f_4 = x_2 - (A - 1)x_3,$$

we have

$$\frac{dV}{dt}\Big|_{(2)} = -[Cx_2^2 + (B - C - A)x_3^2 + (A - 1)x_4^2]. \tag{3}$$

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step4. $V = \int_0^{x_1} \frac{\partial V}{\partial x_1}(x_1,0,0,0)dx_1 + \int_0^{x_2} \frac{\partial V}{\partial x_2}(x_1,x_2,0,0)dx_2$

$\qquad + \int_0^{x_3} \frac{\partial V}{\partial x_3}(x_1,x_2,x_3,0)dx_3 + \int_0^{x_4} \frac{\partial V}{\partial x_4}(x_1,x_2,x_3,x_4)dx_4$

$\qquad = \int_0^{x_1} D x_1^m dx_1 + \int_0^{x_2}[D(x_2+x_1)^m + (B+C)x_2]dx_2$

$\qquad\quad + \int_0^{x_3}[Bx_3 + Cx_2 + D(x_3+x_2+x_1)^m + Ax_2 + (A-1)x_3]dx_3$

$\qquad\quad + \int_0^{x_4}[x_4 + Ax_3 + x_2 - (A-1)x_3]dx_4$

$\qquad = [\frac{D}{m+1}x_1^{m+1}]|_0^{x_1} + [\frac{D}{m+1}(x_2+x_1)^{m+1} + \frac{1}{2}(B+C)x_2^2]|_0^{x_2} + [\frac{1}{2}Bx_3^2$

$\qquad\quad +Cx_2x_3 + \frac{D}{m+1}(x_3+x_2+x_1)^{m+1} + Ax_2x_3 + \frac{1}{2}(A-1)x_3^2]|_0^{x_3}$

$\qquad\quad +[\frac{1}{2}x_4^2 + Ax_3x_4 + x_2x_4 - (A-1)x_3x_4]|_0^{x_4}$

$\qquad = \frac{D}{m+1}x_1^{m+1} + \left\{ [\frac{D}{m+1}(x_2+x_1)^{m+1} + \frac{1}{2}(B+C)x_2^2] - [\frac{D}{m+1}x_1^{m+1}] \right\}$

$\qquad\quad + \left\{ [\frac{1}{2}Bx_3^2 + Cx_2x_3 + \frac{D}{m+1}(x_3+x_2+x_1)^{m+1} \right.$

$\qquad\quad \left. +Ax_2x_3 + \frac{1}{2}(A-1)x_3^2] - [\frac{D}{m+1}(x_2+x_1)^{m+1}] \right\}$

$\qquad\quad +\frac{1}{2}x_4^2 + Ax_3x_4 + x_2x_4 - Ax_3x_4 + x_3x_4$

$\qquad = \frac{D}{m+1}(x_3+x_2+x_1)^{m+1}$

$\qquad\quad +\frac{1}{2}x_4^2 + \frac{1}{2}(A+B-1)x_3^2 + \frac{1}{2}(B+C)x_2^2 + x_4x_3$

$\qquad\quad +x_4x_2 + (A+C)x_3x_2$

$\qquad = \frac{D}{m+1}(x_3+x_2+x_1)^{m+1} + U(x_4,x_3,x_2),$

$\hfill (4)$

where

$$U(x_4,x_3,x_2) = \frac{1}{2}x_4^2 + \frac{1}{2}(A+B-1)x_3^2 + \frac{1}{2}(B+C)x_2^2$$
$$+x_4x_3 + x_4x_2 + (A+C)x_3x_2.$$

Theorem If equation (1) satisfies the following conditions:

$$A \geq 1, \quad B \geq A+C, \quad C > 0, \quad D > 0,$$

then zero solution of equation (1) is stable.

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Proof. First, we prove that Liapunov function (4) is positive definite. Since

$$\frac{D}{m+1}(x_3 + x_2 + x_1)^{m+1} \geq 0,$$

we only need to show the quadratic form

$$U(x_4, x_3, x_2) = \tfrac{1}{2}x_4^2 + \tfrac{1}{2}(A + B - 1)x_3^2 + \tfrac{1}{2}(B + C)x_2^2$$

$$+ x_4 x_3 + x_4 x_2 + (A + C)x_3 x_2$$

is positive definite.

Note that determinant of coefficients of the quadratic form $U(x_4, x_3, x_2)$ is

$$\begin{vmatrix} \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{1}{2}(A + B - 1) & \tfrac{1}{2}(A + C) \\ \tfrac{1}{2} & \tfrac{1}{2}(A + C) & \tfrac{1}{2}(B + C) \end{vmatrix}.$$

It is easy to see that the principal minors of this determinant satisfy

$$\frac{1}{2} > 0, \qquad \begin{vmatrix} \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{1}{2}(A + B - 1) \end{vmatrix} = \frac{1}{4}(A + B - 2) > 0,$$

$$\begin{vmatrix} \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{1}{2}(A + B - 1) & \tfrac{1}{2}(A + C) \\ \tfrac{1}{2} & \tfrac{1}{2}(A + C) & \tfrac{1}{2}(B + C) \end{vmatrix} = \frac{1}{8}[(A + B - 2)(B + C - 1) - (A + C - 1)^2]$$

$$> 0.$$

In fact, from

$$A \geq 1, \quad B \geq A + C, \quad C > 0,$$

we get

$$A + B - 2 \geq A + C - 1 \geq C > 0$$

and

$$B + C - 1 \geq A + C + C - 1 > A + C - 1 > 0.$$

Hence

$$(A + B - 2)(B + C - 1) > (A + C - 1)^2.$$

By virtue of Sylvester theorem, we obtain that the quadratic form $U(x_4, x_3, x_2)$ is positive definite, therefore Liapunov function (4) is positive definite also.

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Next, we prove that the derivative $\frac{dV}{dt}$ of $V(x_1, x_2, x_3, x_4)$ along (2) is negative function.

Since

$$\frac{dV}{dt}|_{(2)} = D(x_3 + x_2 + x_1)^m(\dot{x}_3 + \dot{x}_2 + \dot{x}_1) + x_4\dot{x}_4 + (A + B - 1)x_3\dot{x}_3$$
$$+ (B + C)x_2\dot{x}_2 + \dot{x}_4x_3 + x_4\dot{x}_3$$

$$+ \dot{x}_4x_2 + x_4\dot{x}_2 + (A + C)(\dot{x}_3x_2 + x_3\dot{x}_2)$$

$$= D(x_3 + x_2 + x_1)^m(x_4 + x_3 + x_2) + \dot{x}_4(x_4 + x_3 + x_2)$$

$$+ \dot{x}_3[(A + B - 1)x_3 + x_4 + (A + C)x_2]$$

$$+ \dot{x}_2[(B + C)x_2 + x_4 + (A + C)x_3]$$

$$= D(x_3 + x_2 + x_1)^m(x_4 + x_3 + x_2) + [-Ax_4 - Bx_3 - Cx_2$$

$$- D(x_3 + x_2 + x_1)^m](x_4 + x_3 + x_2) + x_4[(A + B - 1)x_3 + x_4$$

$$+ (A + C)x_2] + x_3[(B + C)x_2 + x_4 + (A + C)x_3]$$

$$= -[Cx_2^2 + (B - C - A)x_3^2 + (A - 1)x_4^2],$$

by condition of theorem, we know that

$$\frac{dV}{dt}|_{(2)} = -[Cx_2^2 + (B - C - A)x_3^2 + (A - 1)x_4^2]$$

is negative function.

By virtue of stability theorem of Liapunov on constant motion, zero solution of equation (2) is stable. Thus it is obtained that zero solution of equation (1) is stable.

References

1. J. A. Walker, L.G. Clark, An integral method of Liapunov function generation for nonlinear autonomous systems, Journal of Applied Mechanics, Sept. (1965), 569–576.

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OSCILLATION OF NEUTRAL DIFFERENCE EQUATIONS WITH "MAXIMA" [g]

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In this paper, a class of neutral difference equations with "maxima" are considered.
Sufficient conditions for oscillation of all solutions are obtained.

1 Introduction

Recently, there has been a lot of activity concerning the oscillatory behavior of solutions of difference equations. See, for example, [1,5-7] and the references cited therein.

Consider the difference equation

$$\triangle[x_n + p_n x_{n-k}] + q_n \max_{[n-l,n]} x_s = 0, \tag{1}$$

where $n \in N(n_0) = \{n_0, n_0 + 1, ..., \}$ and $\triangle$ is the forward difference operator defined by $\triangle x_n = x_{n+1} - x_n$, and $[n - l, n] = [n - l, n - l + 1, ..., n]$.

Throughout the paper we always assume that the following conditions are satisfied:

(H1) k and l are positive integers;

(H2) $\{p_n\}$ is a sequence of real number, for $n \in N(n_0)$;

(H3) $\{q_n\}$ is a sequence of nonnegative real numbers which is not identically zero;

(H4) $\sum_{s=n_0}^{\infty} q_s = +\infty$.

By a solution of Eq.(1), we mean a sequence $\{x_n\}$ of real numbers which is defined for $n \in N(n_0)$. As is customary, a nontrivial solution $\{x_n\}$ of (1) is said to be nonoscillatory if the terms x_n are either eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

Our aim in this paper is to establish the sufficient conditions for oscillstion of all solutions of (1). This work was motivated by the paper Bainov, Petrov and Proicheva [2] in which a detailed analysis of oscillation properties was given for the continuous version of Eq.(1). We shall note that differential equations with maxima occur in the problem of automatic regulation of various real systems [9]. The maxima arise when the regulation law corresponds to

the maximal deviation of the regulable quantity. Also differential equations with maxima are an increasing object investigated by many authors [2-4,8]. However, to the author's knowledge, the corresponding difference equations with maxima, for example, Eq.(1), has not been investigated up to now. On the other hand, Eq.(1) is different from equation

$$\triangle[x_n + p_n x_{n-k}] + q_n \dot{x}_{n-l} = 0. \tag{2}$$

Eq.(1) is nonlinear and in the general case the fact $\{x_n\}$ is a solution of (1) does not imply that $\{-x_n\}$ is also a solution of (1).

2 Main results

Define

$$z_n = x_n + p_n x_{n-k}. \tag{3}$$

Lemma 1 Let conditions (H1)-(H4) hold. Then the following assertions are valid:

(i) If $p_n \leq -1$ and $\{x_n\}$ is an eventually positive solution of (1), then z_n is eventually decreasing and $z_n < 0$ eventually.

(ii) If $p_n \leq -1$ and $\{x_n\}$ is an eventually negative solution of (1), then z_n is an eventually increasing function and $z_n > 0$ eventually.

(iii) If $-1 \leq p_n \leq 0$ and $x_n > 0$ eventually, then $z_n > 0$ eventually.

(iv) If $-1 \leq p_n \leq 0$ and $x_n < 0$ eventually, then $z_n < 0$ eventually.

Proof (i) From the definition of z_n there follows the equality

$$\triangle z_n + q_n \max_{[n-l,n]} x_s = 0. \tag{4}$$

Since $x_n > 0$ eventually, then $\triangle z_n < 0$ and z_n is an eventually decreasing function. Suppose $z_n > 0$ eventually. From (2.3) there follow the inequalities

$$x_n > -p_n x_{n-k} \geq x_{n-k}.$$

From the inequality $x_n > x_{n-k}$ and the fact x_n is an eventually positive function it follows that there exists a constant $m > 0$ such that $x_n > m$ eventually and $\max_{[n-l,n]} x_s > m$ eventually. From (2.4) we obtain the estimate

$$\triangle z_n = -q_n \max_{[n-l,n]} x_s < -m q_n.$$

Summing the last inequality from n_1 to n, where n_1 is a sufficiently large integer, and obtain

$$z_n < z_{n_1} - m \sum_{s=n_1}^{n} q_s. \tag{5}$$

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Passing to the limit in (2.5), from condition H4 it follows that $\lim_{n\to\infty} z_n = -\infty$ which contradicts the assumption that $z_n > 0$ eventually.

(ii) From (2.4) and the inequality $x_n < 0$ eventually it follows that $\triangle z_n$ is an eventually positive function and z_n is an eventually increasing function. Suppose that $z_n < 0$ eventually. From (2.3) and from the condition $p_n \leq -1$ we deduce the inequality $x_n < x_{n-k}$. Hence there exists a negative constant m such that $x_n < m$ eventually and $\max_{[n-l,n]} x_s < m$ eventually. From (2.4) it follows $\triangle z_n \geq -q_n m$. Further, as in the proof of (i) it is shown that $\lim_{n\to\infty} z_n = +\infty$ which contradicts the assumption that $z_n < 0$ eventually. Hence, $z_n > 0$ eventually.

(iii) Suppose that $z_n < 0$ eventually. Then from (2.3) and from the condition $-1 \leq p_n \leq 0$ we obtain

$$x_n < x_{n-k}. \tag{6}$$

Since $x_n > 0$ eventually. then from (2.6) it follows that x_n is a bounded function for $n \in N(n_0)$, hence z_n is a bounded function too. It is immediately verified that $\triangle z_n < 0$ eventually and z_n is an eventually decreasing negative function. Since z_n is a bounded function, then there exists the finite limit $\lim_{n\to\infty} z_n = \alpha(\alpha < 0)$. Let $\beta = \lim_{n\to\infty} \inf x_n$. Suppose that $\beta > 0$. Eventually the inequality $x_n > \frac{\beta}{2}$ is valid, hence $\max_{[n-l,n]} x_s > \frac{\beta}{2}$ eventually. Then as in (i) it is proved that $\lim_{n\to\infty} z_n = -\infty$ which contradicts the fact that z_n is a bounded function.

Thus we proved that $\lim_{n\to\infty} \inf x_n = 0$. There exists a sequence $\{\tau_m\}_{m=1}^{\infty}, \tau_k \in N(n_0), \lim_{m\to\infty} \tau_m = \infty$ and $\lim_{m\to\infty} x_{\tau_m - k} = 0$. From (2.6) it follows that $\lim_{m\to\infty} x_{\tau_m} = 0$. Passing to the limit in the equality $z_{\tau_m} = x_{\tau_m} + p_{\tau_m} x_{\tau_m - k}$ as $m \to \infty$, we obtain that $\lim_{m\to\infty} z_{\tau_m} = 0$ which contradicts the fact that $\alpha = \lim_{n\to\infty} z_n < 0$. Hence $z_n > 0$ eventually.

The proof of (iv) is analogous to the proof of (iii).

From Lemma 1, we immediately obtain

Theorem 2 Let conditions (H1)-(H4) hold and $p_n \equiv -1$. Then each solution of (1) oscillates.

Theorem 3 Let conditions (H1)-(H4) hold. Assume that

$$p_n \leq -1, \qquad k > l, \tag{7}$$

$$\lim_{n\to\infty} \inf \frac{1}{k-l} \sum_{s=n-k+l}^{n-1} \frac{q_s}{\max_{u\in[s-l,s]}\{-p_{u+k}\}} > \frac{(k-l)^{k-l}}{(k-l+1)^{k-l+1}}. \tag{8}$$

Then each solution of equation (1) oscillation.

Proof Suppose that equation (1) has a nonoscillatory solution $\{x_n\}$. Let $x_n < 0$ eventually. From Lemma 1 (ii), it follows that $z_n > 0$ eventually. From

(3) there follow the inequalities

$$z_n < p_n x_{n-k}, \quad x_n < \frac{z_{n+k}}{p_{n+k}}, \quad \max_{[n-l,n]} x_s \leq \max_{[n-l,n]} \frac{z_{s+k}}{p_{s+k}}. \tag{9}$$

Since z_n is an eventually increasing function, then for sufficiently large n the following estimate is valid

$$z_{n+k-l} \leq z_{s+k}, \quad s \in [n-l, n].$$

Then

$$\max_{s \in [n-l,n]} \frac{z_{n+k-l}}{p_{s+k}} \geq \max_{s \in [n-l,n]} \frac{z_{s+k}}{p_{s+k}}$$

$$\frac{-z'_{n+k-l}}{\max_{s \in [n-l,n]}\{-p_{s+k}\}} \geq \max_{s \in [n-l,n]} \frac{z_{n+k}}{p_{s+k}}.$$

From the last inequality and from (2.10) we deduce the inequality

$$\max_{[n-l,n]} x_s \leq \frac{-z_{n+k-l}}{\max_{s \in [n-l,n]}\{-p_{s+k}\}}. \tag{10}$$

From (2.4) and from (2.11) it follows that the eventually positive function z_n satisfies

$$\triangle z_n - \frac{q_n}{\max_{s \in [n-l,n]}\{-p_{s+k}\}} z_{n+k-l} \geq 0. \tag{11}$$

But from (2.9) and [1] it follows that (2.12) has no eventually positive solution. The contradiction obtained shows that Eq.(1) has no eventually negative solutions.

We shall show that it has no eventually positive solutions either. Suppose that this is not true. Let $x_n > 0$ eventually. From Lemma 1(i) it follows that $z_n < 0$ eventually. As above, we obtain

$$\max_{[n-l,n]} x_s \geq \frac{-z_{n+k-l}}{\max_{[n-l,n]}\{-p_{s+k}\}}. \tag{12}$$

From (2.4) and from (2.13) it follows that the eventually negative function z_n satisfies the inequality

$$\triangle z_n - \frac{q_t}{\max_{s \in [n-l,n]}\{-p_{s+k}\}} z_{n+k-l} \leq 0. \tag{13}$$

But from (2.9) and [1] it follows that (2.14) has no eventually negative solution. Hence (1) has no eventually positive solutions. Since in view of what was

proved above it has no eventually negative solutions, then each solution of (1) oscillates.

Theorem 4 Let conditions (H1)-(H4) hold. Assume that

$$-1 \le p_n \le 0, \qquad l \ge k, \qquad (14)$$

$$\lim_{n \to \infty} \inf \frac{1}{k} \sum_{s=n-k}^{n-1} q_s \min_{u \in [s-l,s]} \{-p_u\} \ge \frac{k^k}{(k+1)^{k+1}}. \qquad (15)$$

Then each solution $\{x_n\}$ of (1) oscillates.

Proof Suppose that (1) has a nonoscillatory solution $\{x_n\}$ and let $x_n > 0$ eventually. From Lemma 1, it follows that z_n is an eventually decreasing positive function. Then
$z_n < x_n$ and $\max_{[n-l,n]} z_s < \max_{[n-l,n]} x_s$. From (2.4) and from the last inequality we obtain that

$$\triangle z_n + q_n \max_{[n-l,n]} z_s \le 0.$$

Since z_n is an eventually decreasing function, then

$$\max_{[n-l,n]} z_s = z_{n-l}.$$

Consequently, the eventually positive function z_n satisfies the inequality

$$\triangle z_n + q_n z_{n-l} < 0. \qquad (16)$$

Since $\min_{[n-l,n]} \{-p_u\} \le 1$, then from (2.17) and [1] it follows that (2.18) has no eventually positive solutions. The contradiction obtained shows that (1) has no eventually positive solutions.

We shall show that it has no eventually negative solutions either. Suppose that this is not true. Let $x_n < 0$ eventually. From Lemma 1 it follows that z_n is an eventually negative increasing function. From the inequality $z_n < 0$ eventually and from the definition of z_n there follow the inequalities

$$x_n < -p_n x_{n-k} < -p_n z_{n-k}.$$

$$\max_{[n-l,n]} x_s < \max_{s \in [n-l,n]} \{-p_s z_{s-k}\} \le [\min_{[n-l,n]} \{-p_s\}] z_{n-k}.$$

From (2.4) and from the last inequality we obtain that the eventually negative function z_n satisfies the inequality

$$\triangle z_n + [q_n \min_{[n-l,n]} \{-p_s\}] z_{n-k} > 0. \qquad (17)$$

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From (2.17) and from the known result of [1] it follows that inequality (17) has no eventually negative solutions. Hence (1) has no eventually negative solutions and since in view of what was proved above the equation (1) has no eventually positive solutions either, then each solution of (1) oscillates.

References

1. R.P.Agarwal, Marcel Dekker, New York, 1992.
2. D.Bainov, V.Petrov and V.Proicheva, Revista Mathematica de la Universidad Complutense Madrid, 8, 171-180 (1995).
3. D.Bainov, V.Petrov and V.Proicheva, Dynamic Systems and Application, 4, 137-146 (1995).
4. D.Bainov, V.Petrov and V.Proicheva, Journal of Computational and Applied Mathematics. 83, 237-249 (1997).
5. Goyri and G.Ladas, Clarendon Press, Oxford (1991).
6. W.G.Kelly and A. Peterson, Academic Press, New York (1991).
7. V.Lakshmikantham and D.Trigiante, Academic Press, New York (1988).
8. V.A.Petrov, Communications in Applied Analysis, 2, 129-142 (1998).
9. E.P.Popov, Moscow, 1966(in Russian).

ASYMPTOTIC ANALYSIS OF SOLUTIONS OF SINGULAULY PERTURBED EQUATION

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In this paper, we considered the singularly perturbed problem which is derived from the population model, proved that there existed a unique solution, and obtained the asymptotic solution and its remainder estimation.

1 Introduction

We often encounter the following equation

$$\left(\frac{u^m}{m}\right)_t = u_{xx} - \{u\phi[\int_{-\infty}^{+\infty} k(x-y)u(t,y)dy]\}_x + u^n(u-a)(1-u) \qquad (1)$$

when we study the population problem. Where $u(t,x) \geq 0$ is the population density, $x \in R, t > 0$, $\phi(s)$ is the population rate. $m \geq 1, n \geq 1$ are

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constants. $k(x) = \varepsilon[1 - H(x)]$, H(x) is Heaviside function, $\varepsilon \geq 0$ is a small parameter. $0 < a < 1$ is a constant.

Let $v(t,x) = \int_{-\infty}^{x} u(t,\tau)d\tau, \xi = x - ct$, $v(t,x) = v(\xi)$, then $u(t,x) = u(\xi)$. We set $\alpha = \int_{-\infty}^{+\infty} u(\tau)d\tau$, hence we get $\int_{-\infty}^{+\infty} k(x-y)u(t,y)dy = \varepsilon[\alpha - 2v(\xi)]$, so that we turn (1) into the following equation

$$-cu^{m-1}u_\xi = u_{\xi\xi} - \{u\phi[\varepsilon(\alpha - 2v)]\}_\xi + u^n(u-a)(1-u) \qquad (2)$$

Then, let $\eta = \varepsilon\xi, U(\eta) = u(\xi), V(\eta) = \varepsilon[\alpha - 2v(\xi)]$, we transformate (2) to

$$\varepsilon^2 U_{\eta\eta} - \varepsilon[U\phi(V)]_\eta + U^n(U-a)(1-U) = -c\varepsilon U^{m-1}U_\eta$$

Since we only consider the traveling wave problem, from now on we use the traditional independent variable t instead of η, i.e. the superscript " $\cdot$ " denotes differentiation with respect to t. We rewrite $V = x$, $U = y.\varepsilon U_\eta = z$, then we obtained

$$\begin{cases} \dot{x} = y \\ \varepsilon\dot{y} = z \\ \varepsilon\dot{z} = \varepsilon\phi'(x)y^2 + \phi(x)z - cy^{m-1}z + y^n(y-a)(y-1) \end{cases} \qquad (3)$$

where $\phi'(x)$ denotes differentiation with respect to x. We find the solution that satisfied the following conditions

$$\begin{cases} x(0) = 0 \\ y(0) = \beta \\ z(0) = 0 \end{cases} \qquad (4)$$

The system (3),(4) is a singularly perturbed problem, we proved that there existed a unique solution, and obtained the asymptotic solution and its remainder estimation.

2　The construction of asymptotic solution

We supposed the following conditions hold
(H1) The functions that are given in (1) belong to $C^{N+2}, N \geq 0$ is an integer.
(H2) $\phi(x) - ca^{m-1} < 0$
We denote the second and third formulas of (3) are $\varepsilon\dot{w} = F(x,w,t,\varepsilon)$ where $w = \begin{pmatrix} y \\ z \end{pmatrix}$ is a 2-dimensional vector. So we obtain

$F_w =$
$$\begin{bmatrix} 0 & 1 \\ c(1-m)y^{m-2}z + (n+2)y^{n+1} - (a+1)(n+1)y^n + any^{n-1} & \phi(x) - cy^{m-1} \end{bmatrix}$$

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Let $\varepsilon = 0$, we conclude the solutions of the reduced equations $0 = F(x, w, t, 0)$ are $\bar{w}_1(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\bar{w}_2(t) = \begin{pmatrix} a \\ 0 \end{pmatrix}$, $\bar{w}_3(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The eigenvalues $\bar{\lambda}_i(t), (i = 1, 2)$ of the matrix $\bar{F}_w(t) = F_w(\bar{x}(t), \bar{w}(t), t)$ satisfy $Re\bar{\lambda}_i(t) < 0, (i = 1, 2)$ only when $\bar{w}(t) = \begin{pmatrix} a \\ 0 \end{pmatrix}$. Hence we proved the following Lemma:

Lemma 1: Under (H2), the problem (3),(4) is a stable problem when $\bar{w}(t) = \begin{pmatrix} a \\ 0 \end{pmatrix}$.

Using the boundary function method (see[1]), we find the asymptotic solution for (3),(4) in the following form

$$h(t, \varepsilon) = \bar{h}(t, \varepsilon) + \Pi h(\tau, \varepsilon) \tag{5}$$

where $h = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a 3-dimensional vector, $\tau = \frac{t}{\varepsilon} \geq 0$. $\bar{h}$ and Πh are regular for ε. Let

$$\bar{h}(t, \varepsilon) = \bar{h}_0(t) + \varepsilon \bar{h}_1(t) + \cdots$$

$$\Pi h(\tau, \varepsilon) = \Pi_0 h(\tau) + \varepsilon \Pi_1 h(\tau) + \cdots$$

Substituting (5) into (3),(4) and separating its right by t and τ we get

$$\begin{cases} \varepsilon \frac{d\bar{x}}{dt} + \frac{d\Pi x}{d\tau} = \varepsilon(\bar{y} + \Pi y) \\ \varepsilon \frac{d\bar{y}}{dt} + \frac{d\Pi y}{d\tau} = \varepsilon(\bar{z} + \Pi z) \\ \varepsilon \frac{d\bar{z}}{dt} + \frac{d\Pi z}{d\tau} = \varepsilon\phi'(\bar{x} + \Pi x)(\bar{y} + \Pi y)^2 + \phi(\bar{x} + \Pi x)(\bar{z} + \Pi z) \\ \quad -c(\bar{y} + \Pi y)^{m-1}(\bar{z} + \Pi z) + (\bar{y} + \Pi y)^n(\bar{y} + \Pi y - a)(\bar{y} + \Pi y - 1) \end{cases} \tag{6}$$

and

$$\begin{cases} \bar{x}(0) + \Pi x(0) = 0 \\ \bar{y}(0) + \Pi y(0) = \beta \\ \bar{z}(0) + \Pi z(0) = 0 \end{cases} \tag{7}$$

We expanse the functions which are given in (6),(7) in Taylor series of ε. Equating coefficients of same power of ε in both side of equity, we obtain a sequence of linear boundary value problem. At first we consider the leading term

$$\begin{cases} \frac{d\bar{x}_0}{dt} = \bar{y}_0 \\ 0 = \bar{z}_0 \\ 0 = \phi(\bar{x}_0)\bar{z}_0 - c\bar{y}_0^{m-1}\bar{z}_0 + \bar{y}_0^n(\bar{y}_0 - a)(\bar{y}_0 - 1) \end{cases} \tag{8}$$

$$\begin{cases} \frac{d\Pi_0 x}{d\tau} = 0 \\ \frac{d\Pi_0 w}{d\tau} = \Pi_0 F \end{cases} \tag{9}$$

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and

$$\begin{cases} \bar{x}_0(0) + \Pi_0 x(0) = 0 \\ \bar{y}_0(0) + \Pi_0 y(0) = \beta \\ \bar{z}_0(0) + \Pi_0 z(0) = 0 \end{cases} \tag{10}$$

The boundary-layer function $\Pi_0 h$ must be satisfied the condition $\Pi_0 h(+\infty) = 0$. It is clear that $\Pi_0 x(\tau) = 0$

According to Lemma 1, we conclude that $\bar{h}_0(t) = \begin{pmatrix} at \\ a \\ 0 \end{pmatrix}$.

Hence the second formula of (9) could be written

$$\begin{cases} \frac{d\Pi_0 y}{d\tau} = \Pi_0 z \\ \frac{d\Pi_0 z}{d\tau} = \phi(0)\Pi_0 z - ca^{m-1}\Pi_0 z + a^n(a-1)\Pi_0 y \end{cases} \tag{11}$$

Because the system (11) could be changed to a reducible second order differential equation, it is easy to find the solution of the problem (10),(11). Thus we get the solutions $\bar{h}_0(t)$ and $\Pi_0 h(\tau)$ of (8),(9),(10).Similarly we can get the solutions $\bar{h}_k(t)$ and $\Pi_k h(\tau)$.

3, Existence and uniqueness result

Let

$$\begin{cases} \mu(t,\varepsilon) = x(t,\varepsilon) - X_N(t,\varepsilon) \\ \nu(t,\varepsilon) = w(t,\varepsilon) - W_N(t,\varepsilon) \end{cases} \tag{12}$$

where $\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$ is a 2-dimensional vector,

$$X_N(t,\varepsilon) = \sum_{j=1}^{N} \varepsilon^j [\bar{x}_j(t) + \Pi_j x(\tau)]$$

$$W_N(t,\varepsilon) = \sum_{j=1}^{N} \varepsilon^j [\bar{w}_j(t) + \Pi_j w(\tau)]$$

¿From (12), we get the following

$$\begin{cases} x(t,\varepsilon) = \mu(t,\varepsilon) + X_N(t,\varepsilon) \\ w(t,\varepsilon) = \nu(t,\varepsilon) + W_N(t,\varepsilon) \end{cases} \tag{13}$$

Substituting (13) into (3),(4), we get

$$\begin{cases} \frac{d\mu}{dt} = \nu_1(t,\varepsilon) + Y_N(t,\varepsilon) - \frac{dX_N}{dt} \\ \varepsilon\frac{d\nu}{dt} = F(\mu + X_N, \nu + W_N, t, \varepsilon) - \varepsilon\frac{dW_N}{dt} \end{cases} \tag{14}$$

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and

$$\mu(0,\varepsilon) = 0, \nu(0,\varepsilon) = 0 \tag{15}$$

It follows from the fixed point theorems in Banach space that there exists a unique solution to the system (14),(15) and the following estimation

$$\|\mu(t,\varepsilon)\| \le \sigma\varepsilon^{N+1}, \|\nu(t,\varepsilon)\| \le \sigma\varepsilon^{N+1},$$

where σ is a constant. So we proved the following theorem:

Theorem: Under the conditions (H1),(H2), there exists a unique solution $h(t,\varepsilon)$ to problem (3),(4) and the following estimation

$$h(t,\varepsilon) = \sum_{j=1}^{N} \varepsilon^j [\bar{h}_j(t) + \prod_j h(\frac{t}{\varepsilon})] + o(\varepsilon^{N+1}), (\varepsilon \to 0)$$

holds uniformly for $0 \le t \le T, T > 0$ is a constant.

Acknowledgments

References

1. Z.M. Wang and W.Z. Lin, The Dirichlet Problem for a Quasilinear Singularly Perturbed Second Order System, J. Math. Anal. Appl. **201**, 897-910 (1996).
2. P. Hartman, Ordinary differential equations, Birkhauser, 1982.

DYNAMICS OF 2–D INCOMPRESSIBLE FLOWS

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We study the dynamics of divergence-free vector fields on general 2D Manifolds. Connections to the Hamiltonian vector fields are also indicated. The main motivation is to understand the topological and geometrical structure of the Lagrange dynamics of incompressible fluid flows.

1 Introduction

The main motivation of this Note is to present some results and ideas of a recent new geometric theory developed by the authors on 2–D incompressible flows. The point of view of this theory is to classify the topological structure and its transitions of the *instantaneous* velocity field (i.e. streamlines in the Eulerian coordinates), treating the time variable as a parameter. Based on this viewpoint, our theory contains two main steps: a) the study of the global topological/geometric structure of divergence-free vector fields on 2–D manifolds, and b) transitions of the global structure of the velocity fields as solutions of the Navier-Stokes equations of 2–D incompressible fluid flows as the time t or the Reynolds number changes.

This article along with [2,3,4,6] is in the first direction. In [2,4], we studied the case where M is a compact submanifold of S^2. In this case, the classical Poincaré-Bendixson theorem holds true. Hence in particular, we proved that a divergence-fee vector field is structurally stable with divergence-free vector fields perturbations if and only if (1) v is regular; (2) all interior saddle points of v are self-connected; and (3) each boundary saddle point is connected to boundary saddles on the same connected component of the boundary. These conditions are quite different from those in the classical M. Peixoto[8] structural stability theorem for general vector field with general vector feilds perturbations on two-dimensional compact manifolds; they are (i) the field can have only a finite number of singularities and closed orbits (critical elements) which must be hyperbolic; (ii) there are no saddle connections; (iii) the non wandering set consists of singular points and closed orbits. A direct consequence of the above two sets of necessary and sufficient conditions is that no divergence-free vector field is structurally stable under general C^r vector fields perturbations. Such a drastic change in the stable configurations is explained by the fact that **divergence-free fields preserve volume** and so attractors and sources can never occur for these fields. In particular, this makes it natural the restriction that saddles in the boundary must be connected with saddles on the boundary on the same connected component, in the third condition.

Progress has also been made in the second direction recently in [5,1]. In [5], we proved that when M is a compact submanifold of S^2 for any external forcing in an open and dense subset of $C^\alpha(TM)$ $(0 < \alpha < 1)$, all steady state solutions of the two-dimensional Navier-Stokes equations are structurally stable. In [1], we study the structural bifurcation of one-parameter families of divergence-free vector fields.

The main objective of this article is to study the dynamics of divergence-free vector fields on general 2D compact manifolds. The main results include

1). a limit set theorem and structural classification of divergence-free vector fields, 2). topological structure of ergodic sets of divergence-free vector fields, 3). structural instability and block stability of divergence-free vector fields on 2D compact manifolds with nonzero genus, and 4). structural classfication and stability of Hamiltonian vector fields on 2D compact symplectic manifolds. The proof of these results are length and technical, and will be given in [3,6].

2 Limit Set Theorem and structural classification

Let M be a two dimensional differentiable manifold with boundary ∂M. We always assume that $r \geq 1$ is an integer. Let $C_n^r(TM)$ be the space of all r-th differentiable vector fields v on M such that $v|_{\partial M} \in C^r(T\partial M)$, namely the restriction of any r-th differentiable vector field $v \in C^r(TM)$ on the boundary ∂M is a r-th differentiable vector field of the tangent bundle of ∂M.

A point $p \in M$ is called a singular point of $v \in C_n^r(TM)$ if $v(p) = 0$; a singular point p of v is called nondegenerate if the Jacobian matrix $Dv(p)$ is invertible; v is called regular if all singular points of v are nondegenerate. For convenience, we let

$$D^r(TM)\{v \in C_n^r(TM)| \ \mathrm{div} \ v = 0\},$$

and

$$D_0^r(TM) = \{v \in D^r(TM)| \ v \text{ is regular }\}.$$

Here we assume that M is a Riemannian manifold with the Riemannian metric g. The differential operator div is the divergence operator of a vector field, which can be defined in terms of the Levi-Civita connection.

Let $\Phi(x,t)$ be the orbit passing through $x \in M$ at $t = 0$ of the flow generated by v. An orbit with its end points is called a saddle connection if its α and ω-limit sets are saddle points.

Thanks to the divergence-free conditions, the properties of divergence-free vector fields are quite different from those of general vector fields. We now recall some basic facts of divergence-free vector fields, which will be used hereafter in this article.

1. It is easy to see that for any $v \in D^r(TM)$ an interior non-degenerate singular point of v can either be a center or a saddle, and a nondegenerate boundary singularity must be a saddle.

2. The divergence-free condition eliminates the existence of limiting cycles. Hence for any $v \in D^r(TM)$, and $V \subset M$ be the set consisting of all closed orbits and centers of v. Then V is open.

3. The classical Poincaré–Bendixson theorem holds true for submanifolds of S^2 (or P^2 or K^2). In particular, both the ω-limit and the α-limit sets of a

regular divergence free vector field on a submanifold of S^2 (or P^2 or K^2) must be either a saddle point, or a non limiting cycle closed orbit.

To to obtain a structural classification for divergence-free vector fields on general two-dimensional manifolds, we need a new version of the Poincaré-Bendixson theorem to understand the ω and α limit sets. It is well known that flows on a torus may be nontrivially recurrent, and the ω and α limit sets can be very complicated sets. For instance, the ω and α limit sets of the Cherry flow can have the structure of Cantor sets; see [7]. Thanks to the divergence-free conditions, we have [6]

Theorem 2.1 (Limit Set Theorem) *Let M be a two-dimensional compact manifold with or without boundary and $v \in D^r(TM)$ be a regular vector field. Let $x_0 \in M$ be a regular point of v. Then the α and ω — limit sets $\alpha(x_0)$ and $\omega(x_0)$ must be one of the following types: 1). a closed orbit, but not a limit cycle, 2). a saddle point, or 3). a connected closed domain $\Omega \subset K = \overline{M - \bar{V}}$, with $\partial\Omega$ consisting of saddle connections. Here $V \subset M$ is the open set consisting of the closed orbits of v.*

We then proceed with the structural classification of divergence-free vector fields as follows.

Let $v \in D_0^r(TM)$ be a regula divergence-free vector field. First, let $p \in M$ be a center, then there is an open neighborhood C of p, such that for any $x \in C(x \neq p)$, the orbit $\Phi(x, t)$ is closed. We call the largest neighborhood C of p of this type is called a *circle cell* of p.

Second, let $B \subset M$ be an open set, such that for any $x \in B$, the orbit $\Phi(x, t)$ is closed, and any connected component Σ of ∂B is not a single point. Then B is called a *circle band* of v.

Third, a closed domain $F \subset M$ (i.e. cl(int F) $= F$) is called an *ergodic set* of $v \in D^r(TM)$ if for any $x \in F$ with $\omega(x)$ not a singular point of v, then $\omega(x) = F$.

Then as a corollary of the limit set theorem, we have

Theorem 2.2 (Structural Classification) *Let M be a two-dimensional compact manifold with or without boundary, and $v \in D^r(TM)$ be regular. Then the topological set of orbits of v consists of finite connected components of 1). circle cells and circle bands, 2). ergodic sets, and 3). saddle connections.*

3 Topological structures of ergodic sets

The topological structure of ergodic sets is more complex than that of circle cells and circle bands.

Definition 3.1 Let N be a compact manifold without boundary and with genus $k \geq 0$. A closed domain $\Omega \subset N$ is called a pseudo–manifold with genus

g if

1). Ω is connected and $\partial\Omega$ is homeomorphic to a union of finite number of circles S^1, each of which has finite number of common points with the other, and

2). there exists a submanifold $M \subset N$, such that $\Omega \subset M$ and M is retractable to $\bar{\Omega}$ in N.

The genus *g* of Ω is defined to be the genus of M, and M is called an extended manifold of Ω.

Definition 3.2 Let Ω be an ergodic set of *v*.

1). A saddle point $p \in \Omega$ of *v* is an Ω–interior saddle if there are four orbits connecting *p* in Ω;

2). A saddle point $p \in \Omega$ of *v* is an Ω–boundary saddle if there are only three orbits connecting *p* in Ω, and

3). A saddle point $p \in \Omega$ of *v* is an Ω–exterior saddle point if there are only two orbits connecting *p* in Ω.

The following theorem characterizes the topological structure of ergodic sets.

Theorem 3.1 *Let M be a compact manifold with or without boundary and with genus $k \geq 1$. Let $v \in D^r(TM)$ ($r \geq 1$) be regular and $\Omega \subset M$ be an ergodic set of v. Then Ω is homeomorphic to a pseudo-manifold with or without boundary with the genus g of Ω satisfying*

$$\begin{cases} 1 \leq g \leq k, \text{ if } \Omega \text{ orientable,} \\ 4 \leq g \leq k, \text{ if } \Omega \text{ nonorientable.} \end{cases} \tag{3.1}$$

Moreover, the Euler characteristic of Ω is given by

$$\chi(\Omega) = -s - \frac{b}{2}, \tag{3.2}$$

where s is the number of Ω–interior saddles and b is the number of Ω–boundary saddles of v in Ω. Consequently, the genus of Ω is given by

$$g = \begin{cases} 1 + \frac{1}{2}(s + \frac{1}{2}b - r), \text{ if } \Omega \text{ orientable,} \\ 2 + s + \frac{1}{2}b - r, \quad \text{ if } \Omega \text{ nonorientable,} \end{cases} \tag{3.3}$$

where r is the number of connected components of the boundary of the extended manifold M_1.

When M is a torus with or without boundary, the topological structure of an ergodic set of M is simpler. We denote by $\bigvee_m S^1 \subset T^2$ the connected topological set which consists of m circle S^1 with exactly $m-1$ common points between them, and $\bigvee_m S^1$ encloses m closed disks in T^2.

Theorem 3.2 *Let M be a torus with or without boundary, and $v \in D_0^r(TM)$ $(r \geq 1)$ be regular. Let $\Omega \subset M$ be an ergodic set of v. Then*

1). *Ω is a pseudo-torus (genus $g = 1$) with or without boundary;*

2). *each connected component of $\partial\Omega$ is homeomorphic to $\bigvee_m S^1$ $(m \geq 1)$, and has exactly two saddle orbits in the interior of Ω, int Ω; and*

3). *int Ω does not contain saddle points of v.*

4 Structural instability and block stability on 2-Manifolds with nonzero genus

The structural classification theorem and the structure of the ergodic sets on tori suggest the introduction of a new concept called block stability of divergence-free vector fields on a torus T^2.

Definition 4.1 *Let $M = T^2$. A divergence-free vector field $v \in D^r(TM)$ is called a basic vector field if v is regular and M can be decomposed into blocks as $M = \Omega \cup_{i=1}^K A_i$ with $\Omega \cap A_i = \emptyset$, $A_i \cap A_j = \emptyset$ such that*

a) *Each $A_i \subset M$ is an open invariant submanifold of v with zero genus, and Ω is a compact invariant submanifold of v of genus one;*

b) *The restriction of v on each A_i is a self-connection vector field, and v has exactly one saddle point on each ∂A_i.*

The we have the following main theorem of this article.

Theorem 4.1 *For each basic vector field $v \in D^r(TM)$ having block decomposition $M = \Omega \cup_{i=1}^K A_i$, there is a neighborhood $\mathcal{O} \subset D^r(TM)$ of v such that*

1. *each $v_1 \in \mathcal{O}$ has a block decomposition $M = \Omega^{(1)} \cup_{i=1}^K A_i^{(1)}$ with $\Omega^{(1)}$ and $A_i^{(1)}$ $(1 \leq i \leq K)$ homeomorphic to Ω and A_i respectively. Namely, the block structure of v is stable;*

2. *$v\big|_{A_i}$ is topologically equivalent to $v_1\big|_{A_i^{(1)}}$ for any $v_1 \in \mathcal{O}$; and*

3. *$v\big|_\Omega$ and $v_1\big|_{\Omega^{(1)}}$ are either periodic or ergodic.*

Furthermore, the set of all basic vector fields is open and dense in $D^r(TM)$.

Thanks to this block decomposition, it is easy to see that no divergence-free vector field on tori is structurally stable with divergence-free vector field pertubations, due to the presence of ergodic sets.

5   Hamiltonian dynamics on two-dimensional symplectic manifolds

According to the Hodge decomposition, when $\beta_1(M) \neq 0$, a divergence-free vector field consists of both the Hamiltonian and the harmonic parts. Therefore, the results and theory in Hamiltonian dynamics do not seem to be directly applicable to divergence-free vector fields.

On the other hand, thanks to the Liouville-Arnold invariant tori theorem, there are no ergodic sets for a regular Hamiltonian vector field for a Hamiltonian vector field on a two-dimensional compact symplectic manifold M. Therefore we obtain

Theorem 5.1 *A Hamiltonian H is structurally stable under perturbations of Hamiltonian fileds if and only if 1). $dH^\#$ is regular, and 2). all saddle points are self-connected. Moreover, the set of structurally stable Hamiltonian vector fields is open and dense in $\mathcal{H}^r(TM)$ ($r \geq 1$). Here $\mathcal{H}^r(TM)$ is the space of all C^r Hamiltonian vector fields on M.*

In view of this result for Hamiltonian vector fields, it is easy to see that the instability of an divergence-free vector field is due to the presence of ergodic sets caused by the harmonic perturbations.

References

1. M. GHIL, T. MA, AND S. WANG , *Structural bifurcation of 2-d incompressible flows*, Archive Rational Mechanics and Analysis, submitted, (1999).

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2. T. MA AND S. WANG, *Structural classification and stability of divergence-free vector fields*, Nonlinearity, revised, (1998).

3. ——, *Block structure and block stability of two-dimensional incompressible flows*, submitted, (1998).

4. ——, *Dynamics of incompressible vector fields*, Applied Math. Lett., 12 (1999), pp. 39–42.

5. ——, *The geometry of stream lines of steady states of the navier-stokes equations*, Contemporary Mathematics, Amer. Math. Soc., 238 (1999).

6. ——, *Global Structure of 2-D Incompressible Flows*, Mathematical Modelling and Numerical Analysis, Spacial Issue Dedicated to Roger Temam on the Occasion of his 60th Birthday, submitted, (1999).

7. J. PALIS AND W. DE MELO, *Geometric theory of dynamical systems*, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

8. M. PEIXOTO, *Structural stability on two dimensional manifolds*, Topology, 1 (1962), pp. 101–120.

CLASSIFICATION OF TRANSLATION MAPS OF TORI BY TOPOLOGICAL CONJUGACY

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Let $f_{r_1 \cdots r_n}$ and $f_{s_1 \cdots s_n}$ be two translation maps of torus T^n, $n \geq 1$. Then $f_{r_1 \cdots r_n}$ and $f_{s_1 \cdots s_n}$ are topologically conjugate if and only if there exists an integral square matrix A of order n such that its determinant $|A| = 1$ or -1, and $(r_1, \cdots, r_n)^* A = (s_1, \cdots, s_n)^*$.

1 Introduction

In the theory of dynamical system, the dynamical systems on tori T^n ($n \geq 1$) occupy a specific important position. For example, in order to explain the concepts of almost periodic motion and non-chaotic transitive map, one usually need to know the irrational rotations of the circle S^1 ($\approx T^1$). By giving a flow on torus T^2 which has a singular point, Stepanov has shown that the minimal attractive center and the set of central motions of a flow can be distinct. Markov constructed a C^1 differentiable (but not analytic) flow on T^2, whose minimal set is the whole torus, but whose most of motions are not Liapunov stable (see [1]). By constructing analytic flows on T^2 which are also minimal,

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and also contain Liaponov unstable motions, Ding [2] and Mai [3] obtained answers of the Birkhoff's conjecture respectively.

The above example of Stepanov, Markov, Ding and Mai are all continuous flows on T^2, which are all obtained from uniform continuous flows on T^2 by changing their velocities at different points in distinct sorts of ways. In this paper we will consider uniform discrete flows on torus T^n for any dimension $n \geq 1$. On T^n, uniform discrete flows are generated by translation maps. We will discuss the classification problem of all translation maps of tori by topological conjugacy. Our main result is the following theorem.

Theorem 11 *Let $f_{r_1 \cdots r_n}$ and $f_{s_1 \cdots s_n}$ be two translation maps of torus T^n to itself, $n \geq 1$. Then $f_{r_1 \cdots r_n}$ and $f_{s_1 \cdots s_n}$ are topologically conjugate if and only if there exists an integral square matrix A of order n such that its determinant $|A| = 1$ or -1, and $(r_1, \cdots, r_n)^* A = (s_1, \cdots, s_n)^*$.*

2 Translation Maps of Tori

Denote by $\mathbf{Z}$, $\mathbf{Z}_+$ and $\mathbf{N}$ the sets of integers, nonnegative integers and natural numbers, respectively. Let $\mathbf{R}$ be the real axis, and $\mathbf{R}^n$ be the Euclidean n-space with Euclidean metric d, $n \geq 1$. Then $\mathbf{R}^n$ is also a vector space, and is an additive group, and $\mathbf{Z}^n$ is a subgroup on $\mathbf{R}^n$.

Let $T^n = \mathbf{R}^n/\mathbf{Z}^n$ be the quotient group, with the quotient topology. Then T^n is an analytic manifold with the analytic structure induced by $\mathbf{R}^n$. T^n is called the *standard n-torus*. A space homeomorphic to T^n is called an *n-torus*. Note that T^1 is homeomorphic to the unit circle S^1.

Let $p_n : \mathbf{R}^n \to T^n$ ($= \mathbf{R}^n/Z^n$) be the natural projection. For any $(t_1, \cdots, t_n) \in \mathbf{R}^n$, write $p_n(t_1, \cdots, t_n)$ as $(t_1, \cdots, t_n)^*$. Then $(t_1, \cdots, t_n)^* = \{(t_1 + i_1, \cdots, t_n + i_n) : (i_1, \cdots, i_n) \in \mathbf{Z}^n\}$, which both is a subset of $\mathbf{R}^n$, and is a point in T^n.

For any $(r_1, \cdots, r_n) \in \mathbf{R}^n$, define $F_{r_1 \cdots r_n} : \mathbf{R}^n \to \mathbf{R}^n$ and $f_{r_1 \cdots r_n} : T^n \to T^n$, which are called *translation maps* of $\mathbf{R}^n$ and of T^n respectively, by, for all $(t_1, \cdots, t_n) \in \mathbf{R}^n$,

$$F_{r_1 \cdots r_n}(t_1, \cdots, t_n) = (t_1 + r_1, \cdots, t_n + r_n), \tag{2.1}$$

$$f_{r_1 \cdots r_n}((t_1, \cdots, t_n)^*) = (t_1 + r_1, \cdots, t_n + r_n)^*. \tag{2.2}$$

The classification problem of translation maps of $\mathbf{R}^n$ by topological conjugacy is trivial. In fact, we have

Proposition 3 *Let $\{(r_1, \cdots, r_n), (s_1, \cdots, s_n)\} \subset \mathbf{R}^n - \{(0, \cdots, 0)\}$. Then $F_{r_1 \cdots r_n}$ and $F_{s_1 \cdots s_n}$ are topologically conjugate. Concretely, arbitrarily choose*

an invertible real square matrix B of order n such that $(r_1, \cdots, r_n) \cdot B = (s_1, \cdots, s_n)$, and define a linear transformation $\xi_B : \mathbf{R}^n \to \mathbf{R}^n$ by

$$\xi_B(t_1, \cdots, t_n) = (t_1, \cdots, t_n) \cdot B, \qquad \text{for any } (t_1, \cdots, t_n) \in \mathbf{R}^n. \qquad (2.3)$$

Then $\xi_B F_{r_1 \cdots r_n} = F_{s_1 \cdots s_n} \xi_B$.

However, the classification of translation maps of torus T^n is rather complex.

3 Linear Transformations of Tori

For any $V, W \subset \mathbf{R}^n$, $(a_1, \cdots, a_n) \in \mathbf{R}^n$ and $X, Y \subset T^n$, write

$$V + W = \{(r_1 + s_1, \cdots, r_n + s_n) : (r_1, \cdots, r_n) \in V, \text{ and } (s_1, \cdots, s_n) \in W\},$$

$$V + (a_1, \cdots, a_n) = \{(r_1 + a_1, \cdots, r_n + a_n) : (r_1, \cdots, r_n) \in V\},$$

$$X + Y = \{(r_1 + s_1, \cdots, r_n + s_n)^* : (r_1, \cdots, r_n)^* \in X, \text{ and } (s_1, \cdots, s_n)^* \in Y\}.$$

Denote by $M_n(\mathbf{R})$ (resp. $M_n(\mathbf{Z})$) the set of all real (resp. integral) square matrixes of order n. For any $B \in M_n(\mathbf{R})$ and any $V \subset \mathbf{R}^n$, write

$$VB = V \cdot B = \{(t_1, \cdots, t_n) \cdot B : (t_1, \cdots, t_n) \in V\}.$$

For any $A \in M_n(\mathbf{Z})$, we can define a map $h_A : T^n \to T^n$ by

$$h_A((t_1, \cdots, t_n)^*) = ((t_1, \cdots, t_n) \cdot A)^*, \qquad \text{for all } (t_1, \cdots, t_n) \in \mathbf{R}^n. \qquad (3.4)$$

Such an h_A is called a *linear transformation* (or *linear map*) of T^n. Note that h_A is continuous. Obviously, we have

Lemma 3 *Let $A \in M_n(\mathbf{Z})$. Then h_A is onto if and only if $|A| \neq 0$.*

Lemma 4 *Let $A \in M_n(\mathbf{Z})$, and $k \in \mathbf{N}$. If $|A| = k$ or $-k$, then h_A is a covering map of k-fold.*

Lemma 5 *Let $A \in M_n(\mathbf{Z})$. Then the following six properties are equivalent:*

(a) *the determinant $|A| = 1$ or -1.*

(b) *$|A| \neq 0$, and the inverse matrix $A^{-1} \in M_n(\mathbf{Z})$.*

(c) *$Z^n \cdot A = Z^n$.*

(d) *$(r_1, \cdots, r_n)^* \cdot A = ((r_1, \cdots, r_n) \cdot A)^*$ for some point $(r_1, \cdots, r_n) \in \mathbf{R}^n$.*

(e) *$(r_1, \cdots, r_n)^* \cdot A = ((r_1, \cdots, r_n) \cdot A)^*$ for all point $(r_1, \cdots, r_n) \in \mathbf{R}^n$.*

(f) *h_A is a homeomorphism.* □

4 Proof of Theorem 1

Now we give the proof of Theorem 1.

Proof. ($\Longleftarrow$) If there exists an integral square matrix A of order n such that $|A| = 1$ or -1, and $(r_1, \cdots, r_n)^* A = (s_1, \cdots, s_n)^*$, then, by Lemma 3, h_A is a homeomorphism, where h_A is defined as in (4), and for any $(t_1, \cdots, t_n) \in \mathbf{R}^n$, we have

$$h_A f_{r_1 \cdots r_n}((t_1, \cdots, t_n)^*) = h_A((t_1 + r_1, \cdots, t_n + r_n)^*)$$
$$= ((t_1 + r_1, \cdots, t_n + r_n) \cdot A)^* = ((t_1, \cdots, t_n) \cdot A)^* + ((r_1, \cdots, r_n) \cdot A)^*$$
$$= h_A((t_1, \cdots, t_n)^*) + (s_1, \cdots, s_n)^* = f_{s_1 \cdots s_n} h_A((t_1, \cdots, t_n)^*).$$

This implies that h_A is a topological conjugacy from $f_{r_1 \cdots r_n}$ to $f_{s_1 \cdots s_n}$.

($\Longrightarrow$) If $f_{r_1 \cdots r_n}$ and $f_{s_1 \cdots s_n}$ are topologically conjugate, then there exists a homeomorphism $g : T^n \to T^n$ such that $g f_{r_1 \cdots r_n} = f_{s_1 \cdots s_n} g$. Let $O_n = (0, \cdots, 0)$ be the origin of $\mathbf{R}^n$. Suppose $g(O_n^*) = (-a_1, \cdots, -a_n)^*$. Let $h = f_{a_1 \cdots a_n} g$. Then h is also a homeomorphism from T^n to T^n, and $h(O_n^*) = O_n^*$. Noting $h f_{r_1 \cdots r_n} = f_{a_1 \cdots a_n} g f_{r_1 \cdots r_n} = f_{a_1 \cdots a_n} f_{s_1 \cdots s_n} g = f_{s_1 \cdots s_n} f_{a_1 \cdots a_n} g = f_{s_1 \cdots s_n} h$, we see that h is also a topological conjugacy from $f_{r_1 \cdots r_n}$ to $f_{s_1 \cdots s_n}$. The natural projection $p_n : \mathbf{R}^n \to T^n$ is a covering map. Since $\mathbf{R}^n$ is simply connected, there exists a unique lift $H : \mathbf{R}^n \to \mathbf{R}^n$ of $h p_n : \mathbf{R}^n \to T^n$ relative to p_n such that

$$H(O_n) = O_n, \tag{4.5}$$

see [4, p.230]. Of course, the translation map $F_{r_1 \cdots r_n} : \mathbf{R}^n \to \mathbf{R}^n$ is also a lift of $f_{r_1 \cdots r_n} p_n : \mathbf{R}^n \to T^n$ relative to p_n. Therefore, we have the following commutative diagram.

Similarly, there is a unique lift $H' : \mathbf{R}^n \to \mathbf{R}^n$ of $h^{-1} p_n$ relative to p_n such that $H'(O_n) = O_n$. So $H'H : \mathbf{R}^n \to \mathbf{R}^n$ is a lift of $p_n = h^{-1} h p_n : \mathbf{R}^n \to T^n$ relative to p_n, which satisfies $H'H(O_n) = O_n$. Noting the identity map $id : \mathbf{R}^n \to \mathbf{R}^n$ is the unique lift of p_n relative to p_n itself which preserves O_n to be fixed, we have $H'H = id$. Analogously, we also have $HH' = id$. This implies that $H : \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism, and $H' = H^{-1}$.

For any $(t_1, \cdots, t_n) \in \mathbf{R}^n$, since

$$p_n H((t_1, \cdots, t_n)^*) = h p_n((t_1, \cdots, t_n)^*) = h((t_1, \cdots, t_n)^*)$$

is only a point in T^n, we have $H((t_1, \cdots, t_n)^*) \subset (H(t_1, \cdots, t_n))^*$. Conversely, $H'((H(t_1, \cdots, t_n))^*) \subset (H'H(t_1, \cdots, t_n))^* = (t_1, \cdots, t_n)^*$ also holds. Hence

$$H((t_1, \cdots, t_n)^*) = (H(t_1, \cdots, t_n))^*.$$

Particularly, taking $(t_1, \cdots, t_n) = O_n$, we obtain

$$H(\mathbf{Z}^n) = \mathbf{Z}^n. \qquad (4.6)$$

For any $(i_1, \cdots, i_n) \in \mathbf{Z}^n$, $(t_1, \cdots, t_n) \in \mathbf{R}^n$ and any arc L in $\mathbf{R}^n$ with endpoints being O_n and $(t_1, \cdots, t_n)$, it follows from $p_n H(L + (i_1, \cdots, i_n)) = hp_n(L + (i_1, \cdots, i_n)) = hp_n(L) = p_n H(L)$ and the continuity that $H(L + (i_1, \cdots, i_n)) = H(L) + H(i_1, \cdots, i_n)$, which implies

$$H((t_1, \cdots, t_n) + (i_1, \cdots, i_n)) = H(t_1, \cdots, t_n) + H(i_1, \cdots, i_n). \qquad (4.7)$$

Thus $H|\mathbf{Z}^n : \mathbf{Z}^n \to \mathbf{Z}^n$ is a linear map (but, in general, H itself is not linear), and hence there exists $A_H \in M_n(\mathbf{Z})$ such that

$$H(i_1, \cdots, i_n) = (i_1, \cdots, i_n) \cdot A_H, \qquad \text{for all } (i_1, \cdots, i_n) \in \mathbf{Z}^n. \qquad (4.8)$$

By (6), (8) and Lemma 3 we know $|A_H| = 1$, or -1.
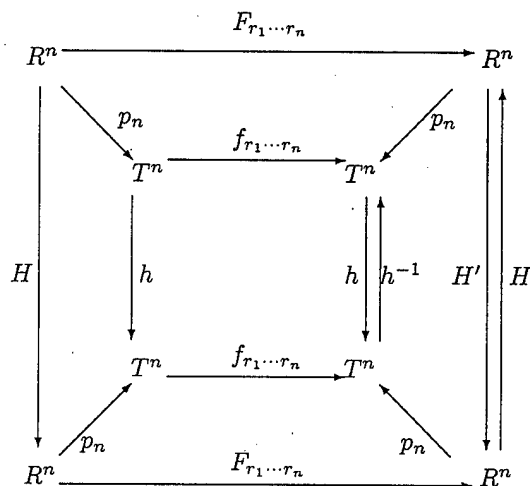
Fig.1. The commutative diagram

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¿From the above commutative diagram we get $p_n H(r_1, \cdots, r_n) = h p_n (r_1, \cdots, r_n) = h((r_1, \cdots, r_n)^*) = h f_{r_1 \cdots r_n}(O_n^*) = f_{s_1 \cdots s_n} h(O_n^*) = f_{s_1 \cdots s_n}(O_n^*) = (s_1, \cdots, s_n)^*$. Hence there exists $(b_1, \cdots, b_n) \in \mathbf{Z}^n$ such that

$$H(r_1, \cdots, r_n) = (s_1, \cdots, s_n) + (b_1, \cdots, b_n). \tag{4.9}$$

For any $t \in [0, \infty)$, write $v_t = (t r_1, \cdots, t r_n)$. Then $v_1 = (r_1, \cdots, r_n)$, $v_0 = H(v_0) = O_n$, and $v_t = t v_1$. From the above commutative diagram we also obtain

$$\begin{aligned}
p_n H(v_t + v_1) &= h p_n(v_t + (r_1, \cdots, r_n)) = h p_n F_{r_1 \cdots r_n}(v_t) \\
&= h f_{r_1 \cdots r_n} p_n(v_t) = f_{s_1 \cdots s_n} h p_n(v_t) = f_{s_1 \cdots s_n} p_n H(v_t) \\
&= p_n F_{s_1 \cdots s_n} H(v_t) = p_n(H(v_t) + (s_1, \cdots, s_n)).
\end{aligned} \tag{4.10}$$

Thus there exists $(b_{t1}, \cdots, b_{tn}) \in \mathbf{Z}^n$ such that

$$H(v_t + v_1) = H(v_t) + (s_1, \cdots, s_n) + (b_{t1}, \cdots, b_{tn}). \tag{4.11}$$

Since $\mathbf{Z}^n$ is a discrete subset of $\mathbf{R}^n$, by the continuity, $(b_{t1}, \cdots, b_{tn})$ is independent from t. In addition, by (9) and (5) we have $(b_{01}, \cdots, b_{0n}) = (b_1, \cdots, b_n)$. Thus

$$(b_{t1}, \cdots, b_{tn}) = (b_1, \cdots, b_n), \qquad \text{for all } t \in [0, \infty). \tag{4.12}$$

Write $w_1 = (s_1 + b_1, \cdots, s_n + b_n)$. Then (11) and (12) yield $H(v_{t+1}) = H(v_t) + w_1$, which leads to

$$H(v_{t+k}) = H(v_t) + k w_1, \qquad \text{for all } t \in [0, \infty) \text{ and } k \in \mathbf{N}.$$

Particularly, since $H(v_0) = v_0 = O_n$, we have $H(v_k) = k w_1$, i.e.

$$H(k r_1, \cdots, k r_n) = (k(s_1 + b_1), \cdots, k(s_n + b_n)), \qquad \text{for all } k \in \mathbf{N}. \tag{4.13}$$

Suppose $(r_1, \cdots, r_n) \cdot A_H = (q_1, \cdots, q_n)$. Then

$$(k r_1, \cdots, k r_n) \cdot A_H = (k q_1, \cdots, k q_n). \tag{4.14}$$

If $(q_1, \cdots, q_n) \neq (s_1 + b_1, \cdots, s_n + b_n)$, then from (13) and (14) we get

$$\lim_{k \to \infty} d(H(k r_1, \cdots, k r_n), (k r_1, \cdots, k r_n) \cdot A_H) = \infty. \tag{4.15}$$

On the other hand, consider the unit cube I^n in $\mathbf{R}^n$, where $I = [0, 1]$. Let $\beta = \max\{d(H(t_1, \cdots, t_n), (t_1, \cdots, t_n) \cdot A_H) : (t_1, \cdots, t_n) \in I^n\}$. Then $\beta < \infty$.

For any $(x_1, \cdots, x_n) \in \mathbf{R}^n$, take $(t_1, \cdots, t_n) \in I^n$ and $(i_1, \cdots, i_n) \in \mathbf{Z}^n$ such that $(x_1, \cdots, x_n) = (t_1 + i_1, \cdots, t_n + i_n)$. Then by (7) and (8) we get

$$d(H(x_1, \cdots, x_n), \ (x_1, \cdots, x_n) \cdot A_H)$$
$$= d(H(t_1, \cdots, t_n) + H(i_1, \cdots, i_n), \ (t_1, \cdots, t_n) \cdot A_H + (i_1, \cdots, i_n) \cdot A_H)$$
$$= d(H(t_1, \cdots, t_n), \ (t_1, \cdots, t_n) \cdot A_H)$$
$$\leq \beta < \infty, \qquad\qquad \text{for all} \ (x_1, \cdots, x_n) \in \mathbf{R}^n, \qquad (4.16)$$

which contradicts (15). Thus we must have $(q_1, \cdots, q_n) = (s_1 + b_1, \cdots, s_n + b_n)$, and hence, by Lemma 3, $(r_1, \cdots, r_n)^* \cdot A_H = ((r_1, \cdots, r_n) \cdot A_H)^* = (q_1, \cdots, q_n)^* = (s_1, \cdots, s_n)^*$. Theorem 1 is proven. $\qquad\square$

Example 1 Suppose $n = 2$, $(r_1, r_2) = (\frac{6}{11}, \frac{5}{11})$, and $(s_1, s_2) = (\frac{10}{11}, \frac{5}{11})$. Take $A = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$. Then $|A| = 1$, and $(r_1, r_2) \cdot A - (s_1, s_2) \in \mathbf{Z}^2$. Thus, by Theorem 1, the translation maps $f_{r_1 r_2} : T^2 \to T^2$ and $f_{s_1 s_2} : T^2 \to T^2$ are topologically conjugate.

Example 2 Suppose $n = 3$, $(r_1, r_2, r_3) = (4\sqrt{3} + 4\sqrt{2}, \ \sqrt{3} - \sqrt{2}, \ -1/3)$, and $(s_1, s_2, s_3) = (8\sqrt{3}, \ -16\sqrt{2} + 2/3, \ 11\sqrt{3} - 5\sqrt{2} + 1/3)$. Take $A = \begin{pmatrix} 1 & -2 & 2 \\ 4 & 8 & 3 \\ 3 & 7 & 2 \end{pmatrix}$. Then $|A| = 1$, and $(r_1, r_2, r_3) \cdot A - (s_1, s_2, s_3) \in \mathbf{Z}^3$. By Theorem 1, the translation maps $f_{r_1 r_2 r_3} : T^3 \to T^3$ and $f_{s_1 s_2 s_3} : T^3 \to T^3$ are also topologically conjugate.

Example 3 If $r_1, \cdots, r_n$ are all rational numbers, and $\{s_1, \cdots, s_n\}$ contains at least an irrational number, then there is no $A \in M_n(\mathbf{Z})$ such that $((r_1, \cdots, r_n) \cdot A)^* = (s_1, \cdots, s_n)^*$, and hence, by Theorem 1, the translation maps $f_{r_1 \cdots r_n} : T^n \to T^n$ and $f_{s_1 \cdots s_n} : T^n \to T^n$ are not topologically conjugate.

More generally, for some $k \in \{1, \cdots, n\}$, if there is no $(a_1, \cdots, a_n) \in \mathbf{Z}^n$ such that $a_1 r_1 + \cdots + a_n r_n - s_k \in \mathbf{Z}$, then $f_{r_1 \cdots r_n}$ and $f_{s_1 \cdots s_n}$ are not topologically conjugate.

References

1. V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equations*, (Princeton Univ. Press, Princeton, 1960).
2. Tong-Ren Ding, *Acta Math. Sinica*, **24**, 64(1981).(in Chinese)
3. Jie-Hua Mai, *Chinese Science Bulletin*, **33**, 544(1988).
4. M. A. Armstrong, *Basic Topology*, (Springer-Verlag, New York, 1983).

EXISTENCE OF BIHARMONIC CURVES AND SYMMETRIC BIHARMONIC MAPS

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The existence of curves and symmetric maps with minimal total tension is proved.
Such curves and maps satisfy a class of fourth order differential equations.

1 Definitions of biharmonic maps and curves

Let N be a Riemannian manifold embedded into the Euclidean space R^m, $m \geq 2$, and Ω a smooth bounded domain in R^n, $n \geq 1$. Given maps $\varphi : \partial\Omega \to N$ and $\psi : \partial\Omega \to T_\varphi N$ (i.e., $\psi(x)$ is tangent to N at $\varphi(x)$ for $x \in \partial\Omega$), we look for an "optimal" map $u : \Omega \to N$ such that

$$u = \varphi, \quad \frac{\partial u}{\partial n} = \psi \text{ on } \partial\Omega, \tag{1.1}$$

where n is the exterior normal direction of $\partial\Omega$. In other words, we look for a "best" way to extend the boundary value φ with the prescribed normal derivative ψ. Typical examples of Ω and N are the unit ball and the unit sphere, respectively. In this case, $\psi : \partial\Omega \to T_\varphi N$ means $\varphi(x) \cdot \psi(x) = 0$ for all $|x| = 1$.

With the given Dirichlet data φ, the most natural extension is perhaps the harmonic map. Recall that a map $u : \Omega \to N$ is harmonic if and only if its tension field $T(u)$ vanishes. In terms of the second fundamental form A of $N \subset R^m$, $T(u)$ can be expressed as

$$T(u) \equiv \Delta u - A(u)(\nabla u, \nabla u), \tag{1.2}$$

where u is considered as a vector valued function from Ω to R^m, Δu is the ordinary Laplacian of u, ∇u is the gradient of u, and $A(u)(\nabla u, \nabla u)$ is understood as the trace of A.

However, with the normal derivative being prescribed, it is easy to see that a harmonic extension does not generally exist. In fact, it was shown in[6] that for *almost all* $\varphi : \partial\Omega \to N$, there is a unique energy minimizing harmonic extension $u : \Omega \to N$; therefore, $\frac{\partial u}{\partial n}$ has been determined by φ. In this paper, we seek an extension u of φ with $\frac{\partial u}{\partial n} = \psi$ that is as close to a harmonic map as possible. Specifically, we consider the *total tension* of u

$$\mathcal{T}(u) = \int_\Omega |T(u)|^2 dx, \tag{1.3}$$

and try to find u as a minimum of $\mathcal{T}$.

Since $A(u)$ is the projection of Δu in the normal space of N at u, we have

$$|T(u)|^2 = |\Delta u|^2 - |A(u)(\nabla u, \nabla u)|^2. \qquad (1.4)$$

Thus the natural class for the extensions is

$$\mathcal{C} = \left\{ u : u \in W^{2,2}(\Omega, N) \text{ and satisfies } (1.1) \right\}, \qquad (1.5)$$

where $W^{2,2}(\Omega, N)$ is the set of all $u : \Omega \to N \subset R^m$ with finite norm $\|u\|_{2,2}$, defined by

$$\|u\|_{2,2}^2 = \int_\Omega \left(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 \right) dx, \qquad (1.6)$$

Following the definition of Eells and Lemaire in [2], we will call a critical point of $\mathcal{T}(u)$ a *biharmonic map*, and when $n = 1$, a *biharmonic curve*. In 1986, Jiang[3] derived the first and second variation formulae of $\mathcal{T}$ and gave some examples of biharmonic maps, which include harmonic maps that are in $W^{2,2}$. However, there is no general existence result for biharmonic maps due to the fact that $\mathcal{T}$ is non-coercive. In Section 2 of this paper, we prove the existence of biharmonic curves. In Section 3, an existence result on axially symmetric biharmonic maps is proved.

Remark 1. A similar energy functional is $\int_\Omega |\nabla^2 u|^2 dx$ (or $\int_\Omega |\Delta u|^2 dx$), which is perhaps more interesting from an analytic point view. Chang, Wang and Yang [1] proved the partial regularity of critical points of $\int_\Omega |\Delta u|^2 dx$. Hardt, Mou and Wang[4] consider the partial regularity of minimizers of $\int_\Omega |\nabla^2 u|^2 dx$ under conditions different from that of [1]. Since these functionals are coercive, the existence of critical points and minimizers follows easily from direct method.

Remark 2. One might be interested in the path in $\mathcal{C}$ with least total curvature $\int_{u[-1,1]} |\kappa_g| ds$. However, such a path might not exist, or when it exists there might be infinitely many. Consider the case of plane curves. It is well known that if $\mathcal{C}$ contains a convex path, then $\int_{u[-1,1]} |\kappa_g| ds$ is constant for all convex curves (as they have same boundary conditions), and so each convex path is a minimum. While if $\mathcal{C}$ contains no convex path, then it could happen that no path would realize the infimum of the total curvature.

Remark 3. One might also be interested in the path of least total squared curvature $\int_{u[-1,1]} |\kappa_g|^2 ds$ of the Willmore type. This quantity might decrease to 0 as the length of $u[-1,1] \to \infty$. Therefore, unless we consider only paths with bounded length, the infimum 0 might never be realized. See [5] and the references therein.

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2 Existence of biharmonic curves

For biharmonic curves $u : [-1, 1] \to N$, the condition (1.1) and definitions (1.5) (1.6) become

$$\{u(-1), u(1), u'(-1), u'(1)\} = \{p_1, p_2, v_1, v_2\} \in N \times N \times T_{p_1} N \times T_{p_2} N \tag{2.7}$$

$$\mathcal{C} = \{u : u \in W^{2,2}([-1, 1], N) \text{ and satisfies } (2.7)\}, \tag{2.8}$$

$$\|u\|_{2,2}^2 = \int_{-1}^1 \left(|u|^2 + |u'|^2 + |u''|^2\right) dt. \tag{2.9}$$

The total tension of $u \in \mathcal{C}$ is

$$\mathcal{T}(u) = \int_{-1}^1 |T(u)|^2 dt = \int_{-1}^1 \left(|u''|^2 - |A(u)(u', u')|^2\right) dt. \tag{2.10}$$

We prove the following existence result.

Theorem 1. *Given boundary data as in (2.7) such that the admissible set $\mathcal{C} \neq \emptyset$, the total tension $\mathcal{T}(u)$ has a minimum in $\mathcal{C}$.*

The proof of this theorem is a standard direct method. The key ingredients are Lemmas 1, 2, which are proved later in this section. We conjecture that Lemmas 1, 2 continue to hold for $n \geq 2$, which would imply the existence of biharmonic maps.

For $u \in \mathcal{C}$ and $q > 0$, we denote $\mathcal{D}_q(u) = \int_{-1}^1 |u'|^q dt$. The following lemmas will be proved later.

Lemma 1. *For every $u \in \mathcal{C}$,*

$$|u(t) - p_1| + |u(t) - p_2| \leq \int_{-1}^1 |u'(s)| ds \leq \sqrt{2} \, \mathcal{D}_2(u)^{1/2}, \tag{2.11}$$

$$\mathcal{D}_2(u) \leq 2\left(|v_1|^2 + |v_2|^2\right) + 4\mathcal{T}(u), \tag{2.12}$$

$$\mathcal{D}_4(u) \leq 2\left(|v_1|^2 + |v_2|^2\right)^2 + 32\mathcal{T}(u)^2. \tag{2.13}$$

Lemma 2. *Suppose $u \in \mathcal{C}$ and $\mathcal{T}(u) \leq K$, then*

$$\|u\|_{2,2}^2 \leq M, \tag{2.14}$$

where M is a constant depending only on K, N and $\{p_1, p_2, v_1, v_2\}$.

Proof of Theorem 1. We use the direct method in calculus of variations. Suppose $\{u^k\}$ is a minimizing sequence such that $T(u^k) \to \inf_{u \in C} T(u)$ as $k \to \infty$. By Lemma 2, the sequence $\{u^k\}$ is bounded in $W^{2,2}$. Therefore, a subsequence exists and weakly converges to some $u \in C$. It is easy to see that $T(u)$ is lower semicontinuous with respect to the weak convergence in $W^{2,2}$. So u is a minimum. $\square$

Proof of Lemma 1. For $t \in [-1, 1]$, by the fundamental theorem of calculus, we have

$$|u(t) - p_1| + |u(t) - p_2| \tag{2.15}$$
$$\leq \int_{-1}^{t} |u'(s)| ds + \int_{t}^{1} |u'(s)| ds = \int_{-1}^{1} |u'(s)| ds$$
$$\leq \sqrt{2} \, \mathcal{D}_2(u)^{1/2}.$$

$$|u(t) - p_1| + |u(t) - p_2| \tag{2.16}$$
$$\leq \int_{-1}^{t} |u'(s)| ds + \int_{t}^{1} |u'(s)| ds = \int_{-1}^{1} |u'(s)| ds$$
$$\leq \sqrt{2} \, \mathcal{D}_2(u)^{1/2}.$$

This proves (2.11). Now using the fact that u' is a tangent vector, which implies that $u' \perp A(u)$, we have

$$|u'(t)|^2 = |v_1|^2 + \int_{-1}^{t} 2u'(s) u''(s) \, ds \tag{2.17}$$
$$= |v_1|^2 + \int_{-1}^{t} 2u'(s) T(u(s)) \, ds$$
$$\leq |v_1|^2 + 2 \int_{-1}^{t} |u'(s)| |T(u(s))| \, ds.$$

Similarly, we have

$$|u'(t)|^2 \leq |v_2|^2 + 2 \int_{t}^{1} |u'(s)| |T(u(s))| \, ds. \tag{2.18}$$

Averaging the above two estimates, we have

$$|u'(t)|^2 \leq \frac{1}{2} (|v_1|^2 + |v_2|^2) + \int_{-1}^{1} |u'(s)| |T(u(s))| \, ds \tag{2.19}$$

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Integrating (2.19) over $[-1, 1]$ and applying Schwarz's inequality to the integral, we get

$$\mathcal{D}_2(u) \leq \left(|v_1|^2 + |v_2|^2\right) + \frac{1}{2}\mathcal{D}_2(u) + 2\mathcal{T}(u). \tag{2.20}$$

Solving for $\mathcal{D}_2(u)$ gives (2.12).

Next squaring the both sides of (2.19), using that $(a+b)^2 \leq 2a^2 + 2b^2$, and then using that $\mathcal{D}_2(u) \leq \sqrt{2}\,\mathcal{D}_4(u)^{1/2}$, we have

$$\begin{aligned}
|u'(t)|^4 &\leq \frac{1}{2}\left(|v_1|^2 + |v_2|^2\right)^2 + 2\mathcal{D}_2(u)\,\mathcal{T}(u) \\
&\leq \frac{1}{2}\left(|v_1|^2 + |v_2|^2\right)^2 + 2\sqrt{2}\mathcal{D}_4(u)^{1/2}\,\mathcal{T}(u) \\
&\leq \frac{1}{2}\left(|v_1|^2 + |v_2|^2\right)^2 + \frac{1}{4}\mathcal{D}_4(u) + 8\mathcal{T}(u)^2
\end{aligned} \tag{2.21}$$

Integrating this estimate over $[-1, 1]$, we get

$$\mathcal{D}_4(u) \leq \left(|v_1|^2 + |v_2|^2\right)^2 + \frac{1}{2}\mathcal{D}_4(u) + 16\mathcal{T}(u)^2. \tag{2.22}$$

Solving for $\mathcal{D}_4(u)$, we get (2.13). $\square$

Proof of Lemma 2. Assuming $u \in \mathcal{C}$ and $\mathcal{T}(u) \leq K$. By Lemma 1, we see that $\|u\|_\infty$ is bounded in terms of K and $\{p_1, p_2, v_1, v_2\}$. It follows that $\|A(u)\|_\infty$ is bounded in terms of K and $\{p_1, p_2, v_1, v_2\}$ and N. Therefore by (2.10),

$$\mathcal{T}(u) \geq \int_{-1}^{1} |u''|^2 dt - \|A(u)\|_\infty\,\mathcal{D}_4(u), \tag{2.23}$$

which implies that

$$\int_{-1}^{1} |u''|^2 dt \leq \mathcal{T}(u) + \|A(u)\|_\infty\,\mathcal{D}_4(u). \tag{2.24}$$

By Lemma 1 again, $\mathcal{D}_4(u)$, and therefore $\int_{-1}^{1} |u''|^2 dt$, is bounded in terms of K, N and $\{p_1, p_2, v_1, v_2\}$. $\square$

3 Existence of symmetric biharmonic maps

In this section we consider the special case when $\Omega = B$, the unit ball in R^n and $N = S^n \in R^{n+1}$, $n \geq 2$. In this case the total Hessian of $u \in W^{2,2}(B, S^n)$ is

$$\mathcal{T}(u) = \int_B \left(|\Delta u|^2 - |\nabla u|^4\right) dx, \tag{3.25}$$

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and the Euler-Lagrange equation satisfied by a biharmonic map $u : B \to S^n$ is

$$\Delta^2 u + 2\nabla \left(|\nabla u|^2 \nabla u \right) + (3\Delta(\nabla u) \cdot \nabla u + |\Delta u|^2)u = 0. \qquad (3.26)$$

To derive this, we first get the Euler equation $\Delta^2 u + 2\nabla \left(|\nabla u|^2 \nabla u \right) = \lambda u$ with a Lagrange multiplier λ. Using the fact $1 = u \cdot u$, we find and simplify λ as

$$\lambda = \left(\Delta^2 u + 2\nabla \left(|\nabla u|^2 \nabla u \right) \right) \cdot u = - \left(3\Delta(\nabla u) \cdot \nabla u + |\Delta u|^2 \right). \qquad (3.27)$$

A map $u : B \to S^n$ is said to be *axially symmetric* if there is a map f: $S^{n-1} \to S^{n-1} \subset S^n$ and a function $\varphi : [0,1] \to [0,\pi]$ such that for $x \in B \backslash \{0\}$,

$$u(x) = (f(\theta) \, \sin\varphi(r), \, \cos \varphi(r)), \qquad (3.28)$$

where $r = |x|$ and $\theta = x/|x|$.

We assume that $f(\theta) : S^{n-1} \to S^{n-1}$ is a harmonic map, that is,

$$\Delta_\theta f + |\nabla_\theta f|^2 f = 0. \qquad (3.29)$$

As for φ, we assume that it satisfies the boundary conditions

$$\varphi(0) = \varphi'(0) = 0, \, \varphi(1) = a, \, \varphi'(1) = b, \qquad (3.30)$$

and that the radial function $\varphi(|x|) : B \to R$ belongs to the space $W^{2,2}(B)$.

We calculate

$$u_r = u_\varphi \varphi' = (f \cos\varphi, \, -\sin\varphi) \, \varphi'$$
$$u_{rr} = u_\varphi \varphi'' - u\varphi'^2$$
$$\Delta_\theta u = -|\nabla_\theta f|^2 (f \, \sin\varphi, \, 0)$$
$$|\nabla u|^2 = \varphi'^2 + \frac{|\nabla_\theta f|^2}{r^2} \sin^2\varphi$$

$$\Delta\varphi = \varphi'' + \frac{n-1}{r} \varphi'$$

$$\Delta u = u_\varphi \Delta\varphi - u\varphi'^2 - \frac{|\nabla_\theta f|^2}{r^2} (f \sin\varphi, 0)$$

$$T(u) = \Delta u + |\nabla u|^2 u = u_\varphi \left\lceil \Delta\varphi - \frac{|\nabla_\theta f|^2}{2r^2} \sin2\varphi \right\rceil$$

$$\mathcal{T}(u) = \int_B |\Delta\varphi - \frac{|\nabla_\theta f|^2}{2r^2} \sin2\varphi|^2 dx \qquad (3.31)$$

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Lemma 3. *If φ is a minimum of (3.31) satisfying (3.30), then the axially symmetric map u defined by (3.28) is a biharmonic map with boundary data*

$$u|_{\partial B} = (f(\theta)\sin a, \cos a), \quad \frac{\partial u}{\partial n}|_{\partial B} = (f(\theta)\cos a, -\sin a)b. \tag{3.32}$$

Proof. Suppose φ is a minimum of (3.31). Consider a variation φ_t of φ with $\frac{d}{dt}\varphi_t|_{t=0} = \eta \in C[0,1]$ with $\eta(0) = \eta(1) = 0$. Let u_t be defined as in (3.28) with φ_t. We have

$$\begin{aligned}
0 &= \frac{d}{dt}\mathcal{T}(u_t)|_{t=0} \tag{3.33}\\
&= 2\int_B \left(\Delta\varphi - \frac{|\nabla_\theta f|^2}{2r^2}\sin 2\varphi\right)\left(\Delta\eta - \frac{|\nabla_\theta f|^2}{r^2}\cos 2\varphi\ \eta\right)dx\\
&= 2\int_B \left[\Delta T(\varphi) - \frac{\alpha}{r^2}\cos 2\varphi\Delta\varphi + \frac{\beta}{4r^4}\sin 4\varphi\right]\eta dx,
\end{aligned}$$

where $T(\varphi) = \Delta\varphi - \frac{\alpha}{2r^2}\sin 2\varphi$, Δ is the Laplacian on $\varphi(|x|)$, and

$$\alpha = \frac{1}{|S^{n-1}|}\int_{S^{n-1}}|\nabla_\theta f|^2 d\sigma, \quad \beta = \frac{1}{|S^{n-1}|}\int_{S^{n-1}}|\nabla_\theta f|^4 d\sigma \tag{3.34}$$

are the average values of $|\nabla_\theta f|^2$ and $|\nabla_\theta f|^4$, respectively. Thus (3.33) implies that

$$\Delta T(\varphi) - \frac{\alpha}{r^2}\cos 2\varphi\Delta\varphi + \frac{\beta}{4r^4}\sin 4\varphi = 0. \tag{3.35}$$

It is not hard to see that this is the same as (2.3) for axially symmetric map. So a critical point φ of (3.31) defines biharmonic map.□

Theorem 2. *Suppose $n \geq 5$, $f: S^{n-1} \to S^{n-1}$ is harmonic, and a, b are two numbers, then there is an axially symmetric biharmonic map $u: B \to S^n$ as in (3.28).*

Proof. The method of proof is again the direct method in calculus of variations. By Lemma 3, it is equivalent to showing that (3.31) has a critical point. The key ingredient here is to show that if $\{\varphi^k\}$ is a minimizing sequence of (3.31), then $\int_B |\Delta\varphi^k|^2 dx$ will be bounded. Indeed, from (3.31) we get

$$\mathcal{T}(u)\,o number\ =\ \int_B |\Delta\varphi - \frac{|\nabla_\theta f|^2}{2r^2}\sin 2\varphi|^2 dx \tag{3.36}$$

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$$\geq \int_B \left\lceil \frac{1}{2}|\Delta\varphi|^2 - \left(\frac{|\nabla_\theta f|^2}{2r^2}\sin2\varphi\right)^2\right\rceil dx$$

$$\geq \int_B \frac{1}{2}|\Delta\varphi|^2 dx - \beta \int_B \sin^2\varphi \cos^2\varphi \, r^{n-5} dx,$$

where β is defined in (3.34). Since $n \geq 5$,

$$\int_B |\Delta\varphi|^2 dx \leq 2T(u) + 2\beta|S^{n-1}| \int_0^1 \sin^2\varphi \cos^2\varphi \, r^{n-5} dr \leq 2T(u) + \frac{2\beta|S^{n-1}|}{n-4}.$$

$$(3.37)$$

This implies that $\{\varphi^k(|x|)\}$ is bounded in $W^{2,2}(B, R^m)$ and so it has a subsequence weakly converging to some φ, which must be a minimum by lower semicontinuity of (3.31). $\square$

References

1. Chang, S-Y. A., L. Wang and P. Yang, A Regularity theory of biharmonic maps, *Communications on Pure and Applied Mathematics*, Vol. LII, 1113-1137 (1999).
2. Eells, J. and L. Lemaire, Selected topics in harmonic maps, CBMS 50, published by American Mathematical Society, 1978.
3. Jiang, G. Y., 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math.*, Series A, 7 (1986), No. 2, 130-144.
4. Hardt, R., L. Mou and C.Y. Wang, Regularity of Hessian minimizers, *Personal Communication*, 1998.
5. Linner, A., Unified representations of nonlinear splines, *Journal of Approximation Theory*, Vol. 84, No. 3, (1996) 315-350.
6. Mou, L., Uniqueness of energy minimizing maps for almost all smooth boundary data, *Indiana University Mathematics Journal*, 40, 1 (1991) 363-392.

STABILITY OF LARGE- SCALE LINEAR SYSTEM WITH TIME-DELAY [h]

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In this paper, the Lyapunov functional and vector Lyapunov function have been respectively used to study stability of large-scale linear systems with time-delay, some new sufficient conditions for stability of such system are obtained. Lastly, a example is given to illustrate and compare our results.

1 Introduction and lemma

Since high dimension system with time-delay is frequently encountered in various engineering systems, the study of Large-scale time-delay system has received considerable attention over the past years. Mori et al.(1981) derived a stability criterion using the comparison method. With the aid of the complex Lyapunov theorem Suh and Bien (1982), Hmamed (1986), and Wang and Song (1989) obtained a sufficient condition for the stability of large-scale systems. Lee et al. (1984) studied the stability problem of time-delay systems via generalized algebraic Riccati equations, Wang et al.(1991) gave a stability criterion using the Lyapunov theorem. Recently, Schoen et al.(1995) gave a Razumikhin stability theorem for uncertain large-scale systems, Nian (1998) obtained a sufficient condition for stability of large-scale interval coefficient system.

In this paper, a general systems

$$\dot{x}_i(t) = A_{ii}x_i(t) + \sum_{j=1,i\neq j}^{m} A_{ij}x_j(t) + \sum_{j=1}^{m} B_{ij}x_j(t - \tau_{ij}(t)), i = 1, 2, \cdots, m \quad (1.1)$$

will be studied. In section 2, a inequality and Lyapunov functional will be used to deduce some sufficient conditions for stability of systems (1). In section 3, the vector Lyapunov function will be used to obtain some criteria for stability of system (1). Lastly, some examples will be give to illustrate our results presented in this paper.

In the sequel, the following lemma will be used.

Lemma (Tokumaru et al.1987) Suppose $C = (c_{ij})_{m \times m}, C = (d_{ij})_{m \times m} \in R^{m \times m}$ are matrices, $x(t)$ is a solution of differential inequality

$$\dot{x}(t) \leq Cx(t) + D\bar{x}(t)$$

where $\bar{x}(t) = [\sup_{t-\tau \leq \theta \leq t} x_1(\theta), \sup_{t-\tau \leq \theta \leq t} x_2(\theta), \cdots, \sup_{t-\tau \leq \theta \leq t} x_m(\theta)]^T$, if $D \geq 0$ and $-(C+D)$ is a M-matrix, then there are exist constant $r > 0$ and $k > 0$, such that

$$x(t) \leq ke^{-rt}, \quad \forall t \geq 0.$$

2 Method of Lyapunove functional

Considering large-scale time-delay system

$$\dot{x}_i(t) = A_{ii}x_i(t) + \sum_{j=1, i \neq j}^{m} A_{ij}x_j(t) + \sum_{j=1}^{m} B_{ij}x_j(t - \tau_{ij}), \quad i = 1, 2, \cdots, m, \quad (2.2)$$

where $x_i(t) \in R^{n_i} (i = 1, 2, \cdots, m)$ is the state vector, $A_{ij}, B_{ij} \in R^{n_i \times n_j} (i, j = 1, 2, \cdots m)$ are constant real matrices, $\tau_{ij} > 0 (i, j = 1, 2, \cdots m)$ denote time-delays in the inter-connections, and $\tau_{ij} \leq \tau$.

Let $V(x) = \sum_{i=1}^{m} [x_i^T(t)P_i x_i(t) + \sum_{j=1}^{m} \int_{t-\tau_{ij}} x_j(s)B_{ij}^T R_i B_{ij}x_j(s)ds]$, where P_i is a solution of the matrix equation $A_{ii}^T P_i + P_i A_{ii} = -Q_i$ and Q_i are given symmetric positive definite matrices. Then

$$\frac{dV(x)}{dt}\Big|_{(2)} \leq \sum_{i=1}^{m} \{x_i^T(t)[A_{ii}^T P_i + P_i A_{ii}]x_i(t)$$

$$+ \sum_{j=1, i \neq j}^{m} [x_j^T(t)A_{ij}^T P_i x_i(t) + x_i^T(t)P_i A_{ij}x_j(t)]$$

$$+ \sum_{j=1}^{m} x_j^T(t - \tau_{ij})B_{ij}^T R_i B_{ij}x_j(t - \tau_{ij}) + \sum_{j=1}^{m} x_i^T(t)P_i R_i^{-1} P_i x_i(t)$$

$$+ \sum_{j=1}^{m} x_j^T(t)B_{ij}^T R_i B_{ij}x_j(t) - \sum_{j=1}^{m} x_j^T(t - \tau_{ij})B_{ij}^T R_i B_{ij}x_j(t - \tau_{ij})\}$$

$$\leq \sum_{i=1}^{m} \{x_i^T(t)[A_{ii}^T P_i + P_i A_{ii}]x_i(t)$$

$$+ \sum_{j=1, i \neq j}^{m} [x_j^T(t)A_{ij}^T P_i x_i(t) + x_i^T(t)P_i A_{ij}x_j(t)]$$

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$$+ \sum_{j=1}^{m} x_j^T(t) B_{ij}^T R_i R_{ij} x_j(t) + \sum_{j=1}^{m} x_i^T(t) P_i R_i^{-1} P_i x_i(t) \}$$

$$= \sum_{i=1}^{m} x_i^T(t) [-Q_i + m P_i R_i^{-1} P_i + \sum_{j=1}^{m} B_{ji}^T R_j B_{ji}] x_i(t)$$

$$+ \sum_{i=1}^{m} \sum_{j=1, i \neq j}^{m} [x_j^T(t) A_{ij}^T P_i x_i(t) + x_i^T(t) P_i A_{ij} x_j(t)].$$

Denote $W_i = -Q_i + m P_i R_i^{-1} P_i + \sum_{j=1}^{m} B_{ji}^T R_j B_{ji}, u_{ij} = -\frac{1}{2}(\|P_i A_{ij}\| + \|P_j A_{ji}\|),$ $i, j = 1, 2, \cdots, m,$

$$U = \begin{pmatrix} \lambda_{max}(W_1) & u_{12} & \cdots & u_{1m} \\ u_{12} & \lambda_{max}(W_1) \cdots & u_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ u_{1m} & u_{2m} & \cdots \lambda_{max}(W_m) \end{pmatrix},$$

then, it follows

$$\frac{dV(x)}{dt}\Big|_{(1)} \leq \sum_{i=1}^{m} x_i^T(t) W_i x_i(t) + \sum_{i=1}^{m} \sum_{j=1, i \neq j}^{m} [x_j^T(t) A_{ij}^T P_i x_i(t) + x_i^T(t) P_i A_{ij} x_j(t)] \tag{2.3}$$

Theorem 1 If $\lambda_{max}(A_{ii}) < 0$ and there exist symmetric positive definite matrices $R_i (i = 1, 2, \cdots, m)$ such that symmetric matrix U is negative definite, then system (2) is asymptotically stable.

Proof From (3), we have

$$\frac{dV_i(x)}{dt}\Big|_{(2)} \leq \sum_{i=1}^{m} \lambda_{max}(W_j) \|x_i(t)\|^2 + \sum_{i=1}^{m} \sum_{j=1, j \neq i}^{m} u_{ij} \|x_i(t)\| \|x_j(t)\|$$

$$= z^T U z < 0, \quad z = (\|x_1(t)\|, \cdots, \|x_m(t)\|)^T.$$

Therefore, system (2) is asymptotically stable.

Considering the following special system

$$\dot{x}_i(t) = A_{ii} x_i(t) + \sum_{j=1}^{m} B_{ij} x_j(t - \tau_{ij}), i = 1, 2, \cdots, m. \tag{2.4}$$

Theorem 2 If $\lambda_{max}(A_{ii}) < 0$ and there exist symmetric positive definite matrices R_i such that the following matrices inequality

$$-Q_i + m P_i R_i^{-1} P_i + \sum_{j=1}^{m} B_{ji}^T R_j B_{ji} < 0, \ i = 1, 2, \cdots, m$$

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are satisfied, then the system (4) is asymptotically stable.

Proof From the above discussing, we have

$$\frac{dV(x)}{dt}|_{(4)} \le x_i^T(t)[A_{ii}^T P_i + P_i A_{ii} + m P_i R_i^{-1} P_i + \sum_{j=1}^m B_{ji}^T R_j B_{ji}]x_i(t) < 0$$

This completes proof of theorem 2.

3 Method of vector Lyapunov function

In this section, the rank decomposition for the interconnection matrices $A_{ij} \in R^{n_i \times n_j}$, i.e.,

$$A_{ij} = A_{ij1} A_{ij2}$$

where $A_{ij1} \in R^{n_i \times r_{ij}}, A_{ij1} \in R^{r_{ij} \times n_j}, r_{ij} \le \text{rank}(A_{ij})$.

Considering time-varying delay system

$$\dot{x}_i(t) = A_{ii}x_i(t) + \sum_{j=1, i \ne j}^m A_{ij}x_j(t) + \sum_{j=1}^m B_{ij}x_j(t - \tau_{ij}(t)) \quad i = 1, 2, \cdots, m \quad (3.5)$$

Suppose $P_1, P_2, \cdots, P_m$ are symmetric positive definite matrices, let $V_i(x) = x_i^T(t)P_i x_i(t)$, we have

$$\lambda_{min}(P_i)x_i^T(t)x_i(t) \le V_i(x) \le \lambda_{max}(P_i)x_i^T(t)x_i(t), \quad\quad (3.6)$$

$$\frac{dV_i(x)}{dt}|_{(6)} \le x_i^T(t)[A_{ii}P_i + P_i A_{ii} + P_i B_{ii1} B_{ii1}^T P_i + \sum_{j=1, i \ne j} P_i A_{ij1} A_{ij1}^T P_i$$

$$+ P_i B_{ij1} B_{ij1}^T P_i]x_i(t) + \sum_{j=1, i \ne j} x_j^T(t)A_{ij2}^T A_{ij2}x_j(t)$$

$$+ \sum_{i=1}^m x_j^T(t - \tau_{ij}(t))B_{ij2}^T B_{ij2}x_j(t - \tau_{ij}(t)).$$

Denote $U_{ii} = A_{ii}P_i + P_i A_{ii} + P_i B_{ii1} B_{ii1}^T P_i + \sum_{j=1, i \ne j}^m [P_i A_{ij1} A_{ij1}^T P_i + P_i B_{ij1} B_{ij1}^T P_i]$,

$$U_{ij} = A_{ij2}^T A_{ij2}(i \ne j), \quad V_{ij} = B_{ij2}^T B_{ij2}, (i, j = 1, 2, \cdots, m).$$

We have

$$\frac{dV_i(x)}{dt}|_{(6)} \le \sum_{j=1}^m x_j^T(t)U_{ij}x_j(t) + \sum_{j=1}^m x_j^T(t - \tau_{ij}(t))V_{ij}x_j(t - \tau_{ij}(t)). \quad (3.7)$$

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Suppose $\lambda_{max}(U_{ii}) < 0$, then

$$\frac{dV_i(x)}{dt}\Big|_{(6)} \leq \sum_{j=1}^{m} \lambda_{max}(U_{ij})x_j^T(t)x_j(t) + \sum_{j=1}^{m} \lambda_{max}(V_{ij})x_j^T(t-\tau_{ij})x_j(t-\tau_{ij}(t))$$

$$\leq \frac{\lambda_{max}(U_{ii})}{\lambda_{max}(P_i)}V_i(x) + \sum_{j=1,i\neq j}^{m} \frac{\lambda_{max}(U_{ij})}{\lambda_{min}(P_i)}V_j(x) + \sum_{j=1}^{m} \frac{\lambda_{max}(V_{ij})}{\lambda_{min}(P_i)}V_j(x(t-\tau_{ij}).$$

Let

$$u_{ii} = \frac{\lambda_{max}(U_{ii})}{\lambda_{max}(P_i)}, u_{ij} = \frac{\lambda_{max}(U_{ij})}{\lambda_{min}(P_i)}(i \neq j),$$

$$v_{ij} = \frac{\lambda_{max}(V_{ij})}{\lambda_{min}(P_i)}, U = (u_{ij})_{m\times m}, V = (v_{ij})_{m\times m}.$$

The following result can be obtained.

Theorem 3 If matrix $-(U+V)$ is a M-matrix, then the system (2) is asymptotically stable.

Proof From the above discussing, the following differential inequality

$$\frac{dV(x(t))}{dt} \leq UV(x(t)) + UV^*(x(t))$$

where $V = (V_1, \cdots, V_m)^T$, $V^*(x) = (\sup_{t-\tau \leq \theta \leq t} V_1(\theta), \cdots, \sup_{t-\tau \leq \theta \leq t} V_m(\theta))^T$ is obtained. By lemma, if matrix $-(U+V)$ is a M-matrix, we can declare that the system (2) is asymptotically stable.

4 Example

Example Considering the following interconnected system

$$\begin{cases} \dot{x}_1(t) = A_{11}x_1(t) + B_{11}x_1(t-\tau_{11}(t)) + B_{12}x_2(t-\tau_{12}(t)) \\ \dot{x}_2(t) = A_{22}x_2(t) + B_{21}x_1(t-\tau_{21}(t)) + B_{22}x_2(t-\tau_{22}(t)) \end{cases}$$

where

$$A_{11} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & -3 & 1 \\ 0 & 1 & -4 \end{pmatrix}, A_{22} = \begin{pmatrix} -3 & 1 \\ -1 & -2 \end{pmatrix}, B_{11} = \epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B_{12} = \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, B_{21} = \epsilon \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}; B_{22} = \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $B_{111} = I_3, B_{112} = B_{11}, B_{121} = I_3, B_{122} = B_{12}, B_{211} = I_2, B_{212} = B_{21}, B_{221} = I2, B_{222} = B_{22}$, we have

$$P_1 = \begin{pmatrix} 0.3716 & -0.0573 & -0.0319 \\ -0.0573 & 0.3419 & 0.0830 \\ -0.0319 & 0.0830 & 0.2707 \end{pmatrix}, P_2 = \begin{pmatrix} 0.3429 & -0.0286 \\ -0.0286 & 0.4857 \end{pmatrix}.$$

From theorem 3 we obtained that the stability condition of this system is $\epsilon \leq 0.5435$.

Let $R_1 = 0.4719I_3, R_2 = 0.4I_2$, from theorem 2 we obtained that the stability condition of this system is $\epsilon \leq 0.8311$. However, by using method of Schoen and Geering(1995) we obtain stability bound, $\epsilon \leq 0.7736$.

References

1. A.Hmamed, Int. J. of Systems Sci. **17**, 1083-1087 (1986).
2. E. B. Lee, S. H. Zak and S. D. Brierley, Int. J. of Control **39**, 1025-1041 (1984).
3. T. Mori, Fukuma,N.,and Kuwahara, M., Int.J. of Control **34**, 1175-1184 (1981).
4. Xiaohong Nian, Control Theory and Application **15**, 134-138 (1998).
5. G.M. Schoen, and H. P. Geering, Int. J. Systems Sci. **26**, 2441-2444 (1995).
6. I. H. Suh, and Z. Bien, IEEE Transactions on Automatic Control **27**, 256-258 (1982).
7. H. Tokumaru, N. Adachi, and T. Amemiya, Proc.of IFAC 6th World Congress, Boston, **Paper 44**, 4 (1975).
8. W. J. Wang, and C.C. Song, Control Theory and Advanced Technology **5**, 315-322 (1991).
9. W. J. Wang, C.C. Song, and C.C. Kao, Int.J.of Systems Sci. **22**, 209-216 (1991).
10. K. Zhou, and P.P. Khargondkar, Systems and Control letters **10**, 17-20 (1988).

ON THE ASYMPTOTIC BEHAVIOR OF CHARACTERISTIC ROOTS OF NEUTRAL EQUATIONS WITH A DELAY

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Consider the following equation

$$\lambda(1 + ce^{-\tau\lambda}) + a + be^{-\tau\lambda} = 0, \tag{1}$$

which is the characteristic equation of the neutral equation

$$\dot{x}(t) + c\,\dot{x}(t - \tau) + ax(t) + bx(t - \tau) = 0, \tag{2}$$

where a, b and c are constants, and $c \neq 0$.
Let

$$\Lambda = \{Re\lambda_i : \lambda_i(i = 1, 2, \cdots) \text{ are all the roots of Eq. (1)}\}$$

Our main results are the following two theorems.

Theorem 1. The set Λ can not reach $\sup\Lambda$, if and only if the parameters of Eq. (1) satisfy either of the following conditions:

$$e^{-a\tau} < c < +\infty, \quad -c(a + \frac{2}{\tau}\ln c) < b < ac, \tag{I}$$

$$-\infty < c < -e^{-a\tau}, \quad ac < b < -c(a + \frac{2}{\tau}\ln(-c)). \tag{II}$$

And $\sup\Lambda = \frac{1}{\tau}\ln c$ if the condition (I) holds, and $sup\Lambda = \frac{1}{\tau}ln(-c)$ if the condition (II) holds.

Theorem 1 is Theorem 2 of §5.2 in [3].

Theorem 2. All roots of Eq. (1) have negative real parts and $\sup\Lambda = 0$ if and only if the parameters of Eq. (1) satisfy: $|c| = 1, a > 0, |b| < a$.

Theorem 2 immediately follows from Theorem 1.

Acknowledgments

References

1. J. K. Hale,Theory of Functional differential Equations,Springer–Verlag,New York,1977.
2. R. Bellman and K. L. Cooke, Differential–difference Equations,Academic Press,New York,1963.
3. Ren Hongshan, On the Accurate Distribution of Characteristic Roots and Stability of Linear Delay Differential Systems, Northeastern Forestry University Press, Harbin, 1999 (in Chinese)

THE QUALITATIVE ANALYSIS OF TWO-SPECIES NONLINEAR COMPETITION SYSTEM WITH PERIODIC COEFFICIENTS

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In this paper we consider the nonlinear relation two-species Lotka-Volterra competition models in which the right-hand sides are periodic in time,the sufficient conditions for the existence of a globally asymptotically stable positive periodic solution are obtained.These conditions also assure that the nonlinear relation competition system is uniformly persistent.

1 Introduction

In paper [1-3],the authors consider following nonautonomous linear periodic system

$$\begin{cases} \dot{x}_1 = x_1[b_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2] \\ \dot{x}_2 = x_2[b_2(t) - a_{21}(t)x_1 - a_{22}(t)x_2] \end{cases} \tag{1.1}$$

and have obtained sufficient conditions for the existence of a globally asymptotically stable positive periodic solution under appropriate conditions.In system (1),between two species is linear relation.

In this paper we consider the nonautonomous system of differential equations

$$\begin{cases} \dot{x}_1 = x_1[b_1(t) - a_{11}(t)x_1^\alpha - a_{12}(t)x_2^\beta] \\ \dot{x}_2 = x_2[b_2(t) - a_{21}(t)x_1^\alpha - a_{22}(t)x_2^\beta] \end{cases} \tag{1.2}$$

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where $b_i(t), a_{ij}(t)(i,j=1,2)$ are continuous positive periodic functions with a common periodic $\omega > o, \alpha, \beta > 0$, this competition system models shows that two species with nonlinear relation live in a ω-periodic environment.

If f is a continuous ω-periodic function defined on $[0,\infty)$, we denote

$$f^L = \min_t f(t), \quad f^M = \max_t f(t), \quad k_1 = \left(\frac{b_1^M}{a_{11}^L}\right)^{\frac{1}{\alpha}}, \quad k_2 = \left(\frac{b_1^M}{a_{22}^L}\right)^{\frac{1}{\beta}},$$

$$\delta_1 = \left(\frac{b_1^L - a_{12}^M k_2^\beta}{a_{11}^M}\right)^{\frac{1}{\alpha}}, \quad \delta_2 = \left(\frac{b_2^L - a_{21}^M k_1^\alpha}{a_{22}^M}\right)^{\frac{1}{\beta}}$$

2 Existence of Periodic Solution

Lemma 1 Both the positive and the nonnegative cone of R^3 are invariant in the sense that if $\Gamma(t) = [x_1(t), x_2(t), x_3(t)]$ is any solution of the system (2) with $\Gamma(0) > 0$ then $\Gamma(t) > 0$, and if $\Gamma(t)$ is any solution of system (2) with $\Gamma(0) \geq 0$ then $\Gamma(t) \geq 0$ when $t \in [0, +\infty)$.

We analyze the system (2) under the following conditions:

$$b_1^L > a_{12}^M k_2^\beta, \quad b_2^L > a_{21}^M k_1^\alpha. \tag{2.3}$$

It follows from Lemma 1 that any solution (2) which has a nonnegative initial condition remains nonnegative.

Thus, we have $\dot{x}_1 \leq x_1(b_1^M - a_{11}^L x_1^\alpha)$ as a consequence of which is

$$0 < x_1(0) \leq \left(\frac{b_1^M}{a_{11}^L}\right)^{\frac{1}{\alpha}} = k_1.$$

We have

$$x_1(t) \leq k_1. \tag{2.4}$$

Since, $\dot{x}_2 \leq x_2[b_2^M - a_{22}^L x_2^\beta]$, as a consequence of which is

$$0 < x_2(0) \leq \left(\frac{b_2^M}{a_{22}^L}\right)^{\frac{1}{\beta}} = k_2.$$

It implies

$$x_2(t) \leq k_2 \tag{2.5}$$

From (5), we have

$$\dot{x}_1 \geq x_1(b_1^L - a_{11}^M x_1^\alpha - a_{12}^M k_2^\beta),$$

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and it follows that

$$x_1(0) \geq \left(\frac{b_1^L - a_{12}^M k_2^\beta}{a_{11}^M} \right)^{\frac{1}{\alpha}} = \delta_1,$$

which implies

$$x_1(t) \geq \delta_1. \tag{2.6}$$

From (4), we have

$$\dot{x}_2 \geq x_2(b_2^L - a_{21}^M k_1^\alpha - a_{22}^M x_2^\beta)$$

and it follows that

$$x_2(0) \geq \left(\frac{b_2^L - a_{21}^M k_1^\alpha}{a_{22}^M} \right)^{\frac{1}{\beta}} = \delta_2$$

which implies

$$x_2(t) \geq \delta_2. \tag{2.7}$$

The condition (3) assures that $\delta_1 > 0$ and $\delta_2 > 0$. It is obvious that

$$0 < \delta_i < k_i \quad i = 1, 2. \tag{2.8}$$

From (4)-(7), we have

Lemma 2 Let $S = \{ x = (x_1, x_2) \in R_+^2, \delta_i \leq x_i \leq k_i, i = 1, 2 \}$, then S is invariant with respect to (2).

We can define a shift operator, also known as a Poincare map

$$\sigma : R^2 \to R^2$$

by $\sigma(x^0) = x(\omega, x^0), x^0 \in R^2$.

The following is one of the main results.

Theorem 1 Suppose that the coefficients of the system (2) satisfy (3). Then system (2) has at least one strictly positive ω-periodic solution.

Proof:: From Lemma 2, the operator σ as defined above maps S into itself, i.e. $\sigma(s) \subset S$. Because the solution of (2) is continuous with respect to the initial value , the operator σ is continuous. It can also be seen that S is a bounded closed convex set in R_+^2. By Brouwers theorem, σ has a fixed point in S. Consequently, there exists at least one strictly positive periodic solution. That the solution is strictly positive is assured by the invariance of the region S. The proof is complete.¶

Suppose $u(t) = [u_1(t), u_2(t)] \in R_+^2$ is a strictly positive ω-periodic solution of (2) as described in Theorem 1, we have the following corollary.

Corollary 2 Let $u_i(t), \delta_i, k_i \quad i = 1, 2$ be defined as above. then

$$\delta_i \leq u_i(t) \leq k_i, \quad i = 1, 2.$$

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3 Global Asymptotic Stability and Uniqueness

Suppose $u(t) = (u_1(t), u_2(t)) \in R_+^2$ is a strictly positive periodic solution of (2) as described in Theorem 1.

Theorem 2 If the coefficients of the system (2) satisfy (3),and

$$a_{11}(t) > a_{21}(t), \quad a_{22}(t) > a_{12}(t), \tag{3.9}$$

then there exists a unique strictly positive solution of the system (2) which is globally asymptotically stable.

Proof: Let $u(t) = (u_1(t), u_2(t)) \in R_+^2$ be a strictly positive periodic solution as described above,and let $x(t) = (x_1(t), x_2(t)) \in R_+^2$ be any solution of (2) with $x_i(0) > 0, i = 1, 2$. Since solution of (2) remain nonnegative,we can let

$$U_i(t) = \ln u_i(t), \quad X_i(t) = \ln x_i(t), \quad i = 1, 2. \tag{3.10}$$

Consider a Liapunov function $V(t)$ defined by

$$V(t) = \sum_{i=1}^{2} |U_i(t) - X_i(t)|, \quad t \geq 0.$$

Calculating the upper right derivative D^+V of $V(t)$ along the solution of (2),we get

$$D^+V(t) \leq \sum_{i=1}^{2} \frac{U_i(t) - X_i(t)}{|U_i(t) - X_i(t)|} [\dot{U}_i(t) - \dot{X}_i(t)].$$

Let $\gamma = \min_t \{a_{11}(t) - a_{21}(t), a_{22}(t) - a_{12}(t)\} > 0$, we get

$$D^+V(t) \leq -\gamma[|e^{\alpha U_1(t)} - e^{\alpha X_1(t)}| + |e^{\beta U_2(t)} - e^{\beta X_2(t)}|]. \tag{3.11}$$

An integration of (11) leads to

$$\sum_{i=1}^{2} |U_i(t) - X_i(t)| + \gamma \int_0^t [|e^{aU_1(s)} - e^{\alpha X_1(s)}| + |e^{\beta U_2(s)} - e^{\beta X_2(s)}|]ds < V(0) < \infty,$$

$$i.e. \sum_{i=1}^{2} |U_i(t) - X_i(t)| + \gamma \int_0^t [|u_1^\alpha(s) - x_1^\alpha(s)| + |u_2^\beta(s) - x_2^\beta(s)|]ds < V(0) < \infty.$$

Therefore $\lim_{t\to\infty} Sup \int_0^t [|u_1^\alpha(s) - x_1^\alpha(s)| + |u_2^\beta(s) - x_2^\beta(s)|]ds \leq \frac{V(0)}{\gamma}$.

Hence $\lim_{t\to\infty} |u_i(t) - x_i(t)| = 0, i = 1, 2$. The proof is complete.¶

Corollary 2 Under the conditions of the Theorem 2, the System (2) is uniformly persistent.

References

1. K. Grpalsamy, J.Austral Math.Soc.Ser.(B) **27**, 66-72 (1985).
2. S. Ahmad, J.Math .Anal.Appl. **127:377-387**, 1987 (.)
3. I. Anine, R. Ortega, J.Math.Anal.Appl. **185:477-489**, 1994 (.)

OSCILLATION OF FIRST ORDER DELAY DIFFERENTIAL EQUATIONS WITH OSCILLATING COEFFICIENTS

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In this paper, some sufficient conditions for oscillation of a first order delay differential equation with oscillating coefficients of the form $x'(t) + p(t)x(t - \tau) = 0$ are established, which improve and generalize some of the known results in the literature.

1 INTRODUCTION

Consider the first order differential equation

$$x'(t) + p(t)x(t - \tau) = 0, \ t \geq t_0 \tag{1.1}$$

where $p(t) : [t_0, \infty) \to \mathbf{R}$ is a right continuous function and τ is a positive constant.

For the case where $p(t) \geq 0$ for $t \geq t_0$, the oscillation of solutions of (1) has been studied by many authors, and some good results are obtained, we refer to [1-4]. For the general case where the coefficient $p(t)$ is allowed to oscillate, the derivative $x'(t)$ of the solution $x(t)$ is oscillatory along with $p(t)$ oscillating, therefore, it is difficult to study the oscillatory behavior of (1), and the corresponding study is relatively scarce, we only find a few papers, for example, [6-8].

In a recent paper [5], Li obtained the following important theorem, which improves many known results.

Theorem A. Suppose that $p(t) \geq 0$ and $\int_t^{t+\tau} p(s)ds > 0$ for $t \geq t_0$ and

$$\int_{t_0}^{\infty} p(t) \ln \left(e \int_t^{t+\tau} p(s)ds \right) dt = \infty. \tag{1.2}$$

Then every solution of Eq.(1) oscillates.

The main aim in this paper is to extend Theorem A to Eq.(1) with oscillating coefficient and improve it for the case where $p(t) \geq 0$. Our results are the following.

Theorem 1. Assume that

(i). there exists a sequence of intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$ such that $b_n \leq a_{n+1}$ and $b_n - a_n \geq 2\tau$ for $n = 1, 2, \cdots$, and that

$$p(t) \geq 0 \text{ for } t \in \cup_{n=1}^{\infty} [a_n, b_n]; \tag{1.3}$$

(ii). $\int_{t_0}^{\infty} \bar{p}(t) \ln \left[e \int_t^{t+\tau} \bar{p}(s)ds + 1 - \text{sign} \left(\int_t^{t+\tau} \bar{p}(s)ds \right) \right] dt = \infty,$

$$\tag{1.4}$$

where

$$\bar{p}(t) = \begin{cases} p(t), & t \in \cup_{n=1}^{\infty} [a_n + \tau, b_n); \\ 0, & t \in [t_0, a_1 + \tau) \cup (\cup_{n=1}^{\infty} [b_n, a_{n+1} + \tau)). \end{cases}$$

Then every solution of Eq.(1) oscillates.

Theorem 2. Assume that $p(t) \geq 0$ for $t \geq t_0$ and

$$\int_{t_0}^{\infty} p(t) \ln \left[e \int_t^{t+\tau} p(s)ds + 1 - \text{sign} \left(\int_t^{t+\tau} p(s)ds \right) \right] dt = \infty. \tag{1.5}$$

Then every solution of Eq.(1) oscillates.

Remark. Theorem 2 removes the restriction $\int_t^{t+\tau} p(s)ds > 0$ for $t \geq t_0$ in Theorem A.

2 PROOFS OF THEOREMS

Proof of Theorem 1. Assume, by way of contradiction, that (1) has an eventually positive solution $x(t)$. Then there exists an integer $k \geq 1$ such that

$$x(t - 2\tau) > 0 \text{ for } t \geq a_k. \tag{2.6}$$

The rest of the proof is divided into three claims:

Claim 1. Show

$$\int_t^{t+\tau} \bar{p}(s)ds \leq 1 \text{ for } t \geq a_k. \tag{2.7}$$

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Indeed, by the definition of $\bar{p}(t)$, we have

$$p(t) \geq \bar{p}(t) \text{ for } t \in \cup_{n=k}^{\infty}[a_n, b_n]. \qquad (2.8)$$

It follows from (1) that

$$x'(t) + \bar{p}(t)x(t - \tau) \leq 0, t \in \cup_{n=k}^{\infty}[a_n, b_n], \qquad (2.9)$$

which implies that $x(t)$ is nonincreasing on $[a_n, b_n]$ for $n \geq k$. We consider the following four possible cases:

Case (i). $t \in \cup_{n=k}^{\infty}[a_n + \tau, b_n - \tau]$. From (9) we obtain

$$x(t) > \int_t^{t+\tau} \bar{p}(s)x(s - \tau)ds \geq x(t) \int_t^{t+\tau} \bar{p}(s)ds,$$

and so

$$\int_t^{t+\tau} \bar{p}(s)ds \leq 1. \qquad (2.10)$$

Case (ii). $t \in \cup_{n=k}^{\infty}(b_n - \tau, b_n]$. Then $t \in (b_n - \tau, b_n]$ for some $n \geq k$, it follows from (10) that

$$\int_t^{t+\tau} \bar{p}(s)ds \leq \int_{b_n-\tau}^{t+\tau} \bar{p}(s)ds = \int_{b_n-\tau}^{b_n} \bar{p}(s)ds \leq 1. \qquad (2.11)$$

Case (iii). $t \in \cup_{n=k}^{\infty}[a_n, a_n + \tau)$. Noting that $t \in [a_n, a_n + \tau)$ for some $n \geq k$, by (10) we have

$$\int_t^{t+\tau} \bar{p}(s)ds = \int_{a_n+\tau}^{t+\tau} \bar{p}(s)ds \leq \int_{a_n+\tau}^{a_n+2\tau} \bar{p}(s)ds \leq 1. \qquad (2.12)$$

Case (iv). $t \in \cup_{n=k}^{\infty}[b_n, a_{n+1}]$. Since $t \in [b_n, a_{n+1}]$ for some $n \geq k$, by the definition of $\bar{p}(t)$ we have

$$\int_t^{t+\tau} \bar{p}(s)ds = \int_{b_n}^{a_{n+1}+\tau} \bar{p}(s)ds = 0. \qquad (2.13)$$

Combining cases (i)-(iv), we have (7).

Claim 2. There exist an integers sequence $\{n_i\}_{i=1}^{\infty}$ and a real numbers sequence $\{\xi_i\}_{i=1}^{\infty}$ such that $k < n_1 < n_2 < \cdots$ and

$$\text{(i). } \xi_i \in (a_{n_i} + \tau, b_{n_i}), i = 1, 2, \cdots; \qquad (2.14)$$

$$\text{(ii)}. \quad \limsup_{i \to \infty} \frac{x(\xi_i - \tau)}{x(\xi_i)} < \infty. \tag{2.15}$$

Indeed, it follows from (4) that $\limsup\limits_{t \to \infty} \int_t^{t+\tau} \bar{p}(s)ds > 0$, which, together with (11),(12) and (13), yields that there exist an integers sequence $\{n_i\}_{i=1}^{\infty}$, a real numbers sequence $\{t_i\}_{i=1}^{\infty}$ and a number $d > 0$ such that $k < n_1 < n_2 < \cdots$ and $t_i \in [a_{n_i} + \tau, b_{n_i} - \tau]$ and $\int_{t_i}^{t_i+\tau} \bar{p}(s)ds \geq 2d, i = 1, 2, \cdots$. Then there exists $\xi_i \in (t_i, t_i + \tau)$ for every $i = 1, 2, \cdots$ such that

$$\int_{t_i}^{\xi_i} \bar{p}(s)ds \geq d \ \text{ and } \ \int_{\xi_i}^{t_i+\tau} \bar{p}(s)ds \geq d. \tag{2.16}$$

By integrating (9) over the intervals $[t_i, \xi_i]$ and $[\xi_i, t_i + \tau]$, we find

$$x(\xi_i) - x(t_i) + \int_{t_i}^{\xi_i} \bar{p}(s)x(s - \tau)ds \leq 0, \tag{2.17}$$

$$x(t_i + \tau) - x(\xi_i) + \int_{\xi_i}^{t_i+\tau} \bar{p}(s)x(s - \tau)ds \leq 0. \tag{2.18}$$

By omitting the first terms in (17) and (18) and by using the nonincreasing nature of $x(t)$ on $[a_{n_i}, b_{n_i}]$ and (16), we get

$$-x(t_i) + dx(\xi_i - \tau) \leq 0 \ \text{ and } \ -x(\xi_i) + dx(t_i) \leq 0,$$

or $x(\xi_i - \tau)/x(\xi_i) \leq \frac{1}{d^2}, i = 1, 2, \cdots$. Thus, (i) and (ii) in Claim 2 hold.

Claim 3. Complete the proof by showing the following

$$\int_{a_k}^{\infty} \bar{p}(t) \ln \left\lceil e \int_t^{t+\tau} \bar{p}(s)ds + 1 - \text{sign}\left(\int_t^{t+\tau} \bar{p}(s)ds \right) \right\rceil dt < \infty, \tag{2.19}$$

which contradicts (4). Indeed, from (1) and (6), we have

$$\frac{x'(t)}{x(t)} + p(t)\frac{x(t - \tau)}{x(t)} = 0, t \geq a_k - \tau. \tag{2.20}$$

Set $\lambda(t) = -x'(t)/x(t)$ for $t \geq a_k - \tau$. Then $\lambda(t) \geq 0$ for $t \in \cup_{n=k}^{\infty}[a_n, b_n]$, and

$$\lambda(t) = p(t) \exp\left(\int_{t-\tau}^{t} \lambda(s)ds \right), t \geq a_k, \tag{2.21}$$

or

$$\lambda(t) \int_t^{t+\tau} \bar{p}(s)ds = p(t)\left(\int_t^{t+\tau} \bar{p}(s)ds \right) \exp\left(\int_{t-\tau}^{t} \lambda(s)ds \right), t \geq a_k. \tag{2.22}$$

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One can easily show that

$$\phi(r)re^x \geq \phi(r)x + \phi(r)\ln(er + 1 - \text{sign } r) \quad \text{for } r \geq 0 \text{ and } x \in \mathbf{R}, \quad (2.23)$$

where $\phi(0) = 0$ and $\phi(r) \geq 0$ for $r > 0$.

We consider the following two possible cases.

Case 1. $t \in \cup_{n=k}^{\infty}[a_n, b_n]$. By the definition of $\bar{p}(t)$ and noting that $p(t)$ is right continuous, it follows that $\bar{p}(t)$ is also right continuous, and so, $\int_t^{t+\tau} \bar{p}(s)ds = 0$ implies that $\bar{p}(t) = 0$. From (22) and (23), we have

$$\lambda(t) \int_t^{t+\tau} \bar{p}(s)ds \geq \bar{p}(t)\left(\int_t^{t+\tau} \bar{p}(s)ds\right)\exp\left(\int_{t-\tau}^t \lambda(s)ds\right)$$

$$\geq \bar{p}(t)\int_{t-\tau}^t \lambda(s)ds + \bar{p}(t)\ln\left\lceil e \int_t^{t+\tau} \bar{p}(s)ds + 1 - \text{sign}\left(\int_t^{t+\tau} \bar{p}(s)ds\right)\right\rceil.$$

Case 2. $t \in \cup_{n=k}^{\infty}[b_n, a_{n+1})$. By the definition of $\bar{p}(t)$, we have $\bar{p}(t) \equiv 0$ and $\int_t^{t+\tau} \bar{p}(s)ds \equiv 0$. It follows that

$$\lambda(t)\int_t^{t+\tau} \bar{p}(s)ds$$

$$= \bar{p}(t)\int_{t-\tau}^t \lambda(s)ds + \bar{p}(t)\ln\left\lceil e \int_t^{t+\tau} \bar{p}(s)ds + 1 - \text{sign}\left(\int_t^{t+\tau} \bar{p}(s)ds\right)\right\rceil.$$

Combining case 1 and 2, we obtain

$$\lambda(t)\int_t^{t+\tau} \bar{p}(s)ds - \bar{p}(t)\int_{t-\tau}^t \lambda(s)ds$$

$$\geq q\bar{p}(t)\ln\left\lceil e \int_t^{t+\tau} \bar{p}(s)ds + 1 - \text{sign}\left(\int_t^{t+\tau} \bar{p}(s)ds\right)\right\rceil, t \geq a_k. \quad (2.24)$$

Then, for $i > 1$,

$$\int_{a_k+\tau}^{\xi_i} \lambda(t)\int_t^{t+\tau} \bar{p}(s)dsdt - \int_{a_k+\tau}^{\xi_i} \bar{p}(t)\int_{t-\tau}^t \lambda(s)dsdt$$

$$\geq \int_{a_k+\tau}^{\xi_i} \bar{p}(t)\ln\left\lceil e \int_t^{t+\tau} \bar{p}(s)ds + 1 - \text{sign}\left(\int_t^{t+\tau} \bar{p}(s)ds\right)\right\rceil. \quad (2.25)$$

By interchanging the order of integration and by using (14), we find

$$\int_{a_k+\tau}^{\xi_i} \bar{p}(t)\int_{t-\tau}^t \lambda(s)dsdt \geq \int_{a_k+\tau}^{\xi_i-\tau} \lambda(t)\int_t^{t+\tau} \bar{p}(s)dsdt.$$

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From this and (25), it follows that

$$\int_{\xi_i-\tau}^{\xi_i} \lambda(t) \int_t^{t+\tau} \bar{p}(s)dsdt$$

$$\geq \int_{a_k+\tau}^{\xi_i} \bar{p}(t)\ln\left\lceil e \int_t^{t+\tau} \bar{p}(s)ds + 1 - \text{sign}\left(\int_t^{t+\tau} \bar{p}(s)ds\right)\right\rceil dt. \qquad (2.26)$$

By (7) and (26), we have

$$\ln\frac{x(\xi_i-\tau)}{x(\xi_i)} \geq \int_{a_k+\tau}^{\xi_i} \bar{p}(t)\ln\left\lceil e \int_t^{t+\tau} \bar{p}(s)ds + 1 - \text{sign}\left(\int_t^{t+\tau} \bar{p}(s)ds\right)\right\rceil dt,$$

$$i = 2, 3, \cdots. \qquad (2.27)$$

Taking the superior limit as $i \to \infty$, we get (19) from (15) and (27), and so the proof is complete.

Proof of Theorem 2. By way of contradiction, that (1) has an eventually positive solution $x(t)$. Then by [7, Lemma 1 and Lemma 2], we have eventually

$$\int_t^{t+\tau} p(s)ds \leq 1 \quad \text{and} \quad \liminf_{t\to\infty}\frac{x(t-\tau)}{x(t)} < \infty.$$

The rest of the proof is similar to that of Claim 3. Here we omit it.

3 Examples

In the final section, we give two examples to demonstrate the advantage of our results that existing ones in the literature.

Example 1. Consider the delay equation

$$x'(t) + a\sin t \cdot x(t - \frac{\pi}{3}) = 0, t \geq 0, \qquad (3.28)$$

where $a^3 > 2(2e)^{1/4}/(2+\sqrt{3})^{\sqrt{3}/2}e \approx 0.360$.

Let $a_n = 2n\pi, b_n = (2n+1)\pi$. Then condition (i) in Theorem 1 holds, and

$$\bar{p}(t) = \begin{cases} a\sin t, & t \in \cup_{n=0}^{\infty}[2n\pi + \frac{\pi}{3}, (2n+1)\pi), \\ 0, & t \in [0, \frac{\pi}{3}) \cup \cup_{n=0}^{\infty}[(2n+1)\pi, 2(n+1)\pi + \frac{\pi}{3}). \end{cases}$$

By direct calculation, we have

$$\int_0^{2\pi} \bar{p}(t)\ln\left\lceil e \int_t^{t+\frac{\pi}{3}} \bar{p}(s)ds + 1 - \text{sign}\left(\int_t^{t+\frac{\pi}{3}} \bar{p}(s)ds\right)\right\rceil dt$$

$$= \frac{a}{2}\ln\frac{(2+\sqrt{3})^{\sqrt{3}/2}ea^3}{2(2e)^{1/4}} > 0,$$

it follows that

$$\int_0^\infty \bar{p}(t)\ln\left\lceil e\int_t^{t+\pi/3}\bar{p}(s)ds+1-\mathrm{sign}\left(\int_t^{t+\pi/3}\bar{p}(s)ds\right)\right\rceil dt=\infty.$$

Therefore, by Theorem 1 we see that every solution of Eq.(28) is oscillatory.

Example 2. Consider the delay equation

$$x'(t)+p(t)x(t-1)=0, t\geq 0, \qquad (3.29)$$

where $1>a>e^{-2/3}$ and

$$p(t)=\begin{cases} a, 0\leq t<3, \\ 0, 3\leq t<5. \end{cases} \quad\text{and}\quad p(t+5)=p(t) \text{ for } t\geq 0$$

Since $p(t)\geq 0$ for $t\geq 0$ and

$$\int_0^5 p(t)\ln\left\lceil e\int_t^{t+1}p(s)ds+1-\mathrm{sign}\left(\int_t^{t+1}p(s)ds\right)\right\rceil dt a\ln(e^2a^3)>0,$$

it follows that

$$\int_0^\infty p(t)\ln\left\lceil e\int_t^{t+1}p(s)ds+1-\mathrm{sign}\left(\int_t^{t+1}p(s)ds\right)\right\rceil dt=\infty.$$

Therefore, by Theorem 2 we see that every solution of Eq.(29) is oscillatory.

On the other hand, $\int_{t-1}^t p(s)ds=0$ for $t\in\cup_{n=1}^\infty[5n-1,5n]$, and $\int_{t-1}^t p(s)ds=a<1$ for $t\in\cup_{n=0}^\infty[5n+1,5n+3]$. For this reason, Theorem A and all know results in the literature can't be applied to Eq.(29).

Acknowledgments

References

1. G. Ladas,*Applicable Anal.* **9**, 93-98 (1979).
2. B. Li,*J.Math.Anal.Appl.* **192**, 312-321 (1995).
3. A. Elbert and I. P. Stavroulakis, *Proc.Amer.Math.Soc.* **123**, 1503-1510 (1995).
4. X. H. Tang and J. H. Shen,.*Math.Anal.Appl.* **217**, 32-42 (1998).
5. Bingtuan Li,*Proc.Amer.Math.Soc.* , 3729-3737 (1996).

6. M. R. Kulenovic and M. K. Grammatikopoulos, *Nonlinear Anal.* **8**, 1043-1054 (1984).

7. J. S. Yu,Z. C. Wang,B. G. Zhang and X. Z. Qian, *Pan.Math.J.* **2**, 59-78 (1992).

8. Y. Domshlak, *"Functional Differential Equations"*, **2**, 59-68 (1994).

THE ASYMPTOTIC THEORY OF FOR SEMILINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS

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This paper deals with the asymptotic theory of initial value problems for semilinear wave equations in two space dimensions. The well-posedness and validity of formal approximations on a long time scale including $0 \leq t \leq T = O(|\epsilon|^{-\sigma})$ ($\sigma > 0, \epsilon \to 0$) and $0 \leq t \leq T = \infty$ are discussed in the classical sense of C^2. These results describe the behavior of long time existence for the validity of formal approximations.

1 Introduction

In this paper an asymptotic theory is established for the following initial value problem of semilinear perturbed wave equation

$$\begin{cases} u_{tt} - \Delta u = \epsilon f(u, \epsilon), & x \in R^2, \quad t > 0, \\ u(0, x, \epsilon) = u_0(x, \epsilon), \quad u_t(0, x, \epsilon) = u_1(x, \epsilon), & x \in R^2, \end{cases} \tag{1.1}$$

where $u(t, x, \epsilon)$ is a real-valued unknown function, $\Delta = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}$, ϵ is a parameter with $0 < |\epsilon| < \epsilon_0 \ll 1$, $f(u, \epsilon)$, $u_0(x, \epsilon)$ and $u_1(x, \epsilon)$ satisfy some assumptions mentioned in Section 2.

In order to make sure what is meant by an interesting aspect,one can consider problem (1) with $\epsilon = 0$ and $\epsilon = 1$. For $\epsilon = 0$, it is easy to prove existence and uniqueness of the solution (see ref.1) in the classical sense. When $\epsilon = 1$, only a local theory can be obtained which states that a unique solution exists for $x \in R^2$ and $0 \leq t \leq T = O(1)$. It can be shown that when $\epsilon \in [-\epsilon_0, \epsilon_0]$, $T = T(\epsilon)$ where $T(\epsilon) \to \infty$ for $\epsilon \to 0$. In the papers ref.1-ref.2, the asymptotic theory for validation of formal approximations of the solutions of initial-boundary value problems for the second order semilinear wave equations in one space dimension with the best order time function $T = O(|\epsilon|^{-1})$ has been presented. But for $x \in R^1$,some open problems and results in ref.1-3

were given on the asymptotic theory of initial value problems for second order nonlinear wave equations. The reason is that asymptotic theory of initial value problems for partial differential equations is more difficult than that of initial-boundary value problems. For nonlinear partial differential equations in high space dimension,as stated in ref.2, little is known about the asymptotic theory. In ref. 3, the asymptotic theory of initial value problems for second order semilinear wave equations in the classical sense of C^2 is presented on the time scale of order $|\epsilon|^{-1}$ in three space dimensions. In this paper, an interesting result is that the asymptotic theory and validation of formal approximations for the second order semilinear wave equation in two space dimensions on the long time scale including $0 \le t \le T = O(|\epsilon|^{-\sigma})$ $(\sigma > 0, \epsilon \rightarrow 0)$ and $0 \le t \le T = \infty$ are established in the classical sense of C^2. These results describe the behavior of long time existence for solution of (1) and the long time validaty for the corresponding formal approximations.

For simplicity, we will denote by C any constants appearing in our paper, which never depends on ϵ.

2 The well-posedness

In order to prove the existence and uniqueness in the classical sense of C^2 for problem (1), by ref.1 we know that the equivalent integral equation for (1) has the following form

$$
\begin{aligned}
u(t,x,\epsilon) = & \{ \tfrac{\partial}{\partial t}[\tfrac{t}{2\pi} \int_{|\xi|<1} u_0(x+t\xi,\epsilon)d\xi] \\
& + \tfrac{t}{2\pi} \int_{|\xi|<1} u_1(x+t\xi,\epsilon)d\xi \} \\
& + \{ \tfrac{\epsilon}{2\pi} \int_0^t (t-\tau) \int_{|\xi|<1} f(u(\tau, x+(t-\tau)\xi,\epsilon),\epsilon)d\xi d\tau \\
& = u^0(t,x,\epsilon) + v^0(t,x,\epsilon),
\end{aligned}
$$

where ξ is a unit vector in R^2 and $d\xi$ is an area element.

Suppose that the nonlinear term $f(u,\epsilon)$ and initial value $u_0(x,\epsilon)$, $u_1(x,\epsilon)$ satisfy following assumptions

(i) $f(u,\epsilon) \in C^2$ with respect to u, $f(0,\epsilon) = f_u(0,\epsilon) = f_{uu}(0,\epsilon) = 0$.

(ii) If $|u(t,x,\epsilon)| < M$, $|v(t,x,\epsilon)| < M$, there exist constants $p > 3$ and $A > 0$ such that

$$
|f(u,\epsilon)| \le A \quad \text{and} \quad |f_{uu}(u,\epsilon) - f_{uu}(v,\epsilon)| \le A|w|^{p-1}|u-v|,
$$

where $w = \max\{|u|, |v|\}$, M and A are independent of ϵ.

(iii) $u_0(x,\epsilon)$ and $u_1(x,\epsilon)$ satisfy

$$
|\partial_x^\alpha u_0(x,\epsilon)|, \quad |\partial_x^\beta u_1(x,\epsilon)| \le \frac{G}{(1+|x|)^{1+k}}, \quad 0 < k < 1,
$$

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where multi-integers α and β satisfy $|\alpha| \leq 2$, $|\beta| \leq 2$, G is independent of ϵ.

Let J_k be given by

$$J_k = \left\{ \begin{array}{ll} (t,x), & (t,x) \in [0,\infty) \times R^2, \quad k > 2/(p-1), \\ (t,x), & (t,x) \in [0,T] \times R^2, \quad 0 < k < 2/(p-1). \end{array} \right.$$

We define $C^2(J_k)$ be the space of all real-valued and twice continuously differentiable functions W on J_k with norm $\|.\|_{J_k}$ given by

$$\|W\|_{J_k} = \sup_{(t,x) \in J_k} [(1 + t + |x|)^k \|W(t,x,\epsilon)\|] < \infty, \tag{2.2}$$

where

$$\|W(t,x,\epsilon)\| = \sum_{0 \leq j + j_1 + j_2 \leq 2} \left| \frac{\partial^{j+j_1+j_2} W(t,x,\epsilon)}{\partial t^j \partial x_1^{j_1} \partial x_2^{j_2}} \right|.$$

By the definition of space $C^2(J_k)$, we know that $C^2(J_k)$ is a Banach space with the norm defined by (3), and for any $u \in C^2(J_k)$, $\|u\|_{J_k}$ is bounded. We shall use the fixed point theorem to prove the existence and uniqueness of solutions to (1) in the space $C^2(J_k)$.

Now we introduce the following two lemmas which can be found in [4].

Lemma 1 *If $0 < k < 1$, then*

$$\frac{t}{2\pi} \int_{|\xi|<1} \frac{d\xi}{(1+|x+t\xi|)^{1+k}} \leq \frac{C}{(1+t+|x|)^k},$$

$$\frac{1}{2\pi} \int_{|\xi| leq 1} \frac{d\xi}{(1+|x+t\xi|)^{1+k}} \leq \frac{C}{(1+t+|x|)^k}.$$

Lemma 2 *Suppose that $u_0(x,\epsilon)$, $u_1(x,\epsilon)$ satisfy (iii), then*

$$\|u^0(t,x,\epsilon)\| \leq \frac{C}{(1+t+|x|)^k} \quad (0 < k < 1).$$

Let the operator Λ be defined as follows

$$\begin{aligned} \Lambda u(t,x,\epsilon) = & \left\{ \frac{\partial}{\partial t} \left[\frac{t}{2\pi} \int_{|\xi|<1} u_0(x+t\xi,\epsilon)d\xi \right] \right. \\ & + \frac{t}{2\pi} \int_{|\xi|<1} u_1(x+t\xi,\epsilon)d\xi \right\} \\ & + \left\{ \frac{\epsilon}{2\pi} \int_0^t (t-\tau) \int_{|\xi|<1} f(u(\tau,x+(t-\tau)\xi,\epsilon),\epsilon)d\xi d\tau \right\} \\ = & \ u^0(t,x,\epsilon) + v^0(t,x,\epsilon). \end{aligned}$$

We can prove that the integral operator Λ is a contractive mapping of itself in the space $C^2(J_k)$. By the same method as [3], we give the following lemma which is the key to obtain our main results.

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Lemma 3 *Suppose that f, u_0, u_1 satisfy assumptions (i)-(iii). For any $u,v \in C^2(J_k)$ and $p > 4$, we have*

$$\|\Lambda u\| \leq \begin{cases} \frac{C}{(1+t+|x|)^k} + \frac{C|\epsilon|\|u\|_{J_k}}{(1+t+|x|)^k}, & k > 2/(p-1), \\ \frac{C}{(1+t+|x|)^k} + \frac{C|\epsilon|(1+t)^{2-k(p-1)}}{(1+t+|x|)^k}\|u\|_{J_k}, & 0 < k < 2/(p-1). \end{cases}$$

and

$$\|\Lambda u - \Lambda v\| \leq \begin{cases} \frac{C|\epsilon|\|u-v\|_{J_k}}{(1+t+|x|)^k}, & k > 2/(p-1), \\ \frac{C|\epsilon|(1+t)^{2-k(p-1)}}{(1+t+|x|)^k}\|u - v\|_{J_k}, & 0 < k < 2/(p-1). \end{cases}$$

The proof is similar to that of Lamma3 in ref.3.

By Lemma 3, we obtain the following main result at once.

Theorem 1 *Suppose that the nonlinear term $f(u, \epsilon)$, initial value $u_0(x, \epsilon)$, $u_1(x, \epsilon)$ satisfy assumptions (i)-(iii) with $0 < |\epsilon| \leq \epsilon_0 \ll 1$, then we have*

(1) If $k > 2/(p-1)$ $(p > 3)$, there exists a unique global C^2 solution to problem (1).

(2) If

$$0 < k < \min\{1, 2/(p-1)\}(p > 3), \ 0 \leq t \leq T = O(|\epsilon|^{-1/(2-kp+k)}),$$

there exists a unique solution $u \in C^2(J_k)$ to problem (1).

3 Validation of formal approximations

Because the initial value problem (1) contains a small parameter ϵ, perturbation methods may be applied for the construction of approximation to the solution. In most perturbation methods for nonlinear problem, a function is constructed that satisfies the differential equation and initial condition up to some order of ϵ (where the parameter ϵ is so small). Such a function is usually called a formal approximation. In order to prove that the formal approximation is an asymptotic approximation (as $\epsilon \to 0$), We have to establish an additional analysis in the space $C^2(J_k)$.

Suppose that on $J_k \times [-\epsilon_0, \epsilon_0]$, the function $v(t, x, \epsilon)$ satisfies

$$\begin{cases} v_{tt} - \Delta v = \epsilon f(v, \epsilon) + |\epsilon|^m c_1(t, x, \epsilon), & m > 1, \\ v(0, x, \epsilon) = u_0(x, \epsilon) + |\epsilon|^{m-1} c_2(x, \epsilon) = v_0(x, \epsilon), & 0 < |\epsilon| \leq \epsilon_0 \ll 1, \quad (3.3) \\ v_t(0, x, \epsilon) = u_1(x, \epsilon) + |\epsilon|^{m-1} c_3(x, \epsilon) = v_1(x, \epsilon), & 0 < |\epsilon| \leq \epsilon_0 \ll 1, \end{cases}$$

where $f(v, \epsilon)$, $u_0(x, \epsilon)$, $u_1(x, \epsilon)$ satisfy assumptions (i)-(iii). Suppose that $c_1(t, x, \epsilon)$, $c_2(x, \epsilon)$ and $c_3(x, \epsilon)$ satisfy following conditions

$$c_1(t, x, \epsilon) \in C^2(J_k), \ \text{and} \ \|c_1(t, x, \epsilon)\| \leq 1/(1 + t + |x|)^{kp}, \quad (3.4)$$

$$|\partial_x^\alpha c_2(x,\epsilon)|, \partial_x^\beta c_3(x,\epsilon)| \leq C/(1+t+|x|)^{k+1}, |\alpha| \leq 3, |\beta| \leq 2, \ 0 < k < 1. \quad (3.5)$$

From Theorem 1 it follows that the initial value problem (3) has a unique solution $v(t,x,\epsilon) \in C^2(J_k)$. On the other hand, the problem (3) can be transformed into the following equivalent equation

$$v(t,x,\epsilon) = \frac{\partial}{\partial t}[\frac{t}{2\pi}\int_{|\xi|<1} v_0(x+t\xi,\epsilon)d\xi] + \frac{t}{2\pi}\int_{|\xi|<1} v_1(x+t\xi,\epsilon)d\xi$$
$$+\frac{\epsilon}{2\pi}\int_0^t(t-\tau)\int_{|\xi|<1}[f(v,(\tau,x+(t-\tau)\xi,\epsilon),\epsilon)$$
$$+|\epsilon|^m c_1(\tau,x+(t-\tau)\xi,\epsilon)]d\xi d\tau$$

If $u \in C^2(J_k)$ is the solution of problem (1), then

$$v(t,x,\epsilon) - u(t,x,\epsilon)$$
$$= \frac{\partial}{\partial t}[\frac{t}{2\pi}\int_{|\xi|<1}|\epsilon|^{m-1}c_2(x+t\xi,\epsilon)d\xi]$$
$$+\frac{t}{2\pi}\int_{|\xi|<1}|\epsilon|^{m-1}c_3(x+t\xi,\epsilon)d\xi$$
$$+\frac{\epsilon}{2\pi}\int_0^t(t-\tau)\int_{|\xi|<1}[f(v(\tau,x+(t-\tau)\xi,\epsilon),\epsilon) \quad (3.6)$$
$$-f(u(\tau,x+(t-\tau)\xi,\epsilon),\epsilon)]d\xi d\tau$$
$$+\frac{\epsilon}{2\pi}\int_0^t(t-\tau)\int_{|\xi|<1}|\epsilon|^m c_1(\tau,x(t-\tau)\xi,\epsilon)d\xi d\tau.$$

By the same idea as that of Lemma 3, we have

$$\|v(t,x,\epsilon) - u(t,x,\epsilon)\|_{J_k} = O(|\epsilon|^{m-1}).$$

Now we get the following asymptotic approximation theorem.

Theorem 2 *Suppose that $v(t,x,\epsilon)$ is the solution of the problem (3), and nonlinear term f, initial data u_0, u_1 satisfy assumptions (i)-(iii). Let $c_1(t,x,\epsilon)$, $c_2(x,\epsilon)$ and $c_3(x,\epsilon)$ satisfy (4) (5). Then for $m > 1$, the formal approximation $v(t,x,\epsilon)$ is an asymptotic approximation (as $\epsilon \to 0$) of the solution $u(t,x,\epsilon)$ of problem (1). Furthermore*

(i)$\|u-v\|_{J_k} = O(|\epsilon|^{m-1})$, for $(t,x) \in [0,+\infty) \times R^2$, if $\frac{2}{p+1} < K < \frac{1}{2}$ $(p > 4)$,

(ii) $\|u-v\|_{J_k} = O(|\epsilon|^{m-1})$ for $x \in R^2$ and $0 \leq t \leq L|\epsilon|^{\frac{-1}{2-k(p-1)}}$, if $0 < k < \min\{1/2, 2/(p-2)\}$, in which $L > 0$ is in dependent of ϵ.

References

1. W. T. Van Horssen, *SIAM J. Appl. Math.*, **48**, 1227 (1988).
2. W. T. Van Horssen and A. H. P. Van Der Burgh, *SIAM J. Appl. Math.*, **48**, 719 (1988).
3. S. Y. Lai and C. L. Mu, *Appl. Math. JCU(B)*, **12**, 321 (1997).
4. K.Kubota,Hokkaido Math. J.,22,123 (1993).

THE UNIQUE NORMAL FORM THEORY

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In this paper, we give a brief introduction to the theory of unique normal forms of ordinary differential equations and introduce some recent results concerning the unique normal form of Bogdanov-Takens singularities.

1 Introduction

It is well known that the classical normal form is computed by using one Lie Bracket and can not give unique normal form for a given equation in general. Here the nonuniqueness means that even the form of the normal form is fixed, the coefficients of the normal form are not uniquely determined by the given equation. Hence the classical normal forms can not be used for formal classification.

Example 4 For equation of Bogdanov-Takens singularities

$$\dot{x} = y + h.o.t, \quad \dot{y} = h.o.t, \tag{1.1}$$

where h.o.t denote higher order terms, by classical normal form theory, its normal form can be taken as

$$\dot{x} = y, \quad \dot{y} = \sum_{k=2}^{\infty} a_k x^k + b_k x^{k-1} y, \tag{1.2}$$

where a_k, b_k are all real constants. But we note that if $a_2 \neq 0$ then b_3 is not uniquely determined by the given equation (1.1) and consequently we can take $b_3 = 0$ in the normal form. See for example [5].

Since the early 80s, many authors considered the refined definition of normal form in order to get unique normal form(also the simplest normal form). In this article, we give brief introduction to the theory of unique normal form and give some recent results on the unique normal form of Bogdanov-Takens singularities.

2 Ushiki's Method

Ushiki[4] gave a new method to compute normal forms for given equations. And then his method is clearly described in [2].

Consider a C^∞ differential system

$$\dot{x} = V(x) = v_1 + v_2 + \ldots + v_k + \ldots, x \in \mathbb{R}^n, \qquad (2.3)$$

where $V(0) = 0, v_k \in H_n^k$, the n dimensional vector valued homogeneous polynomial space of degree k. Let $Y(x)$ be a C^∞ vector field with $Y(0) = 0$ and $\Phi_Y^t(x)$ be the flow generated by Y. Let $y = \Phi_Y^t(x)$. Then Eq.(2.3) is changed to

$$\begin{aligned}\dot{y} &= exp(t * ad(Y))V(x)\\ &= V(x) + t[Y(x), V(x)] + \ldots + t^n/n![Y(x), \ldots [Y(x), V(x)]\ldots] + \ldots, \end{aligned}$$
$$(2.4)$$

where $ad(Y)V(x) = [Y(x), V(x)] = DY(x)V(x) - DV(x)Y(x)$. Denote W^k the k-jet of W at the origin and $W_k = W^k - W^{k-1}$, $\forall k \in \mathbb{N}$.

Theorem 1 ([4],[2]) Let $h_k(t)$ be the $k - th$ order term in Eq.(2.4). If $Y(x)$ satisfies the condition $[Y^k, V^{k-1}]^{k-1} = 0$ then

$$\frac{d}{dt}h_k(t) = [Y^k, V^{k-1} + h_k(t)]_k, \quad h_k(0) = v_k. \qquad (2.5)$$

In [4] and [2] some examples are given, those are normal forms with coefficients being uniquely determined by the given equations up to some finite order.

Example 5 ([4]) Normal form up to order 4 of Eq.(1.1) can be taken as one of the follows:

$$\begin{aligned}(a)&\dot{x} = y, \quad \dot{y} = \delta x^2 + b_2 xy + a_3 x^3 + b_4 x^3 y,\\ (b)&\dot{x} = y, \quad \dot{y} = \delta xy + a_3 x^3 + b_3 x^2 y + a_4 x^4 + b_4 x^3 y, \qquad (2.6)\\ (c)&\dot{x} = y, \quad \dot{y} = w_1 x^3 + w_2 x^2 y + a_4 x^3 + b_4 x^3 y,\end{aligned}$$

where $\delta = 1$ or -1 and $a_3, a_4, b_2, b_3, b_4, w_1, w_2$ are all uniquely determined by Eq.(1.1) and $w_1^2 + w_2^2 = 1$.

Remark 2.1 Note that $Y_1 \neq 0$ is allowed in Ushiki's method. But if we consider only near identity change of variables, then we should assume $Y_1 = 0$. Then $[Y^k, V^{k-1} + h_k(t)]_k = [Y_2, v_{k-1}] + [Y_3, v_{k-2}] + \ldots + [Y_k, v_1]$. Therefore the solution of Eq.(2.5) is

$$h_k(t) = v_k + t\{[Y_2, v_{k-1}] + [Y_3, v_{k-2}] + \ldots + [Y_k, v_1]\}.$$

We may take $t = 1$. Then under the change of variables $y = \Phi_Y^1(x)$ the k-th order term in the result equation is $v_k + [Y_2, v_{k-1}] + [Y_3, v_{k-2}] + \ldots + [Y_k, v_1]$.

[4] and [2] do not explain theoritically why their methods can give unique normal form. But indeed, their methods can give unique normal form, that is in fact solved by [3]. See the section 4 below.

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3 Baider and Sanders's Method

Baider and Sanders [1] defined second order normal form by introducing the second grading. For the normal form of Bogdanov-Takens singularities they consider the further reduction for the first order normal form(i.e., the classical normal form). They wrote the first order normal form in the following form:

$$\dot{x} = X(x), \quad x \in \mathbb{R}^2, \tag{3.7}$$

where

$$X(x) = A_0^1 + \sum_{k=\mu}^{\infty} \alpha_k A_k^{-1} + \sum_{k=\nu}^{\infty} \beta_k B_k^0, 1 \le \mu \le \infty, 1 \le \nu \le \infty, \tag{3.8}$$

and where

$$A_k^l = \begin{pmatrix} \frac{k+1-l}{k+2} x^{l+1} y^{k-l} \\ -\frac{l+1}{k+2} x^l y^{k+1-l} \end{pmatrix} (-1 \le l \le k+1), \quad B_m^n = \begin{pmatrix} x^{n+1} y^{m-n} \\ x^n y^{m+1-n} \end{pmatrix} (0 \le n \le m). \tag{3.9}$$

It is well known that the degree of a monomial in the classical sense is defined by the summation of powers of all variables. But Baider and Sanders gave the new grading function as follows:

$$\delta^{(2)}(A_k^l) = \delta^{(2)}(B_k^l) = 2k + l \times \min(\mu, 2\nu).$$

It is obvious that

$$\delta^{(2)}(A_0^1) = \delta^{(2)}(A_\mu^{-1}) = \mu, \text{ if } \mu \le 2\nu$$

and

$$\delta^{(2)}(A_0^1) = \delta^{(2)}(B_\nu^0) = 2\nu, \text{ if } \mu \ge 2\nu.$$

Note that A_0^1 is a linear term but A_μ^{-1} and B_ν^0 are nonlinear terms in the sense of the classical grading. So the lowest order terms in Eq.(3.7) are

$$A^\mu = A_0^1 + \alpha_\mu A_\mu^{-1}, \text{ if } \mu < 2\nu$$

or

$$X^\nu = A_0^1 + \beta_\nu B_\nu^0, \text{ if } \mu > 2\nu.$$

Then by using one Lie bracket defined by A^μ they reduced Eq.(3.7) to the following so called the second order normal form for the case $\mu < 2\nu$:

$$Y(x) = A^\mu + \sum_{\substack{k>\mu, \\ k(\text{mod}(\mu+2)) \neq \mu-1, \mu}}^{\infty} \alpha_k A_k^{-1} + \sum_{\substack{k \ge \nu, \\ k(\text{mod}(\mu+2)) \neq \mu+1}}^{\infty} \beta_k B_k^0, \tag{3.10}$$

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Furthermore they used two Lie brackets to make further reduction for the second order normal form.

Theorem 2 Eq.(3.7) can be reduced by near identity transformation to

$$\dot{x} = X^{(\infty)} = A^\mu + \alpha_{\mu+\nu} A_{\mu+\nu}^{-1}$$

$$+ \sum_{\substack{k > \mu, \\ k(\mathrm{mod}(\mu+2)) \neq \mu-1, \mu, \mu+1}}^{\infty} \alpha_k A_k^{-1} + \sum_{\substack{k \geq \nu, \\ k(\mathrm{mod}(\mu+2)) \neq \mu+1}}^{\infty} \beta_k B_k^0. \qquad (3.11)$$

And if $\nu(\mathrm{mod}(\mu+2)) \neq 0$ then coefficients in the above equation are all uniquely determined by the original equation.

The case for $\mu > 2\nu$ is omitted here, we refer to their paper [1]. But the case $\mu = 2\nu$ was left as an open problem in [1].

4   Kokubu,Oka and Wang's Method

Kokubu, Oka and Wang[3] combined both ideas of [4] and [1] and proposed the N-th order normal forms by using N Lie brackets and the new grading function.
Definition 4.1 Let

$$D_n = \left\{ \prod_{i=1}^{n} x_i^{l_i} \mathbf{e}_j \;\middle|\; l_i \in \mathbb{Z}^+, x_i \in \mathbb{R} \text{ (or } C), i, j = 1, ..., n \right\},$$

where $\mathbf{e}_j$ is the j-th standard unit vector in R^n (or C^n). Then the function $\delta : D_n \to Z$ defined by

$$\delta \left(\prod_{i=1}^{n} x_i^{l_i} \mathbf{e}_j \right) = \sum_{i=1}^{n} a_i l_i - a_j, \qquad (4.12)$$

where $a_i \in \mathbb{N}$, $i = 1, ..., n$, is called a linear grading function.

In what follows we denote by H_k^δ the homogeneous polynomial splace with respect to the grading δ. Then $[H_j^\delta, H_k^\delta] \subseteq H_{j+k}^\delta$.
Definition 4.2

$$V = V_\mu + V_{\mu+1} + ... + V_{\mu+m} + ...$$

where $V_k \in H_k^\delta$, is called an *N-th order normal form (associated with grading function δ)*, if $V_{\mu+i} \in N_{\mu+i}^{(i)}$ for $\forall 1 \leq i \leq N-1$ and $V_{\mu+j} \in N_{\mu+j}^{(N)}$ for $j \geq N$, where $N_{\mu+k}^{(m)}$ is a complementary subspace to $\mathrm{Im} L_{k-m+1}^{(m)}$ in $H_{\mu+k}$ and where

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$L_k^{(m)} = L_k^{(m)}[V_\mu, V_{\mu+1}, ..., V_{\mu+m-1}]$ for $\forall m \in \mathbb{N}$ is defined as following:

$$L_k^{(1)} : H_k^\delta \to H_{\mu+k}^\delta : Y_k \to [Y_k, V_\mu],$$

$$L_k^{(m)} : \text{Ker} L_k^{(m-1)} \times H_{k+m-1}^\delta \to H_{\mu+m+k-1}^\delta : ((Y_k, Y_{k+1}, \ldots, Y_{k+m-2}), Y_{k+m-1})$$

$$\to [Y_k, V_{\mu+m-1}] + [Y_{k+1}, V_{\mu+m-2}] + \ldots + [Y_{k+m-1}, V_\mu], \quad \forall m > 1.$$

$$(4.13)$$

Definition 4.3

$$V = V_\mu + V_{\mu+1} + ... + V_{\mu+m} + ...$$

is called an *infinite order normal form*, if $V_{\mu+m} \in N_{\mu+m}^{(m)}$ for $\forall m \in \mathbb{N}$, where $N_{\mu+m}^{(m)}$ is a complementary subspace to $\text{Im} L_1^{(m)}$ in $H_{\mu+m}$ and where $L_1^{(m)} = L_1^{(m)}[V_\mu, V_{\mu+1}, ..., V_{\mu+m-1}]$ for $\forall m \in \mathbb{N}$.

Kokubu, Oka and Wang[3] proved that infinite order normal form is unique. They also gave a sufficient condition for an N-th order normal form being infinite order one and hence being unique normal form.

Theorem 3 If there exists an $N \in \mathbb{N}$ such that

$$\text{Im} L_k^{(N+m)} = \text{Im} L_{k+m}^{(N)} \tag{4.14}$$

for any $k, m \in \mathbb{N}$, then the N-th order normal form is an infinite order normal form.

Remark 4.1 If we consider only the near identity transformations and define the grading function δ by the classical degree minus 1, then Ushiki's method in fact can give the infinite order normal form in the sense of [3].

For Eq.(3.7) Kokubu, Oka and Wang[3] solved the special case, $\mu = 2, \nu = 1$, the open problem in [1]. Recently Wang, Li, Huang and Jiang[6] improved the computation method of [3] and solved the open problem under a generic condition([6]).

Consider the following equation:

$$\begin{cases} \dot{x} = y + h.o.t, \\ \dot{y} = \alpha x^\nu y + \beta x^{2\nu+1} + h.o.t., \end{cases} \tag{4.15}$$

where $\nu \in \mathbb{N}, \alpha, \beta \neq 0$ and h.o.t. denote the higher order terms in the sense of the grading function δ defined below.

Define the new grading function $\delta : D_2 \to \mathbb{Z}$ by

$$\delta \begin{pmatrix} x^m y^n \\ 0 \end{pmatrix} = m + n(\nu + 1) - 1, \qquad \delta \begin{pmatrix} 0 \\ x^m y^n \end{pmatrix} = m + n(\nu + 1) - \nu - 1.$$

Theorem 4 If β/α^2 is not an algebraic number, then the first order normal form of Eq.(4.15)(associated with grading function δ) is unique and can be taken as the following form:

$$\dot{x} = y,$$
$$\dot{y} = \alpha x^\nu y + \beta x^{2\nu+1} + b_{2\nu} x^{2\nu} y + \sum_{m=2\nu+2}^{\infty} a_m x^m + \sum_{\substack{n=\nu+1, \\ n(mod(\nu+1))\neq\nu-1,\nu}}^{\infty} b_n x^n y,$$

$$(4.16)$$

where a_m, b_n are all uniquely determined by Equation (4.15).

Acknowledgement

References

1. A. Baider and J. Sanders, J. Diff. Eq. **99**, 205–244 (1992).
2. L.O.Chua and H.Kokubu, Part I:Theory, IEEE.Trans.Circ.Sys.**35**, 863-880 (1988); Part II: applications, ibid.**36**,51-70 (1989).
3. H.Kokubu, H.Oka and D.Wang, J.Diff.Eq. **132**, 293–318 (1996).
4. S. Ushiki, Japan J. Appl. Math. **1**, 1–37 (1984).
5. D. Wang, World Sci., Singapore, 238–247 (1993).
6. D. Wang, J.Li, M.Huang and Y.Jiang, J.D.E. to appear.

ON DISTRIBUTION OF LIMIT CYCLE FOR QUADRATIC SYSTEM IN THE PLANE

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In this paper, we have proved the existence of only four kinds of distribution of limit cycle in E_{200} : (odd, even), (odd, odd), (even, even), (even, odd), whose lower bound at least is (i,j)distribution $(i,j = 0,1)$ respectively. Moreover there is just a distribution of limit cycle in E_{203} : (odd, even), Whose lower bound at least is (1, 0) distribution. We can not construct $(1,k)(k = 2,3,4), (0,k)(k = 1,2,3,4)$ and $(0,k)(k = 2,3,4)$ distribution of limit cycle, even we gave small perturbation for coefficients by the method of Hopf bifurcation.

The paper [1] give out the result for E_{203}: by the method of Hopf bifurcation we can not get (0, 4) and (2, 2) distribution of limit cycles, but theory and practice need us to utilize a inequality consisting of coefficients to describe the characteristic of phase diagram of the system.

In this paper, we give out the result for E_{200}: the quadratic system E_{200} with two normal focus and a unique saddle point at infinity possesses four kinds of distribution of limit cycle and their lower bound.

when we change the normal focus point $(0,0)$ into critical focus point order 3 of the quadratic system. we proved, there is a distribution of limit cycle in E_{203} (odd, even), whose lower bound at least is distribution of (1, 0), For E_{203}, on other distribution of three kinds of limit cycle can' t exist, So using small perturbation for coefficients of the system by Hopf bifurcation, We can not construct $(1, k)(k = 2, 3, 4), (0, k)(k = 1, 2, 3, 4)$ and $(0, k)(k = 2, 3, 4)$ distribution of limit cycle respectively.

1 A Quadratic System With two Normal Focus

We consider the system :

$$\begin{cases} \bar{x} = -y + dx + lx^2 + mxy + ny^2 \\ \bar{y} = x(1 + ax + by) \end{cases} \qquad (E_{200})$$

where two normal focus is $A_0(0, \frac{1}{n}), O_0(0, 0)$. Let :

$$M_0 = [9an - m(b-1)]^2 - 4[(b-1)^2 + 3am][m^2 + 3n(b-1)]$$
$$N_0 = a^2 b^2 + 2d(ab - mb + 2an) + [(a+m)^2 - 4l(n+l)]$$
$$P_0 = lb^2 - abm + a^2 m$$
$$R_0 = (d+m)^2 + 4n(b+n)$$
$$S_0 = d + \frac{m}{n}$$

and we introduce the following notation :

Without less of generality suppose $n > 0$ and $a \neq 0$, From [2] we get the following results:

Lemma 1 : If $P_0 \neq 0$ and $N_0 < 0$, then E_{200} possesses only two real singular point $A(0, \frac{1}{n})$ and $O(0,0)$, and

(1): When $0 < |d| < 2, O(0,0)$ is normal focus, it is unstable when $d > 0$ and it is stable when $d < 0$.

(2): When $R_0 < 0$ and $S_0 \neq 0, A(0, \frac{1}{n})$ is a normal focus, it is unstable when $S_0 = d + m/n > 0$ and it is stable when $S_0 = d + m/n < 0$.

Lemma 2: If $M_0 > 0$, then E 200 has only a unique saddle point at infinity.

Lemma 3 : If $N_0 < 0, R_0 < 0$, then $P_0 < 0$.

Lemma 4: If $a > 0(< 0)$ and $N_0 < 0$, then $L : 1 + ax + by = 0$ is a straight line without contact and slope $k = -a/b > 0(< 0)$, i. e that saddle point at infinity $P(P')$ lies above (below) straight line L on the equator.

From lemma 1 4, we obtain :

Theorem 1: Assume $a > 0$ and $N_00, R_0 < 0, M_0 < 0$.

If $0 < d < 2, S_0 < 0$, then distribution of limit cycle is (odd, even) in E_{200} (see fig 1), Whose lower bound at least is (1, 0) distribution and the total numbers of limit cycles has odd numbers (at least one).

If $-2 < d < 0, S_0 < 0$, then distribution of limit cycle is (odd, odd) in E_{200} (see fig 2), Whose lower bound at least is (1, 1) distribution and the total numbers of limit cycles has even numbers (at least two).

If $0 < d < 2, S_0 < 0$, then distribution of limit cycle is (even, even) in E_{200} (see fig 3), Whose lower bound at least is (0, 0) distribution and the total numbers of limit cycles has even numbers (may be zero).

If $-2 < d < 0, S_0 < 0$, then distribution of limit cycle is (even, add) in E_{200} (see fig 4), Whose lower bound at least is (0,1) distribution and the total numbers of limit cycles has add numbers (may be zero).

When $a0$, we have similar results with $a < 0$, then we have :

Theorem A: The system E_{200} With two normal focus and a unique saddle point at infinity possesses only four kinds of distribution of limit cycle : (odd, even), (odd, odd), (even, even) and (even, odd), whose lower bound at least is (i, j) distribution. $(i, j = 0, 1)$.

A Quadratic system with a critical focus of order 3

Can the system E_{200} keep four kinds of distribution of cycle limit when one or two normal focus of E 200 changes into critical focus? This question relates to E_{200} maximum numbers and relative place of limit cycle.

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In this section, we will consider only the system:

$$\begin{cases} \bar{x} = -y + lx^2 = 5axy + ny^2 \\ \bar{y} = x + ax^2 + (3l + 5nx)xy \end{cases} \qquad (E_{203})$$

In other wards, when $d = 0, m = 5a, b = 3l + 5$, we changes normal focus of E_{200} into critical focus of order 3 of E_{200}. Let :

$$M_3 \equiv -\frac{1}{3}M_0 = 500a^4 + a^2[25(2l + 5n)^2 + 90n(2l + 5n) - 27n^2] + 4n/(2l + 5n)^3,$$

$$N_3 \equiv \frac{1}{2}N_0 = 3a^2 - l(l + 2n), P_3 \equiv -3a^2(5l + 8n) + l(3l + 5n)^2 \qquad (1.1)$$

$$R_3 \equiv 25a^2 + 12n(1 + 2n), S_3 \equiv 5a/n,$$

$$Q_3 = 2a^2 + ln + 2n^2, J_3 = a^2(5l + 6n) - 3(l + 2n)(l + n)^2,$$

$$W_3 = -5a^3Q_3J_3. \qquad (1.2)$$

Similar above lemma 1   4, we have :

Lemma 5: If $P_3 \neq 0, N_3 < 0$, then the E 200 possesses only two real singular point $A(0, \frac{1}{n}), O(0,0)$, and

When $W_3 \neq 0, O(0,0)$ is a critical focus of order 3, it is stable when $W_3 < 0$, it is unstable when $W_3 > 0$.

When $R_3 < 0, a \neq 0, A(0, \frac{1}{n})$ is a normal focus, it is unstable when $a < 0$, it is stable when $a < 0$.

Lemma 6: If $M_3 < 0$, then E_{200} has only a unique saddle point at infinity.

Lemma 7: If $N_3 < 0, R_3 < 0$, then $P_3 < 0$.

Lemma 8: If $a < 0(< 0)$ and $N_3 < 0$, then $L1 + ax + (3l + 5n)y = 0$ is a straight line without contact and slope $k = -a/(3l + 4n) > 0(< 0)$, i. e that saddle point at infinity $P(P')$ lies above (below) straight line L on the equator.

Suppose $a < 0, R < 0$, without losing generality. From lemma 5   8, we obtain :

Theorem 2: If $M_3 < 0$, and one of the following conditions are satisfied,

(1): $R_3 < 0, N_3 < 0, R_3 < 0$ and $W_3 < 0$.

(2): $l < -2n$.

(3): $3l + 5n = b - n$.

Then, distribution of limit cycle is (odd, even) in E_{200}, whose lower bound at least is (1, 0) distribution. If we gave small perturbation for coefficients of E_{200} by Hopf bifurcation, It will possess limit cycles with $(1, k)(k = 1, 2, 3)$ distribution .

From (2), (3), we have :

Lemma 9 : $M_3 < 0, R_3 < 0W_3 < 0$.

Theorem 3: In a Quadratic system E_{203} with a critical focus of order 3 and a unique saddle point at infinity.

(1): distribution of limit cycle (odd, odd) in E_{203} is not exist, Whose lower bound at least (1, 1) distribution is also not exist, i.e We can not construct $(1,k)(k = 2,3,4)$distribution of limit cycle respectively, if we give small perturbation for coefficients of the system by Hopf bifurcation.

(2): distribution of limit cycle (even, even) in E_{203} is not exist, Whose lower bound at least (0, 0) distribution is also not exist, i.e We can not construct $(0,k)(k = 1,2,3)$ distribution of limit cycle respectively, if we give small perturbation for coefficient of the system by Hopf bifucartion.

:distribution of limit cycle (even, odd) in E_{203} is not exist, Whose lower bound at least (0, 1) distribution is also not exist, i.e We can not construct $(0,k)(k = 2,3,4)$ distribution of limit cycle respectively, if we give small perturbation for coefficient of the system by Hopf bifurcation.

Proof: Suppose $a < 0$ without losing generality. From lemma 5 9, we obtain :

(1): If the distribution of the system E_{203} is (odd, odd), then we have : $N_3 < 0, M_3 < 0, R_3 < 0, W_3 < 0, S_3 > 0$, from lemma 9 we have $W_3 < 0$, hence we can not obtain distribution (odd, odd) in E_{203}.

(2) : If the distribution of the system E_{203} is (even, even) or (even , odd), then we have : $S_1 = 5a/n < 0$, but $a < 0$, hence we can not obtain distribution (even, even) or (even, odd).

From theorem 2 3, we obtain :

Theorem B: The system E_{203} with a critical focus of order 3 and a unique saddle point at infinity possesses only kind of distribution of limit cycle (odd, odd), whose lower bound at least is (0, 1) distribution, we can only construct $(1,k)(k = 1,2,3)$ distribution by a small perturbation for coefficient. Other three kinds of distribution (odd, odd) , (even, even), (even, odd) not exist in E_{203}, since their lower bound (1, 1), (0, 0) (0, 0) distribution not exist, So we can' t construct $(1,k)(k = 2,3,4),(0,k)(k = 2,3,4)$, and $(0,k)(k = 1,2,3,4)$ distribution respectively, if we give small perturbation for coefficient of the system by Hopf bifurcation.

References

1. Cai Suilin. Survey of planar quadritic differential systems. Advances in Mathematics **28**, 1 (5 12) 1989.
2. Shi songling. On limit cycle of planar quadratic system, Scientia Sinica ., 734 739 (1980).

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3. Qin Yuanxun, Shi Songling, Cai Surlin. On limit cycle of planar quadratic system, Scientia Sinica (series A) ., 41 50 (1982).

4. Li Chengzhi. Two problems of planar quadratic system. Scientia Sinica (Series A) **471 481**, 1983 (.)

OSCILLATION OF CERTAIN NEUTRAL DIFFERENCE EQUATION OF MIXED TYPE WITH HIGH ORDER

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This paper is mainly concerned with some certain linear neutral difference equations

$$\Delta^i(x_n + ax_{n-m} - bx_{n+k}) = c(qx_{n-g} + px_{n+h}).$$

Sufficient conditions for oscillation of equations (E_i, c) for $i = 2N + 1$, $i = 2N$ and $c = \pm 1$ are established respectively, and the results extend that of S.R.Grace [2]. Where, $a, b, p, q \geq 0$ are real, $m, k, g, h > 0$ are integer and multiples of i, $i = 1, 2, \cdots$.

1 Introduction

This paper is concerned with the oscillation behavior of neutral difference equation of the form,

$$\Delta^i(x_n + ax_{n-m} - bx_{n+k}) = c(gx_{n-g} + px_{n+h}), \qquad (E_i, c)$$

where $c = \pm 1$, i is a positive integer, a, b, p, q are nonnegative real numbers, $g, h, m,$ and k are positive integer and multiples of i. The definitions of oscillation and operator Δ^i for $i = 0, 1, \cdots$ see the papers by I.Györy [1] and S.R.Grace [2].

The analogue of Eq. (E_i, c) in the continuous case is the neutral functional equations

$$\frac{d^i}{dt^i}(x(t) + ax(t-m) - bx(t+k)) = c(qx(t-g) + px(t+h)) \qquad (N_i, c)$$

where i is a positive integer and $c = \pm 1$, a, b, m, k, p, q are nonnegative real numbers, g, h are positive constants. There have been a lot of interests in the oscillation of (E_i, c) and (N_i, c) with different order. S.R.Grace [2] [3] has

established criterion for oscillation of Eq. (N_i, c) where i is even and oscillation of Eq. (E_i, c) where $i = 1, 2, 3$.

Our aim in this article is to establish some conditions involving the coefficients and arguments only, under which all solutions of Eq. (E_i, c) oscillate, where i is a positive integer.

2   Main Results

The following Lemmas are needed in the proof of our main results.

Lemma 6 (2) *Assume that q is positive real number and k is a positive integer and a multiple of i. Then the following statements hold.*

(a) If

$$q > \frac{i^i (k-i)^{k-i}}{k^k} \quad for \quad k > i,$$

then the difference inequality

$$\Delta^i y_n \geq q y_{n+k} \quad for \quad i \geq 1$$

has no eventually positive solution $\{y_n\}$ which satisfies $\Delta^j y_n \geq 0$ eventually, $j = 0, 1, \cdots, i$.

(b) If

$$q > \frac{i^i k^k}{(k+i)^{k+i}} \quad for \quad k \geq 1,$$

then the difference inequality

$$(-1)^i \Delta^i y_n \geq q y_{n-k} \quad for \quad i \geq 1$$

has no eventually positive solution $\{y_n\}$ which satisfies $(-1)^j \Delta^j y_n > 0, \quad j = 0, 1, \cdots i$.

Lemma 7 *Let $\{x_n\}$ be a sequence. If $\Delta x_n > 0$ (< 0) and $\Delta^2 x_n > 0$ (< 0) eventually, then $x_n > 0$ (< 0) eventually, and $\lim_{n \to \infty} x_n = +\infty$ ($-\infty$).*

Theorem 5 *Let $b > 0, \quad h > 2N+1, \quad r = g + k \geq 1$. If*

$$\frac{p}{1+a} > \frac{(2N+1)^{2N+1}(h-2N-1)^{h-2N-1}}{h^h} \qquad H1(a)$$

and

$$\frac{q}{b} > \frac{(2N+1)^{2N+1} r^r}{(r+2N+1)^{r+2N+1}}, \qquad H1(b)$$

then Eq. $(E_{2N+1}, 1)$ is oscillatory.

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Proof. Assume that Eq. $(E_{2N+1}, 1)$ has an eventually positive solution $\{x_n\}$, say $x_n > 0$ for $n \geq n_0 \geq 0$. Let

$$y_n = x_n + ax_{n-m} - bx_{n+k}, \qquad (2.1)$$

then

$$\Delta^{2N+1}y_n = qx_{n-g} + px_{n+h} > 0, \quad \text{for} \quad n \geq n_1 \geq n_0, \qquad (2.2)$$

which implies that $\{\Delta^j y_n\}$ is evetually one sign for $j = 0, 1, \cdots 2N$. Therefore, either (A) $y_n < 0$ evetually, or (B) $y_n > 0$ evetually.

(A) Assume that $y_n < 0$ for $n \geq n_1$. Let

$$0 < v_n = -y_n = bx_{n+k} - ax_{n-m} - x_n \leq bx_{n+k}. \qquad (2.3)$$

There exists $n_2 \geq n_1$ such that

$$x_n \geq \frac{1}{b}v_{n-k} \quad \text{for} \quad n \geq n_2. \qquad (2.4)$$

Using (4) in (2), we have

$$\Delta^{2N+1}v_n + \frac{q}{b}v_{n-(g+k)} \leq 0. \qquad (2.5)$$

The next step we will prove that for sufficient large n, $\{v_n\}$ satisfies that

$$(-1)^i \Delta^i v_n > 0 \quad \text{for} \quad i = 0, 1, \cdots, 2N+1. \qquad (2.6)$$

By (2), we know that $\Delta^{2N+1}v_n < 0$ eventually, which implies that $\Delta^{2N}v_n > 0$ hold eventually, otherwise in view of Lemma 2, $\Delta^{2N}v_n < 0$ eventually contradict with $v_n > 0$ eventually. Therefore there are two possibilities to consider: (i) $\Delta v_n > 0$ eventually, (ii) $\Delta v_n < 0$ eventually. Suppose that (i) holds, there exists $n_3 \geq n_2$ and $c_1 > 0$ such that

$$v_{n-(g+k)} \geq c_1 \quad \text{for} \quad n \geq n_3. \qquad (2.7)$$

Using (7) in (5) and summing from n_3 to $M - 1 > n_3$, we have

$$0 < \Delta^{2N}v_M \leq \Delta^{2N}v_{n_3} - \frac{q}{b}c_1(M - n_3) \longrightarrow -\infty \quad (M \to +\infty),$$

which yields a contradiction. Next suppose that (ii) holds. We can conclude that $\Delta^{2N-1}v_n < 0$ eventually, otherwise $\Delta^{2N-1}v_n > 0$ eventually contradicts with $\Delta v_n < 0$ eventually. Considering $\Delta^{2N-2}v_n$, in view of Lemma 2, it could be nothing but $\Delta^{2N-2}v_n > 0$ eventually, otherwise $\Delta^{2N-2}v_n < 0$ eventually

contradicts with $v_n > 0$ eventually. After using Lemma 2 finite times, we can show that (6) holds.

Then in view of Lemma 1 (b) and H1(b), inequality (5) has no eventually positive solution $\{v_n\}$ satisfying (6), which is a contradiction.

(B) Assume that $y_n > 0$ for $n \geq n_1 > 0$. Let

$$w_n = y_n + ay_{n-m} - by_{n+k}. \tag{2.8}$$

Then

$$\Delta^{2N+1}w_n = qy_{n-g} + py_{n+h}, \tag{2.9}$$

$$\Delta^{2N+1}(w_n + aw_{n-m} - bw_{n+k}) = qw_{n-g} + pw_{n+h}. \tag{2.10}$$

There are two possibilities to consider: (i) $\Delta y_n < 0$ eventually, (ii) $\Delta y_n > 0$ eventually.

Suppose that (i) holds. Because sequence $\{y_n\}$ is decreasing and positive, from (8), we have $w_n < y_n + ay_{n-m} < (1+a)y_{n-m}$. There exists $n_2 > n_1$, such that $y_n > \frac{1}{1+a}w_{n+m}$ for $n \geq n_2$. Therefore from (9), we have

$$\Delta^{2N+1}w_n > py_{n+h} > \frac{p}{1+a}w_{n+h+m}. \tag{2.11}$$

Then we consider three possibilities (I) $w_n < 0$ eventually, (II) $w_n > 0$ and $\Delta w_n < 0$ eventually, and (III) $w_n > 0$ and $\Delta w_n > 0$ eventually. In the case of (I), from the equation

$$\Delta^{2N+1}(y_n + ay_{n-m} - by_{n+k}) = qy_{n-g} + py_{n+h},$$

and (8) (10), it is obvious that the proof is similar to (A) and hence is omitted. Supposed that (II) holds. By (11) and using Lemma 2 finite times, we will have a contradiction. Suppose that (III) holds. If $\Delta^{2N}w_n > 0$ eventually, then in view of Lemma 2, we have

$$\Delta^j w_n > 0 \quad \text{eventually} \quad \text{for} \quad j = 0, 1, \cdots, 2N+1, \tag{2.12}$$

and sequence $\{w_n\}$ is increasing, thus (11) has the form

$$\Delta^{2N+1}w_n > \frac{p}{1+a}w_{n+h}. \tag{2.13}$$

In view of Lemma 2 and H1(a), inequality (13) has no eventually positive solution $\{w_n\}$ satisfying (12), which is a contradiction. If $\Delta^{2N}w_n < 0$ eventually, noticing that $w_n > 0$ and $\Delta w_n > 0$ eventually, there exists $n_3 > 0$ and $c_2 > 0$ such that

$$w_n > c_2 \quad \text{for} \quad n \geq n_3. \tag{2.14}$$

Using (14) in (13) and summing from n_3 to $M - 1 > n_3$, we have

$$0 > \Delta^{2N} w_n > \Delta^{2N} w_{n_3} + \frac{p}{1+a} c_2 (M - n_3) \longrightarrow +\infty \quad (\text{as} \quad M \to +\infty)$$

a contradiction.

Assume that (ii) holds, by (9) and Lemma 2, we know that (12) holds and sequence $\{\Delta^{2N+1} w_n\}$ is increasing. From (10) we obtain (13), which contradicts with condition H1(a) and the proof is completed.

By the same method with the proof of Theorem 1, we can establish the sufficient conditions for oscillation of Eq. (E_i, c) as follows.

Theorem 6 *Let* $b > 0$, $h > 2N > 1$ *and* $s = g - m \geq 1$. *If*

$$\frac{p}{1+a} > \frac{(2N)^{2N}(h - 2N)^{h-2N}}{h^h} \qquad\qquad H2(a)$$

and

$$\frac{q}{1+a} > \frac{(2N)^{2N} s^s}{(s + 2N)^{s+2N}}, \qquad\qquad H2(b)$$

then Eq. $(E_{2N}, 1)$ *is oscillatory.*

Theorem 7 *Let* $b > 0$, $t = h - k > 2N + 1$, $s = g - m \geq 1$. *If*

$$\frac{p}{b} > \frac{(2N + 1)^{2N+1}(t - 2N - 1)^{t-2N-1}}{t^t} \qquad\qquad H3(a)$$

and

$$\frac{q}{1+a} > \frac{(2N + 1)^{2N+1} s^s}{(s + 2N + 1)^{s+2N+1}}, \qquad\qquad H3(b)$$

then Eq. $(E_{2N+1}, -1)$ *is oscillatory.*

Theorem 8 *Let* $b > 0$, $t = h - k > 2N$, $r = g + k \geq 1$. *If*

$$\frac{p}{b} > \frac{(2N)^{2N}(r - 2N)^{r-2N}}{t^t} \qquad\qquad H4(a)$$

and

$$\frac{q}{b} > \frac{(2N)^{2N} r^r}{(r + 2N)^{r+2N}}, \qquad\qquad H4(b)$$

then Eq. $(E_{2N}, -1)$ *is oscillatory.*

Remark 2.1 *The above results can be extended to the case that p and q are variable coefficients, i.e. considering the following equation*

$$\Delta^i (x_n + a x_{n-m} + b x_{n+k}) = c(q_n x_{n-g} + p_n x_{n+h}). \qquad (E_i', c)$$

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We have the following corollary.

Corollary 1 . *Let $b > 0, r, s, t$ be defined as in Theorem 1~Theorem 4. $\{q_n\}$ and $\{p_n\}$ are real nonnegative sequences. Assume that $\lim\limits_{n \to \infty} p_n \geq p$, $\lim\limits_{n \to \infty} q_n \geq q$. If $H_j(a)$ and $H_j(b)$ hold (for $j = 1, 2, 3, 4$), then the results corresponding to Theorem 1 $\sim$ theorem 4 are still valid for Eq. (E_i', c).*

Example 6 *Considering the equation:*

$$\Delta(x_n + h^m x_{n-m} - h^{-k} x_{n+k}) = \frac{(n-m-k)(h-1)+h}{2n+h-g}(h^g x_{n-g} + h^{-h} x_{n+h})$$

where,$i = 1$, m, k are positive integers.

It can easily testified that $x_n = nh^n$ is a nonoscillatory solution of the above equation because of failure of (H1a), where $p_n \to \frac{h-1}{2} h^{-h} = p$.

Remark 2.2 *The conditions of oscillation for Eq. (E_i, c) are just only sufficient other than necessary. It would be valuable and interesting to establish critical conditions for Eq. (E_i, c) to oscillate.*

Example 7 *Consider the defference equation*

$$\Delta^i(x_n + ax_{n-m} - ax_{n+m}) = 2^{i-1}(-1)^{g+i}(x_{n-g} + x_{n+g}) \qquad (L_i, c)$$

where, i is a positive integer, $a > 0$ is a real number, g, m are multiples of i.

It is not difficult to testify that for any arbitrary m, a, g and $i = 1, 2, \cdots$, $x_n = (-1)^n$ is an oscillatory solution of Eq. (L_i, c). Let $0 < a \leq 2^{i-1} - 1$. It can be easily testified that for odd g the conditions of Theorem 1 and Theorem 4 hold,and therefore the Eq. $(L_{2N+1}, 1)$ and Eq. $(L_{2N}, -1)$ are oscillatory, and that for even g the conditions of Theorem 2 and Theorem 3 hold and therefore the Eq. $(L_{2N+1}, -1)$ and Eq. $(L_{2N}, 1)$ are oscillatory. It is also possible that $H_j(a)$ and $H_j(b)$ fail when a is large enough, however Eq. (L_i, c) has a solution of $x_n = (-1)^n$ yet.

References

1. I.Györi, and G.Ladas,*Oscillation Theory of Delay Differential Equations*, Claredon Press. Oxford. 1991.
2. S.R.Grace, *J.Math.Anal.Appl.* **224**, 241(1998).
3. Jurang Yan and Chuanxi Qian, *J.Math.Annl.Appl.* **165**,346(1992).
4. S.R.Grace, *J.Math.Anal.Appl.*, **184**,44(1994).
5. S.R.Grace and B.S.Lalli, *Appl.Math.Comput.*, **51**,119(1992).
6. B.S.Lalli, and S.R.Grace, *Appl. Math. Comput.* **62**,47(1994).

PERSISTENCE AND PERIODIC ORBITS FOR NONAUTONOMOUS DIFFUSION VOLTERRA PREDATOR-PREY SYSTEM WITH UNDERCROWDING EFFECT

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This paper considers a predator-prey system in which one of two species can diffuse between two patches, while the other is confined to one patch and cannot diffuse. It is proved that the system can be made persistent under some appropriate conditions. Further, if the system is a periodic system, it can have a strictly positive periodic orbit which is globally asymptotically stable under the appropriate conditions.

1 Introduction

For the undercrowding effect of the species, Kuno [1] first established this kind of model about a normal Volterra predator-prey system, Bazytin [2] and Chen [3] also studied this kind of models. But all the coefficients in the system they studied are constant. After, all the coefficients in the system Wang [4], [5] also studied are periodic functions. The model [4] we consider in this paper is

$$
\begin{aligned}
\dot{x}_1 &= a_1(t)x_1^2(K_1(t) - x_1) - b(t)x_1 y + D_1(t)(x_2 - x_1), \\
\dot{x}_2 &= a_2(t)x_2^2(K_2(t) - x_2) + D_2(t)(x_1 - x_2), \\
\dot{y} &= y(-c(t) + d(t)x_1 - \alpha(t)y).
\end{aligned}
\tag{1}
$$

Here x_1 and y are population density of species X and Y in patch 1, and x_2 is density of species X in patch 2. Species Y is confined to patch 1 while the species X can diffuse between two patches. $a_i(t)$, $K_i(t)$, $D_i(t)(i = 1, 2)$, $b(t)$, $c(t)$, $d(t)$, $\alpha(t)$ are all continuous and strictly positive functions. $D_i(t)(i = 1, 2)$ are diffusive coefficients of species X. We define for real function $f(t)$: $f^U = \sup_{[0,\infty)} \{f(t)\}, f^L = \inf_{[0,\infty)} \{f(t)\}$. To use this definition, we need all the coefficients to satisfy

$$
\begin{aligned}
&\min_{i=1,2}\{a_i^L, K_i^L, D_i^L, b^L, c^L, d^L, \alpha^L\} > 0, \\
&\max_{i=1,2}\{a_i^U, K_i^U, D_i^U, b^U, c^U, d^U, \alpha^U\} < \infty
\end{aligned}
\tag{2}
$$

2 Main Results

From (1) and (2), by the same method as [4,6], we can prove the following lemmas.

Lemma 1 $R_+^3 = \{(x_1, x_2, y)|x_i > 0(i = 1, 2), \quad y > 0\}$ *is a positive invariant set of system (1).* We will focus our discussion in R_+^3 with respect to biological meaning. And this also ensures the solution with positive initial value to be positive for all time. Now we construct an ultimately bounded region of system (1). We can let it be

$$H_0 = \{(x_1, x_2, y)|0 < x_i < M(i = 1, 2), \quad 0 < y < y^U\}.$$

Here, M and y^U are selected as

$$M > M^*, y^U > y^{U^*}, \tag{3}$$

where M^* and y^{U^*} are defined as

$$M^* = \max\{K_1^U, K_2^U, \frac{c^L}{d^U}\}, \tag{4}$$

$$y^{U^*} = \frac{d^U M - c^L}{\alpha^L}. \tag{5}$$

Lemma 2 *If system (1) satisfies assumption (2), then bounded set H_0 is a positive invariant set of system (1), and for each solution $\{x_1(t), \ x_2(t), \ y(t)\}$ satisfying $x_i(0) > 0(i = 1, 2), \quad y(0) > 0$, there exists $T > 0$; if $t \geq T$, we have $\{x_1(t), x_2(t), y(t)\} \in H_0$.* We will construct a compact region of the positive orthant which has positive distance from coordinate hyperplanes. We need all the coefficients to satisfy

$$\max\{K_1^U, K_2^U\} < \frac{c^L}{d^U} + \frac{(a_1^L K_1^L)^2 \alpha^L}{4 a_1^U b^U d^U}, \tag{6}$$

$$\frac{a_1^L}{a_1^U} < \frac{K_2^L}{K_1^L}, \tag{7}$$

$$\frac{c^U}{d^L} < \min\{K_2^L, \frac{a_1^L K_1^L}{2 a_1^U}\} = m^*. \tag{8}$$

Theorem 9 *Suppose that the system (1) satisfies assumption (2) and (6)–(8). Then there exists a compact region $H_1 \subset R_+^3$ such that for each solution $\{x_1(t), x_2(t), y(t)\}$ which satisfies $x_i(0) > 0(i = 1, 2), \quad y(0) > 0$, there exists $T > 0$; if $t \geq T$, we have*

$$\{x_1(t), x_2(t), y(t)\} \in H_1.$$

That is to say, system (1) is persistent.

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proof Suppose that $\{x_1(t), x_2(t), y(t)\}$ is a solution of (1) which satisfies

$$x_i(0) > 0 (i = 1, 2), \quad y(0) > 0.$$

According to Lemma2, we know that each trajectory with positive initial value will ultimately be bounded in H_0, so, without loss of generality, we can assume that this solution satisfies

$$\{x_1(t), x_2(t), y(t)\} \in H_0 \quad for \quad t \geq 0.$$

From the first two equations of system (1), we have

$$\dot{x}_1 \geq x_1(-b^U y^U + a_1^L K_1^L x_1 - a_1^U x_1^2) + D_1(t)(x_2 - x_1)$$

$$\dot{x}_2 \geq a_2(t) x_2^2 (K_2^L - x_2) + D_2(t)(x_1 - x_2)$$

From (5) and (6), we know $(a_1^L K_1^L)^2 - 4 a_1^U b^U y^{U*} > 0$ holds. Also from Lemma 2, we obtain that y^U can be chosen close enough to y^{U*} such that $(a_1^L K_1^L)^2 - 4 a_1^U b^U y^U > 0$ holds. So, from (7), we can choose m as $0 < m_1(y^U) < m < m^*$, where $m_1(y^U)$ satisfies the following equation in x_1: $-a_1^U x_1^2 + a_1^L K_1^L x_1 - b^U y^U = 0$. There exists $T_1 > 0$; if $t \geq T_1$, we have $\min\{x_1(t), x_2(t)\} \geq m$. According to system (1), we have $\dot{y} \geq y(-c^U + d^L m - \alpha^U y)$. From (8), we say that m can be chosen close to m^* such that the inequality $-c^U + d^L m > 0$ holds. we let

$$\widehat{y}(m) = \frac{d^L m - c^U}{\alpha^U},$$

So we can choose y^L as $0 < y^L < \widehat{y}(m)$ such that $y(t) \geq y^L (t \geq T_2 > 0)$. Finally, we let $H_1 = \{(x_1, x_2, y) | m \leq x_i \leq M (i = 1, 2), y^L \leq y \leq y^U\}$. Then H_1 is a bounded compact region in R_+^3 which has positive distance from coordinate hyperplanes. Let $T = \max\{T_1, T_2\}$. We have $\{x_1(t), x_2(t), y(t)\} \in H_1$, if $t \geq T$. □ We suppose that all the coefficients in system (1) are continuous and positive $\omega-$periodic functions. then the system (1) is an $\omega-$ period system for this case, and the coefficients will naturally satisfy assumption(2). We denote the unique solution of periodic system (1) for initial value $Z^0 = \{x_1^0, x_2^0, y^0\}$:

$$Z(t, Z^0) = \{x_1(t, Z^0), x_2(t, Z^0), y(t, Z^0)\}, \quad for \quad t > 0, \quad Z(0, Z^0) = Z^0.$$

Now define Poincaré transformation $A : R_+^3 \rightarrow R_+^3$ is

$$A(Z^0) = Z(\omega, Z^0).$$

Here, ω is the period of periodic system(1). In this way, the existence of periodic orbit of system(1) will be equal to the existence of the fixed point of A.

Theorem 10 *If ω−periodic system (1) satisfies (6)-(8), then there is at least one strictly positive periodic orbit of (1).*

proof If assumption (6)-(8) is satisfied, then from Theorem 1 we know that the compact region $H_1 \subset R_+^3$ is a positive invariant set of system(1), and H_1 also is a closed bounded convex set. So we have

$$Z^0 \in H_1 \Longrightarrow Z(t, Z^0) \in H_1,$$

and $Z \in H_1$, thus $AH_1 \subset H_1$. The operator A is continuous because the solution is continuous about the initial value. Using the fixed point theorem of Brouwer [4], we can obtain that A has at least one fixed point in H_1, then there exists at least one strictly positive ω−periodic orbit of system (1). □

Theorem 11 *Suppose that the ω−periodic system (1) satisfies (6)-(8) and*

$$
\begin{aligned}
2m^* a_1^L &> a_1^U K_1 U + d^U + \frac{D_2^U}{m^*}, \\
2m^* a_2^L &> a_2^U K_2^U + \frac{D_1^U}{m^*}, \\
\alpha^L &> b^U.
\end{aligned}
\tag{9}
$$

Then the system(1) has a unique strictly positive ω−periodic orbit which is globally asymptotically stable.

proof Suppose $Z(t) = \{x_1(t), x_2(t), y(t)\}$ is a solution of (1) with $x_i(0) > 0 (i = 1, 2), y(0) > 0$; according to Lemma1, we let

$$\tilde{x}_1(t) = \ln x_i(t), (i = 1, 2), \tilde{y}(t) = \ln y(t),$$

$$\tilde{u}_i(t) = \ln u_i(t), (i = 1, 2), \tilde{v}(t) = \ln v(t).$$

Here, $U(t) = \{u_1(t), u_2(t), v(t)\}$ is strictly positive ω−periodic solution of (1), and its existence is ensured by Theorem 2. By Theorem 1, we know that each solution with positive initial conditions will be ultimately bounded in H_1. Consider the Lyapunov function

$$V(t) = \sum_{i=1}^{2} |\tilde{x}_i(t) - \tilde{u}_i(t)| + |\tilde{y}(t) - \tilde{v}(t)|.$$

Now we calculate and estimate the upper right derivation of $V(t)$ along the solution of (1).

$$
\begin{aligned}
D^+ V(t) \leq &-(2ma_1^L - a_1^U K_1^U - d^U)|x_1(t) - u_1(t)| \\
&-(2ma_2^L - a_2^U K_2^U)|x_2(t) - u_2(t)| \\
&-(\alpha^L - b^U)|y(t) - v(t)| + \tilde{D}_1(t) + \tilde{D}_2(t).
\end{aligned}
$$

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where

$$\tilde{D}_i(t) = \begin{cases} D_i(t)(\frac{x_j(t)}{x_i(t)} - \frac{u_j(t)}{u_i(t)}), & x_i(t) > u_i(t), \\ D_i(t)(\frac{u_j(t)}{u_i(t)} - \frac{x_j(t)}{x_i(t)}), & x_i(t) < u_i(t), \end{cases} \quad i \neq j; i, j = 1, 2.$$

We can obviously obtain

$$\tilde{D}_i(t) \leq \frac{D_i^U}{m}|x_j(t) - u_j(t)|, i \neq j; i, j = 1, 2.$$

So we get

$$\begin{aligned} D^+V(t) \leq & -(2ma_1^L - a_1^U K_1^U - d^U - \frac{D_2^U}{m})|x_1(t) - u_1(t)| \\ & - (2ma_2^L - a_2^U K_2^U - \frac{D_1^U}{m})|x_2(t) - u_2(t)| \\ & - (\alpha^L - b^U)|y(t) - v(t)|. \end{aligned}$$

From the proof of Theorem 1, we know that y^U can be close to y^{U^*} sufficiently, while m can be close to m^* sufficiently if M is close to M^* sufficiently. So according to the assumption (9), we can choose appropriate M such that for m which is sufficiently close to m^*, we have

$$\begin{aligned} 2ma_1^L &> a_1^U K_1^U + d^U + \frac{D_2^U}{m}, \\ 2ma_2^L &> a_2^U K_2^U + \frac{D_1^U}{m}, \\ \alpha^L &> b^U. \end{aligned}$$

So there exists $\beta > 0$ such that

$$D^+V(t) \leq -\beta(\sum_{i=1}^{2}|x_i(t) - u_i(t)| + |y(t) - v(t)|), \quad t \geq T, \tag{10}$$

where $\beta = \min\{2ma_1^L - a_1^U K_1^U - d^U - \frac{D_2^U}{m}, 2ma_2^L - a_2^U K_2^U - \frac{D_1^U}{m}, \alpha^L - b^U\}$. An integration of (10) leads to

$$V(t) + \beta \int_T^t (\sum_{i=1}^{2}|x_i(s) - u_i(s)| + |y(s) - v(s)|)ds \leq V(T) < +\infty,$$

as a consequence of which we have

$$\lim_{t \to +\infty} \sup \int_T^t (\sum_{i=1}^{2}|x_i(s) - u_i(s)| + |y(s) - v(s)|)ds \leq \frac{V(T)}{\beta} < +\infty. \tag{11}$$

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It follows from (11) that

$$\lim_{t \to +\infty} |x_i(t) - u_i(t)| = 0 (i = 1, 2), \lim_{t \to +\infty} |y(t) - v(t)| = 0. \quad \square$$

1. E.Kuno, Advances in ecological research, **16** (1987)252.
2. A.D.Bazytin, Mathematical biophysics of interacting populations, *Nauka, Moscow,* (1985) *(in Russian).*
3. L.S.Chen and H.Li, The Mathematical behavior of a Volterra predator-prey model with undercrowding effect, *Ann. of Diff. Eqs.* **11** (1995), 14-19.
4. J.K.Wang, Analysis of the periodic Volterra Predator-Prey model with undercrowding effect, *Ann. of Diff. Eqs.* **14**(1998), 68-73.
5. J.K.Wang, The mathematical behavior of a nonautonomous Voltera Predator-Prey system with undercrowding effect. *Ann. of Diff. Eqs.* **14** (1998), 307-312.
6. G.Z.Zeng, et al, Persistence and periodic orbits for two-species nonautonomous diffusion Lotka-Volterra models, *Mathl.Comput. Modeling* **20**(1994), 69-80.

COMPETING PREDATORS FOR A PREY IN A CHEMOSTAT MODEL WITH TWO-NUTRIENTS AND DELAY

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In this paper, a chemostat model of n predators competing for a prey in two-nutrients with delay is considered. The key feature of our model is the incorporation of two-nutrients on two trophic levels and the incorporation of time delay. Based on the technique of Razumikhin, we obtain the sufficient conditions for global asymptotic stability of the extinct equilibrium.

1 Introduction

The paper of Hsu *et al* ([1]) was the first paper to give a mathematical analysis of chemostat model on n competing microorganisms for a single nutrient. Since this paper, many papers have been written on chemostat model. In those

papers, lots of paper dealt with single nutrient([2-4]), however, microorganisms often depend on not only one nutrient, but depend on many nutrients. Hence, it is necessary to study chemostat model on two-nutrients.

Under condition of single trophic level for nutrients, Ballyk *et al* ([5,6]) considered the model of single-species growth on two nutrients, got the sufficient conditions for stability of the extinct equilibrium and survival equilibrium, but they didn't consider species-specific time lag due to lapses between the uptake of nutrients by cells and the incorporation of those nutrients as biomass(i.e., growth). Based on the paper [5,6], paper [7] lead this delay into model, obtained the sufficient conditions for uniform persistence and the existence of periodic solution. For two trophic level, paper [4] considered the model on single nutrient. In this paper, a chemostat model on n competing predators for a prey in two-nutrients is considered. The key feature of the model is the incorporation of two-nutrients on two trophic level and the incorporation of delay.

This paper is organized as follows. In the next section, we describe the model on n competing predators for a prey in two-nutrients and obtain some preliminary results. In section 3, we discuss the global stability of extinct equilibrium, get some sufficient conditions.

2 The model and the preliminary results

In this paper, we consider the following model:

$$
\begin{cases}
S'(t) = (S^0 - S(t))D - P_1(S(t), R(t))x(t), \\
R'(t) = (R^0 - R(t))D - P_2(S(t), R(t))x(t), \\
x'(t) = -Dx(t) + \alpha P(S(t-\tau), R(t-\tau))x(t-\tau) - \sum_{i=1}^{n} f_i(x(t))y_i(t), \\
y_i'(t) = -Dy_i(t) + \beta_i f_i(x(t-\tau))y_i(t-\tau), \qquad i = 1, 2, \cdots, n.
\end{cases}
\tag{1}
$$

where D denotes the dilution rate; S^0, R^0 represent the input nutrient concentration rate of S and R, respectively; $S(t)$, $R(t)$, $x(t)$, $y_i(t)$ are the nutrient concentration of S, R and the microorganism concentration of x, y_i at time t, $i = 1, 2, \cdots, n$.

The functions $P_1(S, R)$ and $P_2(S, R)$ represent the rate of consumption of resources S and R, respectively. It is generally assumed that

$$
P_1(S, R) = \frac{m_S k_R S}{k_S k_R + k_R S + k_S R}, \qquad P_2(S, R) = \frac{m_R k_S R}{k_S k_R + k_R S + k_S R}.
$$

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where m_S is the maximal growth rate of species x on resource S in the absence of resource R, and k_S is the corresponding half-saturation constant. The constant m_R and k_R are similarly defined. The function $P(S, R)$ will represent the rate of conversion of nutrient to biomass of population x as a function of the concentrations of resource S and R in the culture vessel. Since nutrients S and R are perfectly substitutable, we take

$$P(S, R) = \frac{m_S k_R S + m_R k_S R}{k_S k_R + k_R S + k_S R}.$$

$f_i(x)$ represents the consumptive function of predator y_i on prey x, here, we take

$$f_i(x) = \frac{c_i x}{a_i + x}, \quad i = 1, 2, \cdots, n.$$

where c_i is the maximal growth rate of predator y_i on prey x, a_i is the corresponding half-saturation constant. $0 < \alpha, \beta_i \leq 1$ represent the rate of change from nutrients(or prey) to biomass, $\tau > 0$ is the lag time.

Let Banach space of continuous-functions mapping be $C = C([-\tau, 0], R^{n+3})$ (with supermum norm), the nonnegative cone is defined by $C^+ = C([-\tau, 0], R_{n+3}^+)$. For the meaning of biology, we take initial date for system (1) is $(0, \varphi)$, $\varphi \in C^+$ and $\varphi(0) > 0$. Denote the solution of system (1) through $(0, \varphi)$ as $(S(t), R(t), x(t), y_1(t), \cdots, y_n(t))$.

In this paper, we assume that resource S is superior to resource R in the sense that $m_S \geq m_R$, when the inequality is strict, the partial derivatives of $P(S, R)$ satisfy the following conditions:

$$\frac{\partial P}{\partial S}(S, R) > 0, \quad \text{for all } (S, R) \in int R_+^2,$$

$$\frac{\partial P}{\partial R}(S, R) > 0, \quad \text{for all } R > 0, 0 < S < S^c, \quad (2)$$

$$\frac{\partial P}{\partial R}(S, R) < 0, \quad \text{for all } R > 0, S > S^c.$$

where $S^c = \frac{m_R k_S}{m_S - m_R}$. If $m_S = m_R$, then we define $S^c = \infty$.

Define

$$\lambda = \begin{cases} \dfrac{k_S \dfrac{D}{\alpha}}{m_S - \dfrac{D}{\alpha}}, & \text{if } m_S > \dfrac{D}{\alpha}; \\ \infty, & \text{otherwise.} \end{cases}$$

where λ is obtained by solving the equation $P(S, 0) = \frac{D}{\alpha}$ when $m_S > \frac{D}{\alpha}$. Thus, λ represents the breakeven concentrations for resource S when none of the resource R is available.

Theorem 1 For system (1), all solutions through $(0, \varphi)$, $\varphi \in C^+$ and $\varphi(0) > 0$, are nonnegative, and system (1) is dissipative.

Proof If $S(t_1) = 0$ for some $t_1 > 0$, then the first equality of (1) imply that $S'(t_1) = S^0 D > 0$, thus, $S(t)$ must remain positive for $t > 0$. As the similar reason, $R(t)$ remain positive for $t > 0$. If there exist some $t_1 > 0$, such that $x(t) > 0$, $t \in (0, t_1)$, but $x(t_1) = 0$, thus, $x(t_1 - \tau) \geq 0$, so we have $x'(t_1) = \alpha P(S(t_1 - \tau), R(t_1 - \tau))x(t_1 - \tau) \geq 0$, from this we obtain that $x(t)$ always remain nonnegative. As the same reason, $y_i(t)$ always remain nonnegative, too.

Let $Z(t) = S(t) + R(t) + \frac{1}{\alpha}x(t + \tau) + \sum\limits_{i=1}^{n} \frac{1}{\beta_i}y_i(t + 2\tau)$, then

$$Z'(t) = -DZ(t) + D(S^0 + R^0),$$

thus,

$$Z(t) = (Z(0) - S^0 - R^0)e^{-Dt} + S^0 + R^0 \to S^0 + R^0.$$

Being $(S(t), R(t), x(t), y_1(t), \cdots, y_n(t))$ remain nonnegative, we obtain that all solutions of (1) are ultimately bounded, i.e., system (1) is disspative. $\square$

3 Extinction

For convenience, we are adopting the convention that, for each $a \in R$, $\hat{a}$ denotes the constant function $\hat{a}(u) = a$, $u \in [-\tau, 0]$. It is clear that $E_0 = (\hat{S}^0, \hat{R}^0, \hat{0}, \hat{0}, \cdots, \hat{0})$ is always the equilibrium of (1), called extinct equilibrium. In the section, we study the stability of E_0.

Lemma 1(Hayes[8]) If $A, B \in R$, then all roots ω of

$$Ae^\omega + B - \omega e^\omega = 0$$

have negative real parts if and only if $A < 1$ and $A < -B < \sqrt{\alpha^2 + A^2}$, where α is the root of $\alpha = A \tan \alpha$, $\alpha \in (0, \pi)$. (If $A = 0$, then take $\alpha = \frac{\pi}{2}$).

Theorem 2 If $P(S^0, R^0) < \frac{D}{\alpha}$, then E_0 is local asymptotically stable.

Proof The system linearized of system (1) around E_0 is

$$
\begin{cases}
S'(t) = -DS(t) - P_1(S^0, R^0)x(t), \\
R'(t) = -DR(t) - P_2(S^0, R^0)x(t), \\
x'(t) = -Dx(t) + \alpha P(S^0, R^0)x(t - \tau), \\
y_i'(t) = -Dy_i(t), \qquad i = 1, 2, \cdots, n.
\end{cases}
\tag{3}
$$

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the characteristic equation associated with (3) is $H(\lambda) \equiv det\Delta(\lambda) = 0$, where

$$\Delta(\lambda) = \begin{bmatrix} \lambda + D & 0 & P_1(S^0, R^0) & 0 & 0 & \cdots & 0 \\ 0 & \lambda + D & P_2(S^0, R^0) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda + D - \alpha P(S^0, R^0)e^{-\tau\lambda} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda + D & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda + D \end{bmatrix}$$

Upon simplification, the characteristic equation becomes

$$H(\lambda) = (\lambda + D)^{n+2}(\lambda + D - \alpha P(S^0, R^0)e^{-\tau\lambda}) = 0.$$

Denoting the two factors of $H(\lambda)$ by $H_1(\lambda)$ and $H_2(\lambda)$, respectively. It is clear that $H(\lambda) = 0$ if and only if $H_i(\lambda) = 0, i = 1, 2$. The location of the roots of the quasi-polynomial $H_2(\lambda)$ is accomplished by using the lemma 1.

Let $\omega = \tau\lambda$, so that ω and λ have real parts of the same sign. Multiplying both sides of $H_2(\lambda) = 0$ by $-\tau e^\omega$ yields

$$H_2^*(\omega) \equiv -D\tau e^\omega + \alpha\tau P(S^0, R^0) - \omega e^\omega = 0.$$

H_2^* is the form to which lemma 1 applies for $A = -D\tau$ and $B = \alpha\tau P(S^0, R^0)$. So $A < 1$, $-B < 0$, also, $A < -B$ if and only if $P(S^0, R^0) < \frac{D}{\alpha}$, then all roots ω of $H_2^*(\omega) = 0$ have negative real parts. Thus, all roots λ of $H_2(\lambda) = 0$ have negative real parts. So we conclude that E_0 is local asymptotically stable if and only if $P(S^0, R^0) < \frac{D}{\alpha}$. □

Now, based on the technique of Razumikhin and the idea of [10], we discuss the global stability of E_0.

Lemma 2 For every $\varphi \in C^+$ and $\varphi(0) > 0$, all solutions of system (1) satisfy $\limsup\limits_{t\to\infty} S(t) \leq S^0$, $\limsup\limits_{t\to\infty} R(t) \leq R^0$.

Proof We give the proof in the case of $S(t)$, the same argument can be modified to prove the result on $R(t)$.

Let $S^\infty = \limsup\limits_{t\to\infty} S(t)$, and suppose that $S^\infty > S^0$. Next we consider two possible case: either $S(t) \to S^\infty$ or $\liminf\limits_{t\to\infty} S(t) < S^\infty$.

i) If $S(t) \to S^\infty$, by the first equality of (1), we obtain

$$\limsup\limits_{t\to\infty} S'(t) \leq \frac{1}{2}(S^0 - S^\infty)D < 0,$$

which would imply that $S(t) \to -\infty$, a contradiction.

ii) If $\liminf\limits_{t\to\infty} S(t) < S^\infty$, then by the fluctuation lemma([9]), there is a sequence $t_n \uparrow \infty$ such that when $n \to +\infty$ have $S(t_n) \to S^\infty$ and $S'(t_n) = 0$. So by the first equality of (1), we obtain

$$S'(t_n) = (S^0 - S(t_n))D - P_1(S(t_n), R(t_n))x(t_n),$$

then

$$S(t_n) = S^0 - \frac{P_1(S(t_n), R(t_n))x(t_n)}{D} \leq S^0,$$

which lead to a contradiction.

Thus, we conclude that $\limsup\limits_{t\to\infty} S(t) \leq S^0$. $\square$

Theorem 3 If one of the following conditions satisfied:

i) $m_S \leq \dfrac{D}{\alpha}$;

ii) $m_S > \dfrac{D}{\alpha}$, $m_R \geq \dfrac{D}{\alpha}$ and $P(S^0, R^0) < \dfrac{D}{\alpha}$;

iii) $m_S > \dfrac{D}{\alpha} > m_R$, $S^0 < \lambda$

then for every solution of system (1) though initial date $(0, \varphi)$, $\varphi \in C^+$ and $\varphi(0) > 0$, we have $\lim\limits_{t\to\infty} x(t) = 0$ and $\lim\limits_{t\to\infty} y_i(t) = 0, i = 1, 2, \cdots, n$.

Proof By the third equality of (1), we have

$$x'(t) \leq -Dx(t) + \alpha P(S(t-\tau), R(t-\tau))x(t-\tau).$$

Consider following system

$$\begin{cases} S'(t) = (S^0 - S(t))D - P_1(S(t), R(t))x(t), \\ R'(t) = (R^0 - R(t))D - P_2(S(t), R(t))x(t), \\ x'(t) = -Dx(t) + \alpha P(S(t-\tau), R(t-\tau))x(t-\tau), \\ y'_i(t) = -Dy_i(t) + \beta_i f_i(x(t-\tau))y_i(t-\tau), \qquad i = 1, 2, \cdots, n. \end{cases} \quad (4)$$

Under one of three conditions, we first proof every solution of (4) through $(0, \varphi)$, $\varphi \in C^+$ and $\varphi(0) > 0$, exist $T_1(\varphi) > 0$, such that $P(S(t), R(t)) < \dfrac{D}{\alpha}, t \geq T_1$.

i) when $m_S \leq \dfrac{D}{\alpha}$, if there exist $(S, R) \in R_+^2$, such that $P(S, R) \geq \dfrac{D}{\alpha}$, then

$$(m_S - \frac{D}{\alpha})k_R S + (m_R - \frac{D}{\alpha})k_S R \geq \frac{D}{\alpha}k_S k_R.$$

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Thus, at least one of the inequalities, $m_S \geq \frac{D}{\alpha}$ or $m_R \geq \frac{D}{\alpha}$, must be strict, contradiction. So for every $(S, R) \in R_+^2$, we have $P(S, R) < \frac{D}{\alpha}$ at this case;

ii) when $m_S > \frac{D}{\alpha}$, $m_R \geq \frac{D}{\alpha}$. By $P(S^0, R^0) < \frac{D}{\alpha}$, we have

$$(m_S - \frac{D}{\alpha})k_R S^0 + (m_R - \frac{D}{\alpha})k_S R^0 < \frac{D}{\alpha}k_S k_R,$$

thus,

$$S^0 < \frac{\frac{D}{\alpha}k_S k_R - (m_R - \frac{D}{\alpha})k_S R^0}{(m_S - \frac{D}{\alpha})k_R} \tag{5}$$

$$\leq S^c,$$

so, for every $(0, \varphi)$, $\varphi \in C^+$ and $\varphi(0) > 0$, by lemma 2, there exists $T_1(\varphi) \geq 0$, such that $S(t) \leq S^0$, $R(t) \leq R^0$, $t \geq T_1$. By (2) and (5), when $t \geq T_1$, we have $P(S(t), R(t)) \leq P(S^0, R^0) < \frac{D}{\alpha}$;

iii) when $m_S > \frac{D}{\alpha} > m_R$, $S^0 < \lambda$. Every $R^0 > 0$, we have $P(\lambda, R^0) < \frac{D}{\alpha}$ and $P(\lambda, 0) = \frac{D}{\alpha}$. By lemma 2, there exist $T_1(\varphi) \geq 0$, such that $S(t) \leq S^0$, $R(t) \leq R^0$, $t \geq T_1$. By (2), if $S^0 < S^c$, then $P(S(t), R(t)) \leq P(S^0, R^0) < P(\lambda, R^0) < \frac{D}{\alpha}$, $t \geq T_1$; if $S^0 > S^c$, then $P(S(t), R(t)) \leq P(S^0, 0) < P(\lambda, 0) = \frac{D}{\alpha}$, $t \geq T_1$.

Choose $1 < r < \min\{\frac{D}{\alpha P(S^0, R^0)}, \frac{D}{\alpha P(S^0, 0)}\}$. Let $q(y) = ry$, then once $x(t + \theta) < q(x(t))$, $\theta \in [-\tau, 0]$, we have

$$\begin{aligned} x'(t) &= -Dx(t) + \alpha P(S(t-\tau), R(t-\tau))x(t-\tau) \\ &< [\alpha r P(S(t-\tau), R(t-\tau)) - D]x(t), \end{aligned} \tag{6}$$

from before discuss, we know $x'(t) < 0$, $t \geq T_1 + \tau$.

Let

$$\overline{x}(t) = \max\{x(t + \theta) \mid -\tau \leq \theta \leq 0\},$$

when $t \geq T_1 + \tau$, we suppose $\overline{x}(t) = x(t + \theta_0)$, then either $\theta_0 < 0$ or $\theta_0 = 0$. If $\theta_0 < 0$, suppose when $\theta_0 < \theta \leq 0$, $x(t + \theta_0) > x(t + \theta)$, so it is clear that for very small $h > 0$, we have $\overline{x}(t + h) \leq x(t + \theta_0) = \overline{x}(t)$, thus $\overline{x}'(t) \leq 0$; if $\theta_0 = 0$, then $x(t + \theta) \leq x(t) < q(x(t))$, $\theta \in [-\tau, 0]$, by (6), $x'(t) < 0$, so for very small $\zeta > 0$, we have $\overline{x}(t + \zeta) = x(t + \theta_0) = \overline{x}(t)$, thus $\overline{x}'(t) = 0$.

In brief, when $t \geq T_1 + \tau$, we always have $\overline{x}'(t) \leq 0$, so

$$x(t) \leq \max\{x(t) \mid 0 \leq t \leq T_1 + \tau\} \equiv \eta.$$

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Next, we prove for every $0 < \beta < \eta$, there exist $T(\varphi) \geq T_1 + \tau$, such that $x(t) < \beta, t \geq T$.

Take positive integer N such

$$\beta + (N-1)d \leq \eta < \beta + Nd,$$

where $d = \min\{q(y) - y \mid \beta \leq y \leq \eta\}$. Now, we show that when $t \geq T_1 + 2\tau$, the solutions of system (4) can't always satisfy $x(t) \geq \beta + (N-1)d$, otherwise,

$$q(x(t)) \geq x(t) + d \geq \beta + Nd > \eta \geq x(t + \theta), \ \theta \in [-\tau, 0],$$

by (6), we have

$$x'(t) \leq \gamma,$$

where $\gamma = \max_{\beta \leq x \leq \eta} \{x(\alpha r P(S^0, R^0) - D), x(\alpha r P(S^0, 0) - D)\} < 0$. Integrating from $T_1 + 2\tau$ to t, we have $x(t) < x(T_1 + 2\tau) + \gamma(t - T_1 - 2\tau)$, thus, $x(t) \to -\infty$, a contradiction. So there must exist $T_2 \geq T_1 + 2\tau$, such that $x(T_2) < \beta + (N-1)d$. Next, we prove when $t \geq T_2$, always have

$$x(t) < \beta + (N-1)d. \tag{7}$$

If (7) can't be established, there must exist $t^* > T_2$, such that $x(t^*) = \beta + (N-1)d$ and $x'(t^*) \geq 0$. However, being

$$q(x(t^*)) \geq x(t^*) + d = \beta + Nd > \eta \geq x(t^* + \theta), \ \theta \in [-\tau, 0],$$

by (6), $x'(t^*) < 0$, contradiction. So (7) is remain.

As the same method, there exist $T_3 \geq T_2 + \tau \geq T_1 + 3\tau$, such that $x(t) < \beta + (N-2)d, t \geq T_3$.

Continually done, we obtain that there exist $T = T_N \geq T_1 + (N-1)\tau$, such that for $t \geq T$, we always have $x(t) < \beta$. Since β is arbitrary, we conclude that $\lim_{t \to \infty} x(t) = 0$.

For the last equality of (4), since $\lim_{t \to \infty} x(t) = 0$, so $\lim_{t \to \infty} f_i(x(t - \tau)) = 0$, then, it is easy to prove $\lim_{t \to \infty} y_i(t) = 0, i = 1, 2, \cdots, n$.

By normal comparison theorem, we have $\lim_{t \to \infty} x(t) = 0$ and $\lim_{t \to \infty} y_i(t) = 0$ for every $\varphi \in C^+$ and $\varphi(0) > 0$ in system (1), $i = 1, 2, \cdots, n$. □

Corollary Under condition of theorem 3, E_0 is global asymptotically stable.

Proof From theorem 3, we have $\lim_{t \to \infty} x(t) = 0$ and $\lim_{t \to \infty} y_i(t) = 0, i = 1, 2, \cdots, n$. By (1) and theorem 2, it is easy to prove $\lim_{t \to \infty} S(t) = S^0$ and $\lim_{t \to \infty} R(t) = R^0$. Thus, E_0 is global asymptotically stable. □

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Remark In theorem 3, we have $\lim\limits_{t\to\infty} x(t) = 0$ and $\lim\limits_{t\to\infty} y_i(t) = 0$, $i = 1, 2, \cdots, n$. Being x represent prey, y_i represent predator, we can think that extinction of predator(y_i) due to the lack of prey(x) at this time, this phenomenon is general in nature.

Acknowledgments

References

1. S.B.Hsu, S.P.Hubbell, P.Waltman, A mathematical theory for single nutrient competition in continuous cultures of micro-organisms, *SIAM J. Appl. Math.*, **32**, *366-383 (1977)*.

2. G.S.K.Wolkowicz, Zhiqi Lu, Global dynamics of a mathematical model of competition in the chemostat: general response functions and differential death rates, *SIAM J. Appl. Math.*, **52**, *222-223 (1992)*.

3. S.F.Ellermeyer, Competition in the chemostat: global asymptotic behavior of a model with delayed response in growth, *SIAM J. Appl. Math.*, **54**, *456-465 (1994)*.

4. Feng Yang, H.I.Freedman, Competing predators for a prey in a chemostat model with periodic nutrient input, *J. Math. Biol.*, **29**, *715-732 (1991)*.

5. M.M.Ballyk, G.S.K.Wolkowicz, Exploitative competition in the chemostat for two perfectly substitutable resources, *Math. Biosci.*, **118**, *127-180 (1993)*.

6. M.M.Ballyk, G.S.K.Wolkowicz, An examination of the thresholds of enrichment: a resource-based growth model, *J. Math. Biol.*, **33**, *435-457 (1995)*.

7. Wang kaifa, Uniform persistence and periodic solution of chemostat model, *J. Southwest China Normal University (Nature Science)* , **22**, *591-598 (1997)*.

8. R.Bellman, K.L.Cooke, Differential-Difference equations, *Academic Press, New York, 1963*.

9. W.M.Hirsch, H.Hanisch, J.P.Gabriel, Differential equation models of some parasitic infections, Methods for the study of asymptotic behavior, *Comm. Pure Appl. Math.*, **38**, *733-752 (1985)*.

10. Wang wendi, Persistence function in the models with time delays,*J. Southwest China Normal University (Nature Science)*, **16**, *426-431 (1991)*.

QUALITATIVE STUDIES ON SOME CHEMOTACTIC DIFFUSION SYSTEMS

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Chemotaxis and diffusivity (also called motility) of cells are known to affect their growth. We study qualitatively some chemotactic diffusion systems with emphasis on the effects of chemotaxis and motility on the steady state population, spatial distribution, competition-exclusion and stable co-existence.

1 Introduction

Here we are interested in the effects of diffusivity and chemotaxis on the growth of cells. Diffusivity of cells is also called *motility* in some engineering literature. Chemotaxis is the oriented movement of cells in response to the concentration gradient of chemical substances in their environment. It is "anti-diffusion".

The effects of diffusion on the dynamics of reaction diffusion equations has long been the subject of intensive studies. It is essentially true that for single equations, diffusion exhibits the smoothing effect. On the other hand, for a class of activator-inhibitor reaction diffusion systems, we have the celebrated *Turing* (1950) phenomenon: a large difference in the diffusion rates of the activator and inhibitor destablizes constant steady states and creates stable nonconstant ones. For more discussions on the effects of diffusion, see a nice expository paper of Ni [11].

The history of mathematical studies on chemotaxis has not been very long: the first mathematical models were proposed by Keller and Segal [7] in 1970. They were trying to model the phenomenon called *chemotaxis collapse*: chemotactic cells, initially distributed almost homogeneously, aggregate at several spots in finite time, as in the case of slime mold amoebae in response to a chemical released by the population itself. New models continue to be proposed to date [1,2,6,13,14]. Mathematical and rigorous analysis for this phenomenon has been carried out [3,5,10,12]. Spiky stationary patterns are rigorously constructed for the Keller-Segal model, and give an alternative explanation for aggregation [11].

All the results mentioned above do *not* concern the effects of diffusion and chemotaxis on the *growth* of cells. On the other hand, this was originally observed experimentally and numerically by physiologists [15,8,9]: when several species of cells compete for a limited resource, the species with smaller diffusion

rate and larger chemotaxis rate enjoys better growth, even when the other species have superior growth kinetics; in the special case when all the species involved have identical growth kinetics, the species with the smallest diffusion rate and largest chemotaxis rate wipes out the populations of all the others.

So far there has been no mathematical proof of this phenomenon, except in the case of a Lotka-Voterra competition model (nonchemotactic) which is "monotone" (i.e., the comparison principle applies) [4]. But systems often are not monotone (or have variational structures)–this is the case for all chemotaxis models, sometimes even in the absence of chemotaxis. Thus new tools have to be invented. There are many good mathematical problems such as this in this field. Their solutions may lead to break-throughs in nonlinear analysis.

To illustrate the general idea in modeling chemotaxis, we consider a species of cells which respond chemotactically to a chemical. Let $u(x,t)$ and $v(x,t)$ be the concentration and density of the chemical and cells, respectively. Assuming Fick's law, the random diffusive flux is given by $-D_1 \nabla u$, where $D_1 > 0$ is assumed to be a constant. The cell flux is assumed to be the sum of the random diffusive flux and chemotactic flux, with the latter parallel to ∇u, so it takes the form $-D_2 \nabla v + \chi v \phi'(u) \nabla u$, where $D_2 > 0$ and χ are constants, and $\phi'(u) > 0$. D_2 (the *motility* of cells) measures the ability of cells to diffuse randomly; χ (positive if the chemical is an attractant and negative if it is a repellant) is called the *chemotaxis coefficient* and measures the magnitude of cell-response to the chemical. $\phi(u)$ is called the *sensitivity function* – the sensitivity of cells to the chemical may vary with the level of chemical concentration. Conservation of mass leads to the following system:

$$(1) \qquad \begin{cases} u_t = D_1 \Delta u + k(u,v), \\ v_t = \nabla \cdot (D_2 \nabla v - \chi v \nabla \phi(u)) + h(u,v), \end{cases}$$

where $k(u,v)$ is the creation-degradation rate of the chemical and $h(u,v)$ is the birth-death rate of the cells.

To elucidate the effects of cell motility and chemotaxis on population growth, Lauffenburger, Aris and Keller [8] investigated a single bacterial population in a 1D medium of finite length with growth limited by a nutrient diffusing from an adjacent phase not accessible to the bacteria. Their model is

$$(2) \qquad \begin{cases} u_t = u_{xx} - f(u)v, \ 0 \le x \le 1, \ t > 0, \\ v_t = (\lambda v_x - \chi v(\phi(u))_x)_x + (kf(u) - \theta)v, \\ u_x(0) = 0, \ u_x(1) = h(1 - u(1)), \\ \lambda v_x - \chi v(\phi(u))_x = 0 \ \text{ at } \ x = 0, 1. \end{cases}$$

Here u is the concentration of the substrate and v the density of the bacteria; $f(u)$ is the consumption rate of the substrate per cell; the term $(kf(u) - \theta)v$ in

the v-equation represents that the bacteria have a *Malthusian* (or exponential) growth with $kf(u)$ and θ measuring the respective birth and death rates. The boundary condition for u at $x = 1$ reflects the fact that the substrate is diffusing into the medium through that point. Notice that the boundary condition for v is nonlinear. From biological considerations, f and ϕ should satisfy $f(0) = 0$, $f'(u) > 0$, $\phi'(u) > 0$. Typical choices for f and ϕ are: $f(u) = au/(b + u)$, $\phi(u) = u$, $\phi(u) = \log(c + u)$, $\phi(u) = u/(1 + cu)$, etc.

The existence of steady states of (2) was studied by Zeng [16] who proved (i) if $\theta \geq kf(1)$, the only steady state of (2) is the trivial one: $(u, v) = (1, 0)$; (ii) if $0 < \theta < kf(1)$, then (2) has a positive steady state (u, v). Numerical simulations of these states (with $\phi(u) = u$, χ proportional to λ and $f(u) =$ step function) led to the following observations [8]: (a) Random motility λ may lead to decreased population (at least in the non-chemotactic case $\chi = 0$) and (b) the chemotaxis coefficient χ acts to increase population size.

(2) is a single species case which does not include the competitor. Motivated by the above observations, I have worked on this case, obtaining results supporting them on one hand, discovering, on the other hand, that if the cells have Logistic growth, the opposite of (b) is true: large chemotaxis rate is detrimental to the steady state population of cells. Y. Wu and I studied the case of two species competing for the same resource. Theses results are described in the folllowing sections.

2   Single Species

I proved the following theorems(to appear in *SIAM J. Math. Anal.*). Theorem 1 concerns the behavior of steady states of (2) with small or large motility λ and chemotaxis coefficient χ.

Theorem 12 *Suppose $0 < \theta < kf(1)$. Let $(u(x), v(x))$ be a positive steady state of (2).*

(i) *Allow the chemotactic coefficient $\chi \geq 0$ to be dependent on the motility λ in any fashion. Then as $\lambda \to 0$, $u \to$ constant c uniformly on $[0, 1]$ where $kf(c) = \theta$, v concentrates and blows up at $x = 1$, i.e., v converges to zero uniformly outside any left neighborhood of $x = 1$ and $v(1) \to \infty$; Moreover, the total population of cells $\int_0^1 v(x)dx \longrightarrow kh(1 - c)/\theta$. (Thus v converges to a multiple of the δ-function centered at $x = 1$.)*

(ii) *Allow λ and χ to be dependent on one another, and assume that as either $\chi \to \infty$ or $\lambda \to \infty$, $\lambda/\chi \to 0$ (so χ is relatively large). Then the conclusion in (i) is true.*

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(*iii*) *Suppose* $\lambda/\chi \to \infty$ *as* $\chi \to \infty$ *or as* $\lambda \to \infty$ *(so* χ *is relatively small).*
Then $u(x) \to$ *some function* $u_\infty(x)$, $v(x) \to$ *some constant* v_∞ *in*
$C^1([0,1])$, *where* u_∞ *and* v_∞ *are uniquely determined by*

$$(4) \qquad \begin{cases} u''_\infty = f(u_\infty)v_\infty, \ x \in [0,1], \\ u'_\infty(0) = 0, \ u'_\infty(1) = h(1 - u_\infty(1)), \\ \text{constraint} : \int_0^1 (kf(u_\infty(x)) - \theta)v_\infty dx = 0. \end{cases}$$

(*iv*) *Suppose* $\lambda/\chi \to$ *constant* $a > 0$ *as* $\chi \to \infty$ *or as* $\lambda \to \infty$ *(so* λ *and*
χ *are of the same order). Then after passing to a subsequence* $u(x) \to$
$u_\infty(x)$, $v(x) \to v_\infty(x) \equiv (\text{const. } M)\exp(\phi(u_\infty)/a)$ *in* $C^1([0,1])$, *where*
u_∞ *and* v_∞ *satisfy (4).*

Parts (i) and (ii) of this theorem imply that small motility and large chemo-
taxis (compared to motility) have the same effect on the distribution as well
as the total population of bacteria. (This was not observed formally or nu-
merically before, to the best of our knowledge.) Furthermore, by (i) and (ii)
we can deduce that the total cell population (for fixed λ or χ) $\int_0^1 v(x)dx <$
$\lim_{\lambda \to 0} \int_0^1 v(x)dx = \lim_{\chi \to \infty} \int_0^1 v(x)dx$. Thus the total population for fixed λ
or χ is less than that for λ small or χ large (relative to λ). (iii) and (iv)
imply that if λ is large and at least at the same order of χ, then u and v
are close to profiles that can be determined. (i), (iii) and (iv) also imply
$\lim_{\lambda \to \infty} \int_0^1 v(x)dx < \lim_{\lambda \to 0} \int_0^1 v(x)dx$. These support the suggestions of Lauffen-
burger *et al* [8] that the decreased motility leads to increased population size
and that chemotaxis acts to increase this effect.

We see that for small λ or large χ (compared to λ), the bacteria concentrate
in a small neighborhood of $x = 1$, which is a "fertile zone". This promotes the
growth of bacteria if they have a Malthusian growth, as in (2) but may not do
so if the growth is logistic. This prompted me to study the steady-states of
(2) with $(kf(u) - \theta)v$ replaced by a logistic growth term $(kf(u) - \theta - \beta v)v$.

Theorem 13 (*i*) *A positive steady state* $(u(x), v(x))$ *exists if and only if* $0 <$
$\theta < kf(1)$ *(which will be assumed in the rest of this theorem).*

(*ii*) *Allow* λ *to be dependent on* χ *with* $\lambda/\chi \to 0$ *as* $\chi \to \infty$. *Then as* $\chi \to \infty$,
we have $u(x) \to 1$ *uniformly on* $[0,1]$, v *concentrates at* $x = 1$ *(i.e.,*
$v \to 0$ *outside any left neighborhood of* $x = 1$, *while* $v(1)$ *is bounded away*
from 0); Furthermore, $\lim_{\chi \to \infty} \int_0^1 v(x)dx = 0$.

(*iii*) *Suppose* $\lambda/\chi \to \infty$ *as* $\chi \to \infty$ *or as* $\lambda \to \infty$. *Then the conclusion of (iii)*
of Theorem 1 holds with the constraint in (4) replaced by $\int_0^1 (kf(u_\infty(x)) -$
$\theta - \beta v_\infty)v_\infty dx = 0$.

(iv) *Suppose $\lambda/\chi \to$ constant $a > 0$ as $\chi \to \infty$ or as $\lambda \to \infty$. Then the conclusion of (iv) of Theorem 1 holds with the modification as above.*

(v) *Allow χ to be dependent on λ with $\chi/\sqrt{\lambda}$ remaining bounded as $\lambda \to 0$. Then as $\lambda \to 0$, u and v converge to functions u_0 and v_0, respectively, where $kf(u_0) = \theta + \beta v_0$ and u_0 is the unique solution of*

$$\begin{cases} u_0'' = f(u_0)(kf(u_0) - \theta)/\beta, \\ kf(u_0) > \theta, \ x \in (0,1), \\ u_0'(0) = 0, \ u_0'(1) = h(1 - u_0(1)). \end{cases}$$

In particular, $\liminf_{\lambda \to 0} \int_0^1 v(x)dx > 0$. *This is true in the special case* $\phi(u) = u$ *requiring only* $\chi \to 0$ *as* $\lambda \to 0$.

Part (i) of the above theorem reveals that in sharp contrast to the *Malthusian* case, if the bacteria have logistic growth, then large chemotaxis coefficient χ (compared to motility λ) is *detrimental* to the growth of bacteria. Part (v) indicates that small motility λ no longer has exactly the same effect on the distribution and the bacterial population as for large chemotaxis χ : v is not concentrating at $x = 1$, and the total population $\int_0^1 v(x)dx$ is not diminishing for shrinking λ. I conjecture that in the logistic case $\lim_{\lambda \to 0} \int_0^1 v(x)dx$ is less than $\int_0^1 v(x)dx$ for any fixed $\lambda > 0$, in contrast to the *Malthusian* case. Parts (iii) and (iv) imply that large motility λ has the same effect on u and v in the logistic case as in the *Malthusian* case.

The next result concerns the global existence and boundedness of time-dependent solutions, and stability of steady states for both *Malthusian* and logistic cases.

Theorem 14 *Consider the full system (2) with $(kf(u)-\theta)v$ replaced by $(kf(u)-\theta-\beta v)v$, where $\beta \geq 0$ is a constant (so that both Malthusian and Logistic cases are covered).*

(i) *If the initial value $(u(x,0), v(x,0))$ is H^1-smooth and $0 \leq u(x,0) \leq 1$, $v(x,0) \geq 0$, then (2) (with the modification indicated above) has a unique global positive solution $(u(x,t), v(x,t))$ with $0 \leq u \leq 1$ and v being bounded for all $t \geq 0$.*

(ii) *If $\theta \geq kf(1)$, then as $t \to \infty$, $u(\cdot, t) \to 1$, $v(\cdot, t) \to 0$ in $L^\infty([0,1])$ norm, i.e. the trivial steady state $(1, 0)$ is globally asymptotically stable; moreover, the convergence rate is exponential if $\theta > kf(1)$, and algebraic if $\theta = kf(1)$ and $\beta > 0$. If $0 < \theta < kf(1)$, then the trivial steady state $(1, 0)$ is unstable.*

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(iii) For $\theta \in (0, kf(1))$ but close enough to $kf(1)$, the positive steady state of (2) is unique and asymptotically stable.

Part (i) of the above theorem implies that there is no possibility of a chemotactic collapse. In (iii), the uniqueness and asymptotic stability of positive steady states for the full range of the (bifurcation) parameter $\theta \in (0, kf(1))$ are not obtained, though we conjecture that these should be true. (Y. Lou and later J. Shi pointed out that if $\chi = 0$, then we do have uniqueness for $\theta \in (0, kf(1))$.) These properties of positive steady states are important in understanding the effects of motility and chemotaxis. The uniqueness would ensure that the functions

$$(5) \qquad \lambda \rightarrow \int_0^1 v(x)dx, \quad \chi \rightarrow \int_0^1 v(x)dx$$

are single-valued. We then can proceed to study the **monotonicity** of these functions. It seems that the monotone dependence of the integrals (5) on parameters has never been studied analytically for any diffusion systems (with or without chemotaxis).

The global stability of positive steady states would imply that the behavior of steady states represents that of the time dependent solutions for large time. Unfortunately, part (iii) of Theorem 3 covers steady states only for θ close to $kf(1)$. In fact, it has been an outstanding open problem for many "classical" (non-chemotaxis) diffusion systems to obtain uniqueness and stability of steady states for the full range of parameters.

3 Competition between Two Species

To better understand the effects of motility and chemotaxis on population growth, Y. Wu and I studied a model of two species of bacteria competing for the same nutrient, where the growth kinetics of both species are identical but their motility and chemotaxis coefficients are different. The interest is in the possibility of "competition exclusion" and stable coexistence, attributable solely to the motility and chemotaxis. Let the competing species have density function w and to focus solely on the effect of motility and chemotaxis assume that both species have the same consumption rate of the substrate, and the same birth and death rates. The model is

$$(6) \quad \begin{cases} u_t = u_{xx} - f(u)(v + w), \ 0 \le x \le 1, \ t > 0, \\ v_t = (\lambda_1 v_x - \chi_1 v(\phi(u))_x)_x + (kf(u) - \theta)v, \\ w_t = (\lambda_2 w_x - \chi_2 w(\phi(u))_x)_x + (kf(u) - \theta)w, \\ \lambda_1 v_x - \chi_1 v(\phi(u))_x = 0 = \lambda_2 w_x - \chi_2 w(\phi(u))_x, \ at \ x = 0, 1, \\ u_x = 0 \ at \ x = 0, \ u_x = h(1 - u) \ at \ x = 1. \end{cases}$$

Malthusian growth is assumed for both species. We are primarily interested in finding the ranges for motility and chemotaxis parameters $\lambda_1, \lambda_2, \chi_1,$ and χ_2 so that one species can wipe out the other, or they coexist in a stable equilibrium.

Parts (i) and (ii) of Theorem 3 can be easily generalized to the current case, only now the trivial steady state is $(u, v, w) = (1, 0, 0)$. In particular, if $\theta \geq kf(1)$ the only nonnegative steady of (6) is the trivial one. If $0 < \theta < kf(1)$, then the result for the existence in the single species case implies that (6) has two *semitrivial* steady states $(\underline{u}(x), \underline{v}(x), 0)$ and $(\overline{u}(x), 0, \overline{w}(x))$. Let $d_1 > 0$ and $\chi_1 \geq 0$ be fixed. We are interested in the conditions on the pair (d_2, χ_2) for the stability/instability of the semitrivial states and for coexistence of two species (i.e., existence of positive steady states).
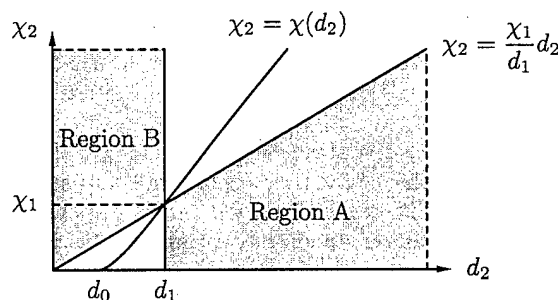
Theorem 15 (*i*) *For (d_2, χ_2) in the shaded regions (excluding (d_1, χ_1) and χ_2-axis), the only nonnegative steady states of (6) are the trivial one $(1, 0, 0)$, and the semitrivial ones $(\underline{u}, \underline{v}, 0)$ and $(\overline{u}, 0, \overline{w})$. Moreover, in the shaded Region A, all $(\overline{u}, 0, \overline{w})$ are unstable, and if $(\underline{u}, \underline{v})$ is asymptotically stable with respect to the dynamics of system (2) (known to be true if θ is close to $kf(1)$), then so is $(\underline{u}, \underline{v}, 0)$ with respect to that of (6). The same is true in Region B if we exchange $(\overline{u}, 0, \overline{w})$ and $(\underline{u}, \underline{v}, 0)$.*

(ii) *There exists a continuous function $\chi(d_2)$, strictly increasing on $[d_0, \infty]$ (where $0 < d_0 \leq d_1$ with the equality holding only if $\chi_1 = 0$) with $\lim_{d_2 \to \infty} \chi(d_2) = \infty$. For (d_2, χ_2) below the graph of $\chi(d_2)$, all $(\underline{u}, \underline{v}, 0)$ are asymptotically stable provided θ is close to $kf(1)$ (closeness independent of d_2 and χ_2); for (d_2, χ_2) above the graph, all $(\underline{u}, \underline{v}, 0)$ are unstable.*

(*iii*) *For each fixed d_2 (and d_1 and χ_1) with $d_0 < d_2 \neq d_1$, local bifurca-*

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tion of positive stable steady states of (6) occurs near $(\chi_2, u, v, w) = (\chi_2(d_2), \underline{u}, \underline{v}, 0)$, provided θ is close to $kf(1)$ (closeness independent of d_2 and χ_2).

(iv) *The branch of positive steady states mentioned in (iii) can be extended and it reaches, at a finite value of $\chi_2 > \chi(d_2)$, the branch of semitrivial steady states $(\overline{u}, 0, \overline{w})$.*

As we can see, some of the results rely on the stability of positive steady states of (2) which has not been proved except in the case when θ is close to $kf(1)$. And even when we have this condition, our proof of the stability of bifurcating solutions mentioned in (iii) is already very long and technical.

We believe that there should exist a curve $\chi = \overline{\chi}(d_2)$ passing through the point (d_1, χ_1) and staying above the curve $\chi = \chi(d_2)$, such that (i) for (d_2, χ_2) between these two curves, there exists a globally asymptotically stable positive steady state(thus we have stable coexistence); (ii) for (d_2, χ_2) not between these two curves, we have competition exclusion: below the curve $\chi = \chi(d_2)$, the semitrivial steady state $(\underline{u}, \underline{v}, 0)$ is globally asymptotically stable; above the curve $\chi = \overline{\chi}(d_2)$, $(\overline{u}, \overline{v}, 0)$ is globally asymptotically stable. The nonexistence of positive steady states and the local stability/instability of other steady states suggest this. However, as mentioned before the difficulty we encounter here is common and new ideas are needed.

Acknowledgments

References

1. E. Ben-Jacob, Contemporary Phys. **38**, 205 (1997).
2. M. Brenner, L. Levitov and E. Budrene, Biophys. J. **74**, 1677 (1998).
3. S. Childress and J. Perkus, Math. Biosci. **56**, 217 (1981).
4. J. Dockery, V. Hutson, K. Mischaikow and M. Pernarowski, J. Math. Biol. **37**, 61 (1998).
5. M. Herrero and J. Velazquez, Math. Annalen **306**, 583 (1996).
6. R. Goldstein, Phys. Rev. Let. **77**, 775 (1996).
7. E. Keller and L. Segel, J. Theoret. Biol. **26**, 399 (1970).
8. D. Lauffenburger, R. Aris and K. Keller, J. Biophys. Soc. **40**, 209 (1982).
9. D. Lauffenburger and P. Calcano, Biotech. & Bioeng **25**, 2103 (1983).
10. H. Levine and B. Sleeman, SIAM J. Appl. Math. **57**, 683 (1997).

11. W.-M. Ni, Notices of AMS **45**, 9 (1998).

12. T. Nagai, T. Senba and K. Yoshida , Funkcial. Ekvac. **40**, 411 (1997).

13. H. Othmer and A. Stevens, SIAM J. Appl. Math. **57**, 1044 (1997).

14. C. Oss, *et. al.*, J. Theor. Biol. **181**, 203 (1996).

15. W. Pilgram and F. Williams, Can. J. Microbiol. **22**, 1771 (1976).

16. B. Zeng, Math. Appl. **3**, 78 (1990).

NONOSCILLATION FOR SECOND ORDER LINEAR IMPULSIVE DIFFERENTIAL EQUATION WITH DELAY

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We established a group of necessary and sufficient conditions for nonoscillation of a second order linear impulsive differential equation with delay.

1   Introduction

Recently, Huang ([1]) investigated the oscillatory and nonoscillatory behaviors of a second order linear impulsive differential equation with the form:

$$u'' = -p(t)u(t), \ t \geq 0; \tag{1.1}$$

where $p(t) = \sum_{n=1}^{\infty} a_n \delta(t - t_n)$ and $\delta(t)$ is a δ-function, i.e,

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)dt = \int_{-\varepsilon}^{\varepsilon} \delta(t)\varphi(t)dt = \varphi(0)$$

for all $\varphi(t)$ being continuous at $t = 0$. Under the assumptions that

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots, \text{ and } \lim_{n \to \infty} t_n = \infty,$$

and $a_n > 0$ for all $n \in N$, he obtained a group of necessary and sufficient conditions for nonoscillation of (1.1).

In this paper, we consider the equation with the form

$$u'' = -p(t)u(t - \tau),\ t \geq 0. \tag{1.2}$$

Here $p(t)$ and $\{t_n\}$ are defined as above. We shall pay attention on the influence of τ on the oscillatory and nonoscillatory behaviors of the given equation. We obtain a conclusion that if the impulsive points $\{t_n\}$ and the delay τ have the relationship: $t_{n+1} - t_n \geq \tau$ for all $n \geq n^*$, then a group of refined necessary and sufficient conditions for nonoscillation still hold for equation (1.2). It can be seen in the following sections that one gets the main theorem of [1] from Theorem 3.1 in this paper by setting $\tau = 0$.

A function $u(t)$ is called the solution of equation (1.2) if it is continuous on $[-\tau, +\infty]$ and it is a linear function on every interval $[t_n, t_{n+1}]$ such that

$$u'(t_n^+) = u'(t) = u'(t_{n+1}^-) \text{ for } t_n < t < t_{n+1},\ n = 0, 1, 2, \cdots,$$

$$u'(t_n^-) - u'(t_n^+) = \lim_{\varepsilon \to 0} \int_{t_n-\varepsilon}^{t_n+\varepsilon} p(t)u(t - \tau)dt = u(t_n - \tau)a_n \text{ for all } n \in N.$$

$$\tag{1.3}$$

(1.2) is said to be nonoscillatory if all its nontrivial solutions are nonoscillatory, and to be oscillatory if all its nontrivial solutions are oscillatory.

2   Two Lemmas

In this paper, we always assume that there is a positive integer $n^* \in N$ (N is the set of positive integers) such that τ satisfies $t_{n+1} - t_n \geq \tau$ for all $n \geq n^*$. For each $n \in N$ we define the sequence $\{P_k^n\}_{k=1}^\infty$ by induction:

$$P_1^n = a_{n+1}\left(t_{n+1} - t_n - \tau\right),$$
$$P_{k+1}^n = \frac{a_{n+k+1}}{a_{n+k}}\left(\frac{P_k^n + \tau a_{n+k}}{1 - P_k^n} + a_{n+k}\left(t_{n+k+1} - t_{n+k} - \tau\right)\right), \tag{2.4}$$

thus we have $P_{k+1}^n = \infty$ provided $P_k^n = 1$. Denote $S_n = \sup\{P_k^n |\ k = 1, 2, \cdots\}$. We note that $S_n \leq 1$ implies $0 < P_k^n < 1$ for all $k \in N$.

Lemma 8 *If $S_{n_0} \leq 1$ for some $n_0 \in N$ ($n_0 > n^*$), then $S_{n_0+1} \leq 1$ and therefore $S_n \leq 1$ for all $n \geq n_0$.*

Lemma 9 *Suppose that $0 < \alpha_k \leq \beta_k$, $\lambda_k > 0$ for all $k \in N$. Define by*

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induction

$$p_1 = \alpha_1 \lambda_1, \ q_1 = \beta_1 \lambda_1,$$

$$p_{k+1} = \frac{\alpha_{k+1}}{\alpha_k} \left(\frac{p_k + \tau \alpha_k}{1 - p_k} + \alpha_k \lambda_{k+1} \right),$$

$$q_{k+1} = \frac{\beta_{k+1}}{\beta_k} \left(\frac{q_k + \tau \beta_k}{1 - q_k} + \beta_k \lambda_{k+1} \right), \ k = 1, 2 \cdots.$$

If $0 < q_k < 1$ for all $k \in N$, then

$$0 < p_k \le \frac{\alpha_k}{\beta_k} q_k < 1, \tag{2.5}$$

for all $k \in N$.

3 Main Result and Its Proof

Theorem 16 *The following statements are equivalent:*
   (i) There is $n_0 \in N, n_0 > n^*$ such that $S_{n_0} \le 1$.
   (ii) There is $n_0 \in N, n_0 > n^*$ such that $S_n \le 1 \ (n \ge n_0)$
   (iii) Equation (1.2) is nonoscillatory.
   (iv) Equation (1.2) has a nonoscillatory solution.

Proof. It is clearly that (i) implies (ii) from lemma 1. Next we want to show that (ii) implies (iii).

Suppose that (ii) holds. Then there is $n_0 \in N$, $n_0 > n^*$ such that

$$0 < P_k^n < 1, \ \text{for all} \, n \ge n_0, k \ge 1. \tag{3.6}$$

Let $u(t)$ be a nontrivial solution of equation (1.2) and $u(t)$ is oscillatory. Then without loss of generality, there exist $n : n \ge n_0$ and $\bar{t} \in [t_n, t_{n+1})$ such that $u(\bar{t}) = 0, u'(\bar{t}^+) > 0$.

By (1.3), we have

$$u'\left(\bar{t}^+\right) - u'\left(t_{n+1}^+\right) = u'\left(t_{n+1}^-\right) - u'\left(t_{n+1}^+\right) = u(t_{n+1} - \tau)a_{n+1}. \tag{3.7}$$

There are two subcases:
   (1) If $t_{n+1} - \tau \le \bar{t}$, then $u\left(t_{n+1} - \tau\right) \le 0$.
   (2) If $t_{n+1} - \tau > \bar{t}$, then $u\left(t_{n+1} - \tau\right) > 0$.
   In both cases, we could show that

$$u'\left(t_{n+1}^+\right) \ge \left(1 - P_1^n\right) u'\left(\bar{t}^+\right) = \left(1 - P_1^n\right) u'\left(t_n^+\right) > 0. \tag{3.8}$$

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Thus we derive from (3.3) and (3.2) that

$$\frac{P_1^n + \tau a_{n+1}}{1 - P_1^n} u'\left(t_{n+1}^+\right) \geq \left(u\left(t_{n+1} - \tau\right) + \tau u'\left(\vec{t}^+\right)\right) a_{n+1}.$$

Therefore we obtain

$$P_2^n u'\left(t_{n+1}^+\right) = \frac{a_{n+2}}{a_{n+1}}\left[\frac{P_1^n + \tau a_{n+1}}{1 - P_1^n} + a_{n+1}\left(t_{n+2} - t_{n+1} - \tau\right)\right] u'\left(t_{n+1}^+\right)$$
$$\geq a_{n+2}\left[u\left(t_{n+1} - \tau\right) + \tau u'\left(\vec{t}^+\right) + u'\left(t_{n+1}^+\right)\left(t_{n+2} - t_{n+1} - \tau\right)\right]$$
$$= u\left(t_{n+2} - \tau\right) a_{n+2}.$$

$$(3.9)$$

Now, we can show

$$u'\left(t_{n+k}^+\right) \geq \left(1 - P_k^n\right) u'\left(t_{n+k-1}^+\right) > 0,$$
$$P_{k+1}^n u'\left(t_{n+k}^+\right) \geq u\left(t_{n+k+1} - \tau\right) a_{n+k+1} \qquad (3.10)$$

by induction (we omit the details). Note that (3.5) contracts with the assumption that $u(t)$ is oscillatory, then (ii) implies (iii).

It is obvious that (iii) implies (iv). Now we want to show that (iv) implies (i). Suppose that $u(t)$ is a nonoscillatory solution, so there is t_{n_0} $(n_0 > n^*)$ such that $u(t) > 0$ for $t \geq t_{n_0}$. We claim that

$$u'\left(t_{n_0+k}^+\right) > 0 \text{ for all } k = 0, 1, 2, \cdots. \qquad (3.11)$$

If we can show that

$$0 < P_k^{n_0}, \ u'\left(t_{n_0+k}^+\right) \leq \left(1 - P_k^{n_0}\right) u'\left(t_{n_0+k-1}^+\right), \ k \in N, \qquad (3.12)$$

then combining with (3.6), we obtain (iv)$\Rightarrow$(i). In fact, one could show (3.7) and

$$P_{k+1}^{n_0} u'(t_{n_0+k}^+) \leq u(t_{n_0+k+1} - \tau) a_{n_0+k+1}, \quad k \in N \qquad (3.13)$$

by induction, but we omit it. Thus we complete the proof.

Corollary 2 *Let $q(t) = \sum_{n=1}^{\infty} b_n \delta\left(t - t_n\right)$. Suppose that $0 < a_n \leq b_n$, for large $n \in N$ $(n > n^*)$. Then we have the following conclusions:*

(i) If equation

$$u'' = -q(t)u(t - \tau) \qquad (3.14)$$

is nonoscillatory, then (1.2) is nonoscillatory.

(ii) If (1.2) is oscillatory, then (3.9) is oscillatory.

Example 8 *Consider the equation (1.2). Let $t_n = nT + n\tau$ $(0 \leq \tau < \frac{1}{2}T)$ and $b_n = \frac{1}{8n(n+1)T}$. If there is $n_0 \in N$ such that $a_n \leq b_n$ for $n \geq n_0$, then equation (1.2) is nonoscillatory.*

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References

1. C. Huang, *J. Math. Anal. Appl. 214(1997)*.
2. Drumi Bainov and Pavel Simeonov, *Oscillation Theory of Impulsive Differential Equations (International Publications, 1998)*.
3. K. Gopalsamy and B. G. Zhang, *J.Math.Anal.Appl. 139(1989)*.
4. Aimin Zhao and Juran Yan, *J.Math.Anal.Appl. 201(1996)*.

EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR A CLASS OF HIGHER DIMENSIONAL NON-AUTONOMOUS SYSTEM WITH DELAY

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In this paper, the sufficient conditions for the existence and uniqueness of the periodic solutions of a higher dimensional system with delay

$$\dot{x}(t) = A(t)x(t) + g(t, x(t-r)), x \in R^n$$

are given by Schauder's fixed point theorem. Some results in [1-4] are extended and improved.

1   Basic Knowledge

Considering a higher dimensional non-autonomous system with delay

$$\dot{x}(t) = A(t)x(t) + g(t, x(t-r)), x \in R^n \qquad (1)$$

where $r \in R$, $A(t) \in C[R, R^{n \times n}]$, $g(t, x) \in C[R \times R^n, R^n]$, and there is a constant $\omega > 0$, it makes $A(t + \omega) = A(t)$, $g(t + \omega, x) = g(t, x)$. Besides we assume homogeneous linear system

$$\dot{x}(t) = A(t)x, x \in R^n \qquad (2)$$

has a non-trivial ω-periodic solution. In the above condition and $r = 0$, in [1-4] they ever studied the existence of ω-periodic solution in system (1). In this paper, the author discusses the existence and uniqueness of ω-periodic solution under the condition of $r \neq 0$, and extended and improved these results in [1-4]. And the author also draws some new conclusions.

Let $B_\omega = \{u(t) : u(t) \in C(R, R^n), u(t + \omega) = u(t)\}$, for $u(t) \in B_\omega$, we define its norm $\|u\| = \sup_{0 \le t \le \omega} |u(t)|$, it is easy to know B_ω is a Banach space. We define an operator in B_ω

$$T[u](t) = \int_0^\omega G(t - k\omega, \tau)\, g(\tau, u(\tau - r))d\tau \ \ (k\omega \le t \le (k+1)\omega, k = 0, \pm 1, \pm 2, \cdots),$$

here $G(t, \tau)$ is a Green matrix defined as following

$$G(t, \tau) = \begin{cases} X(t)(E + D)X^{-1}(\tau), & 0 \le \tau \le t \le \omega \\ X(t)DX^{-1}(\tau), & 0 \le t < \tau \le \omega \end{cases}$$

where $X(t)$ is the fundamental solution matrix of system (2) with $X(0) = E$, and $D = (X^{-1}(\omega) - E)^{-1}$. we can prove that the following Lemma are true, the proof is omitted.

Lemma 1 (i) Operator $T : B_\omega \to B_\omega$ is a completely continuous operator.

(ii) The fixed point of operator T is equivalent to the ω-periodic solution of system (1).

Let $a \in R^n, \varphi(t) = T[a](t)$, it is clear that $\varphi(t) \in B_\omega$. let $S_R(\varphi) = \{u(t) : u \in B_\omega, \|u - \varphi\| \le R\}$. It is easy to know that $S_R(\varphi)$ is a bounded closed convex subset in B_ω. Its boundary is $\partial S_R = \{u(t) : u(t) \in B, \|u - \varphi\| = R\}$. Define the nuclear conservative contraction projection of B_ω to $S_R(\varphi)$ is

$$f(u) = \begin{cases} u, & \text{if } u \in S_R(\varphi) \\ \dfrac{R(u - \varphi)}{\|u - \varphi\|} + \varphi, & \text{if } u \notin S_R(\varphi) \end{cases}$$

Lemma 2 (i) f is a continuous operator from B_ω to $S_R(\varphi)$.

(ii) If $f(u) \in S_R^0(\varphi) = \{u(t) : u \in B_\omega, \|u - \varphi\| < R\}$, then $f(u) = u$.

Lemma 3 $fT : B_\omega \to S_R(\varphi)$ is a completely continuous operator.

2 Main Results

The following discussions are based on which all the above conditions are satisfied.

Theorem 1 If there Exists $R > 0$, it makes $\sup_{\substack{|x| \le R \\ 0 \le t \le \omega}} |g(t, x)| = K(t)$, and $\int_0^\omega K(\tau)d\tau \le \frac{R}{\Gamma}$, then system (1) has at least one ω-periodic solution, where $\Gamma = \max_{0 \le t, \tau \le \omega} |G(t, \tau)|$.

Proof Let $S_R = \{u(t) : u(t) \in B_\omega, \|u\| \le R\}, \forall u \in S_R$, we have $u(t - r) \in$

S_R, so $\forall t \in [0, \omega]$, we have

$$\begin{aligned}
|T[u](t)| &= |\int_0^\omega G(t,\tau)g(\tau, u(\tau-r))d\tau| \\
&\leq \Gamma \int_0^\omega |g(\tau, u(\tau-r))|d\tau \\
&\leq \int_0^\omega K(\tau)d\tau \leq R,
\end{aligned}$$

so $\|T[u](t)\| \leq R, T\{S_R\} \subset S_R$. Because S_R is a bounded closed convex subset in B_ω, and also from (i) in Lemma 1, we know $T : S_R \to S_R$ is a completely continuous. By Schauder's fixed point, T has a fixed point in S_R. By (ii) in Lemma 1, so system (1) has a ω-periodic solution in S_R.

Corollary 1 If $\overline{\lim}_{|x| \to +\infty} \frac{|g(t,x)|}{|x|} = K(t)$ for $t \in [0, \omega]$ is uniformly true, and $\int_0^\omega K(\tau)d\tau < \frac{1}{\Gamma}$, then system (1) has at least one ω-periodic solution.

Proof Since $\int_0^\omega K(\tau)d\tau < \frac{1}{\Gamma}$, so there exists $\varepsilon_0 > 0$ such that

$$\int_0^\omega K(\tau)d\tau + \varepsilon_0\omega < \frac{1}{\Gamma}, \tag{3}$$

and $\overline{\lim}_{|x| \to +\infty} \frac{|g(t,x)|}{|x|} = K(t)$ for $t \in [0, \omega]$, then for above $\varepsilon_0 > 0$, there exists $M_1 > 0$ such that $\frac{|g(t,x)|}{|x|} \leq K(t) + \frac{\varepsilon_0}{2}$ $(0 \leq t \leq \omega)$ for $|x| \geq M_1$, that is, $|g(t,x)| \leq (K(t) + \frac{\varepsilon_0}{2})|x|$ $(0 \leq t \leq \omega)$. hence $\forall R \geq M_1$, we have

$$\sup_{M_1 \leq |x| \leq R} |g(t,x)| \leq (K(t) + \frac{\varepsilon_0}{2})R \quad (0 \leq t \leq \omega) \tag{4}$$

Let $\beta = \sup_{\substack{0 \leq t \leq \omega \\ |x| \leq M_1}} |g(t,x)|$, since $\lim_{R \to +\infty} \frac{\beta}{R} = 0$, thus for above $\varepsilon_0 > 0$, there exists $M_2 > 0$, when $R > M_2$, we have $\frac{\beta}{R} < \frac{\varepsilon_0}{2}$. So when $R \geq \max\{M_1, M_2\}$, and from (4), we obtain

$$\begin{aligned}
\frac{1}{R}\sup_{|x| \leq R} |g(t,x)| &\leq \frac{1}{R}(\sup_{|x| \leq M_1}|g(t,x)| + \sup_{M_1 \leq |x| \leq R}|g(t,x)|). \\
&\leq \frac{\beta}{R} + (K(t) + \frac{\varepsilon_0}{2}) \\
&\leq K(t) + \varepsilon_0 \quad (0 \leq t \leq \omega).
\end{aligned}$$

Therefore

$$\sup_{|x| \leq R} |g(t,x)| \leq (K(t) + \varepsilon_0)R \quad (0 \leq t \leq \omega),$$

and from (3), we have $\int_0^\omega R(K(\tau) + \varepsilon_0)d\tau = R(\int_0^\omega K(\tau)d\tau + \varepsilon_0\omega) < \frac{R}{\Gamma}$. By Theorem 1, we know this Corollary is true.

Corollary 2 If there exists $R > 0$, it makes $r(R) = \max_{\substack{0 \leq t \leq \omega \\ |x| \leq R}} |g(t,x)| \leq \frac{R}{\omega\Gamma}$, then system (1) has at least one ω-periodic solution.

Proof For $R > 0$, let $K(t) = \sup_{|x| \leq R} |g(t,x)|$ $(0 \leq t \leq \omega)$, it is obvious that $K(t) \leq \frac{R}{\omega \Gamma}$ $(0 \leq t \leq \omega)$, therefore $\int_0^\omega K(\tau)d\tau \leq \frac{R}{\Gamma}$. By theorem 1, we know the Corollary is true.

In system (1), let $r = 0$, from Corollary 1 and 2, we get the same conclusions in [1-2], In Corollary 1, let $K(t)$ also satisfied: $\sup_{0 \leq t \leq \omega} K(t) < \frac{1}{\Gamma \omega}$, we get the result in [3](see [1]).

Example Considering a periodic system of two-dimension

$$\dot{x}(t) = A_1(t)x(t) + g_1(t, x(t-r)), \tag{5}$$

and suppose (5) is satisfied all the conditions for $\omega = 2\pi$ in the first section in this paper. Let $g(t,x) = \frac{|x|\sin t}{4\sqrt{2}\Gamma} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Next we prove that there exists a 2π-periodic solution of (5). For $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, let its module $|x| = \sqrt{x_1^2 + x_2^2}$.

Because $\sup_{|x| \leq R} |g(t,x)| = \frac{|\sin t|}{4\Gamma} R = K(t)$ $(0 \leq t \leq 2\pi)$,

$$\int_0^{2\pi} K(\tau)d\tau = \frac{R}{4\Gamma} \int_0^{2\pi} |\sin\tau|d\tau = \frac{R}{2\Gamma} \int_0^\pi \sin\tau d\tau = \frac{R}{\Gamma},$$

thus from theorem 1, we know has at least one 2π-periodic solution in system (5). However, $\max_{\substack{0 \leq t \leq 2\pi \\ |x| \leq R}} |g_1(t,x)| = \frac{R}{4\Gamma} > \frac{R}{2\pi\Gamma}$, so the above result cann't be got by Corollary 2. And also $\overline{\lim}_{|x| \to +\infty} \frac{|g_1(t,x)|}{|x|} = \frac{|\sin t|}{4\Gamma}$ for $t \in [0, 2\pi]$ is uniformly true. But $\int_0^{2\pi} \frac{|\sin\tau|d\tau}{4\Gamma} = \frac{1}{\Gamma}$, so the results can not be deduced by Corollary 1.

Corollary 3 If $\overline{\lim}_{n \to \infty} \frac{1}{n} \int_0^\omega \sup_{|x| \leq n} |g(\tau,x)|d\tau < \frac{1}{\Gamma}$, then has at least one ω-periodic solution in system (1).

Proof Because $\lim_{|x| \to +\infty} \frac{1}{n} \int_0^\omega \sup_{|x| \leq n} |g(\tau,x)|d\tau < \frac{1}{\Gamma}$, then there exists natural number n_0, it makes $\frac{1}{n_0} \int_0^\omega \sup_{|x| \leq n_0} |g(\tau,x)|d\tau < \frac{1}{\Gamma}$. If we let $K(t) = \sup_{|x| \leq n_0} |g(t,x)|$ $(0 \leq t \leq \omega)$, we have $\int_0^\omega K(\tau)d\tau < \frac{n_0}{\Gamma}$, by Theorem 1, we know this Corollary is true.

Theorem 2 suppose that conditions of Theorem 1 or Corollary 2 is satisfied, and the system (1) also satisfied: $\forall u_1(t), u_2(t) \in S_R$, we have

$$|g(t, u_1(t-r)) - g(t, u_2(t-r))| \leq \alpha_{u_1 u_2}(t)|u_1(t-r) - u_2(t-r)| \quad (0 \leq t \leq \omega),$$

and $\Gamma \int_0^\omega \alpha_{u_1 u_2}(\tau)d\tau < 1$, Then system (1) has a unique ω-periodic solution in S_R.

Proof The existence is obvious. In order to prove the uniqueness, we

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prove at first T is the contraction projection in S_R. $\forall u_1, u_2 \in S_R$,

$$
\begin{aligned}
\|T[u_1](t) - T[u_2](t)\| &= \| \int_0^\omega G(t,\tau)(g(\tau, u_1(\tau - r)) - g(t, u_2(\tau - r)))d\tau \| \\
&\leq \Gamma \int_0^\omega \alpha_{u_1 u_2}(\tau)|u_1(\tau - r) - u_2(\tau - r)|d\tau \\
&\leq \|u_1 - u_2\| \Gamma \int_0^\omega \alpha_{u_1 u_2}(\tau)d\tau \\
&< \|u_1 - u_2\| \quad (\text{if } \|u_1 - u_2\| \neq 0)
\end{aligned}
$$

If system has two different ω-periodic solutions $u_{10}(t)$ and $u_{20}(t)$ in S_R, then $\|u_{10} - u_{20}\| = \|T[u_{10}] - T[u_{20}]\| < \|u_{10} - u_{20}\|$, which is contradicted.

From the same reason, we can prove:

Theorem 3   For system (1), suppose that conditions of Corollary i (i=1,3,4)

is satisfied, and it still satisfied: $\forall u_1, u_2 \in B_\omega$, we have

$$
|g(t, u_1(t - r)) - g(t, u_2(t - r))| \leq \alpha_{u_1 u_2}(t)|u_1(t - r) - u_2(t - r)| \quad (0 \leq t \leq \omega),
$$

and $\Gamma \int_0^\omega \alpha_{u_1 u_2}(\tau)d\tau < 1$, then there exists a unique ω-periodic solution of (1).

Next we discuss the existence and uniqueness of the ω-periodic solutions of system (1) on the super spherical area $S_R(\varphi)$.

Theorem 4   If there exists $S_R(\varphi)$, such that $\forall u \in \partial S_R(\varphi)$, we have

$$
|g(t, u(t - r)) - g(t, a)| \leq \beta_u(t)|u(t - r) - a| \quad (0 \leq t \leq \omega),
$$

$\Gamma \int_0^\omega \beta_u(\tau)d\tau \overset{\triangle}{=} \theta(u) \leq 1$, and also $\|u - a\| \leq R$, then system (1) has at least one ω-periodic solution in $S_R(\varphi)$.

Proof   From Lemma 2, we know $fT : S_R(\varphi) \to S_R(\varphi)$ is a completely continuous operator, and $S_R(\varphi)$ is a bounded closed convex subset in B_ω. By Scauder's fixed point theorem, we know fT has a fixed point $u_0(t)$ in $S_R(\varphi) : f\{T[u_0](t)\} = u_0$.

Next we prove $u_0(t)$ is a fixed point of T. For this purpose we prove at first $T\{\partial S_R(\varphi)\} \subset S_R(\varphi)$, Since $\forall u \in \partial S_R(\varphi), \forall t \in [0, \omega]$, we have

$$
\begin{aligned}
|T[u](t) - \varphi(t)| &= | \int_0^\omega G(t,\tau)(g(\tau, u(\tau - r)) - g(t, a))d\tau | \\
&\leq \Gamma \int_0^\omega \beta_u(\tau)|u(t - r) - a|d\tau \\
&\leq \Gamma \|u - a\| \int_0^\omega \beta_u(\tau)d\tau \\
&= \theta(u)\|u - a\| \leq R,
\end{aligned}
$$

so $\|T[u](t) - \varphi(t)\| \leq R$, and $T[u](t) \in S_R(\varphi)$.

Because $u_0 \in S_R(\varphi)$, if $u_0 \in \partial S_R(\varphi)$, from above we have $T[u](t) \in S_R(\varphi)$, By the definition of f, we have $u_0 = f\{T[u_0](t)\} = T[u_0](t)$. And if $u_0 = f\{T[u_0](t)\} \in S_R^0(\varphi)$. By (ii) of Lemma 2, we have $u_0 = f\{T[u_0](t)\} = T[u_0](t)$. Therefore u_0 is the fixed point of T. By (ii) of Lemma 1, we know

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there exists ω-periodic solution in $S_R(\varphi)$.

Theorem 5 If there exists $S_R(\varphi)$, Such that

(i) $\forall u_1, u_2 \in S_R(\varphi)$, we have

$$|g(t, u_1(t-r)) - g(t, u_2(t-r))| \leq \beta_{u_1 u_2}(t)|u_1(t-r) - u_2(t-r)| \quad (0 \leq t \leq \omega),$$

and $\Gamma \int_0^\omega \beta_{u_1 u_2}(\tau)d\tau \overset{\triangle}{=} \theta(u_1, u_2) < 1$.

(ii) $a \in S_R(\varphi)$, and $\forall u \in \partial S_R(\varphi)$, if $\theta(u, a) \geq \frac{1}{2}$, we have

$$\|u - a\| \leq \frac{1 - \theta(u, a)}{\theta(u, a)} R,$$

then there exists a unique ω-periodic solution of (1).

Proof Since $a \in S_R(\varphi)$, from (i), $\forall u \in S_R(\varphi)$, we have $|g(t, u) - g(t, a)| \leq \beta_{u_1 u_2}|u(t) - a| \quad (0 \leq t \leq \omega)$, and $\theta(u, a) < 1$.

Since $0 < \theta(u, a) < \frac{1}{2}$, $\frac{1 - \theta(u, a)}{\theta(u, a)} > 1$, so $\|\varphi - a\| \leq R < \frac{1 - \theta(u, a)}{\theta(u, a)} R$. Therefore from (ii), $\forall u \in \partial S_R(\varphi)$, If $\theta(u, a) \neq 0$, we have $\|\varphi - a\| \leq \frac{1 - \theta(u, a)}{\theta(u, a)} R$, therefore

$$\begin{aligned}
|T[u](t) - \varphi(t)| &\leq \Gamma \int_0^\omega |g(\tau, u(\tau - r)) - g(\tau, a)|d\tau \\
&\leq \Gamma \int_0^\omega \beta_{ua}(\tau)|u(t-r) - a|d\tau \\
&\leq \theta(u, a)\|u - a\| \\
&\leq \theta(u, a)[\|u - \varphi\| + \|\varphi - a\|] \\
&\leq \theta(u, a)R + (1 - \theta(u, a))R = R,
\end{aligned}$$

thus $\|T[u](t) - \varphi(t)\| \leq R, T\{\partial S_R(\varphi)\} \subset S_R(\varphi)$. From the proof of Theorem 2, it is easy to know that system (1) there exists ω-periodic solution in $S_R(\varphi)$.

Now we prove the uniqueness, $\forall u_1, u_2 \in S_R(\varphi), \forall t \in [0, \omega]$, from condition (i), we have

$$\begin{aligned}
|T[u_1](t) - T[u_2](t)| &\leq \Gamma \int_0^\omega |g(\tau, u_1(\tau - r)) - g(\tau, u_2(\tau - r))|d\tau \\
&\leq \Gamma \int_0^\omega \beta_{u_1 u_2}(\tau)|u_1(t-r) - u_2(t-r)|d\tau \\
&\leq \theta(u_1, u_2)\|u_1 - u_2\|,
\end{aligned}$$

so $\|T[u_1](t) - T[u_2](t)\| \leq \theta(u_1, u_2)\|u_1 - u_2\| < \|u_1 - u_2\| \quad$ (if $\|u_1 - u_2\| \neq 0$). Therefore the fixed point of T in $S_R(\varphi)$ is unique. System (1) there exists a unique ω-periodic solution in $S_R(\varphi)$.

If $r = 0$, from this theorem we can induce the results in [4] (see Corollary 4 in [2]).

Acknowledgments

References

1. R. Reissig, G. Sanson, R. Cinti, *Nonlinear differential Equations of Higher Order* (Noordhoff International Publishing Leyden, 72-86, 1974.)
2. X. F. Wu, *Journal of Mathematics (Chinese)***16**,485-489 (1996).
3. R. Conti,*Equazioni Differenziali Ordinarie Condizioni Lineari Generali, Atti Accad. Naz. Incei Rend. Cl. Sci. Fis. Mat. Natur.***261**, 636-640 (1959).
4. A. Halang, *Qualitative Theory of Differential Equations: Lyapunov Stability, Oscillations, System with Deviating Argument, Ed. Acad. Pop.* (Romine,Bucharest, 1963, Translation: Academic Press, New York, London, 1967).

TRAVELLING WAVES FOR A QUASILINEAR BURGERS-TYPE EQUATION

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1 Introduction

The simplest discontinuous solution of hyperbolic conserved system

$$u_t + f(u)_x = 0 \tag{1.1}$$

are the shock wave solution defined by

$$u(x,t) = \begin{cases} u_L & x < st \\ u_R & x > st \end{cases} \tag{1.2}$$

and u_L, u_R and s satisfy Rankine-Hugoniot conditions

$$-s(u_R - u_L) + f(u_R) - f(u_L) = 0. \tag{1.3}$$

Consider the viscous conserved system

$$u_t + f(u)_x = \mu[B(u)u_x]_x. \tag{1.4}$$

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the admissible shock of (1.1)-(1.3) is the limit of the smooth travelling front (viscous shock profile) $U((x - st)/\mu)$ as $\mu \to 0$, if some entropy conditions are satisfied and $B(u)$ is admissible [3,5].

The existence and stability of viscous shock profile for various viscous conserved system has been considered by many authors [3,4,5,6,7,8].

The typical viscous conserved equation is Burger's equation:

$$u_t + uu_x = u_{xx}.$$

The stability of travelling front has been obtained in [3,4].

Recently, A.Kurganov etc.[1,2] considered a new variant of the burgers equation with a genuinely nonlinear saturating diffusion

$$u_t + f(u)_x = \mu[Q(u_x)]_x \tag{1.5}$$

with $Q(s) = Q_1(s) = \frac{s}{\sqrt{1+s^2}}, Q(s) = Q_2(s) = \frac{s}{1+s^2}$ resp.

The numerical results in [1,2] showed that some travelling fronts may be stable.

In this paper, we consider the the existence and stability of the smooth travelling waves for (1.5) with $Q(s) = Q_2(s)$.

2 The Existence of Travelling Fronts

Consider the following quasilinear Burgers-type equations

$$u_t + f(u)_x = [Q(u_x)]_x \tag{2.1}$$

with

$$f(u) = \frac{u^2}{2}, \quad Q(u_x) = \frac{u_x}{(1 + u_x^2)}. \tag{2.2}$$

Let $u(x, t) = U(x - ct)$ be a travelling wave solutions of (2.1) and (2.2) connecting u_1 and 0, then $U(x - ct)$ satisfies the following boundary value problem:

$$-cU' + (U^2/2)' = [U'/(1 + (U')^2)]', \quad \xi \in R \tag{2.3}$$

$$U(-\infty) = u_1, \quad U(+\infty) = 0 \tag{2.4}$$

Integrating the equation (2.3) from ξ to $+\infty$, we have

$$-cU + \frac{U^2}{2} = \frac{U_\xi}{1 + (U_\xi)^2}, \quad \text{and } u_1 = 2c. \tag{2.5}$$

If $u_1 < 2$, we can show that there exists travelling front solution $U(x - ct)$ satisfying (2.3) and (2.4) with $c = \frac{u_1}{2}$.

and

$$(1 - U_\xi^2)/(1 + U_\xi^2)^2 \geq c_0 > 0, \tag{2.6}$$

with c_0 only depends on u_1.

Theorem 1. For any $0 < u_1 < 2$, there exists a travelling front solution $U(x - ct)$ connecting u_1 and 0 with speed $c = u_1/2$, satisfying (2.7).

3 The Stability of Travelling Front Solutions

In this section , we consider the following initial value problem of (2.1) and (2.2),

$$\begin{cases} u_t + uu_x = \frac{1 - u_x^2}{(1 + u_x^2)^2} u_{xx} \\ u|_{t=0} = u_0(x). \end{cases} \tag{3.1}$$

Introducing new variable $\xi = x - ct$, (3.1) can be written as

$$\begin{cases} u_t - cu_\xi + uu_\xi = \frac{1 - u_\xi^2}{(1 + u_\xi^2)^2} u_{\xi\xi} = [Q(u_\xi)]_\xi \\ u|_{t=0} = u_0(\xi). \end{cases} \tag{3.2}$$

Define $w(\xi, t) = u(\xi, t) - U(\xi)$, then $w(\xi, t)$ satisfies

$$\begin{cases} w_t - cw_\xi + (Uw)_\xi + ww_\xi = [Q(U_\xi + w_\xi) - Q(U_\xi)]_\xi \\ w|_{t=0} = u_0(\xi) - U(\xi). \end{cases} \tag{3.3}$$

Rewrite (3.3) as

$$w_t = \mathcal{L}_U w + g(w, w_\xi, w_{\xi\xi}), \tag{3.4}$$

with

$$\mathcal{L}_U w = \frac{\partial}{\partial \xi}(a_0(\xi)w_\xi) - \frac{\partial}{\partial \xi}(Uw) + cw_\xi, \tag{3.5}$$

$$a_0(\xi) = (1 - U_\xi^2)/(1 + U_\xi^2)^2,$$

and

$$\|g(w, w_\xi, w_{\xi\xi})\|_{L_2(R)} \leq C\|w\|_{H^2(R)}^2, \quad \text{if } \|w\|_{H^2(R)} \text{ small.}$$

Define weighted space

$$X_a = \{w/v_a(\xi)w(\xi) \in L_2(R)\}, v_a(\xi) = \exp(a\xi) + \exp(-a\xi),$$

$$D(A) = \{w \in X_a, v_a(\xi)w(\xi) \in H^2(R)\}.$$

Obviously

$$\|g(w, w_\xi, w_{\xi\xi})\|_{X_a} \leq C\|w\|_{D(A)}^2, \quad \text{if } \|w\|_{D(A)} \text{ small.} \tag{3.6}$$

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Define operator $A : D(A) \to X_a$, and $Aw = \mathcal{L}_U w$.

Lemma 3.1. *For any* $0 < u_1 < 2, 0 < a < \frac{u_1}{2}$, *A is a sectorial operator in the weighted space X, and*

$$\sup \mathrm{Re} \{\sigma_{ess}(A)\} < -c_0 < 0. \tag{3.7}$$

Lemma 3.2. For $\mathrm{Re}\lambda \geq -\sigma_0$, *the eigenvalue of A must be real and*

$$\sup\{\mathrm{Re}\{\sigma_p(A) \setminus \{0\}\}\} < 0,$$

and 0 is a simple eigenvalue of A.

Theorem 3.1. *For any* $0 < a < c/2$,

$$\sup\{\sigma(A) \setminus \{0\}\} < 0, \text{with} \ D(A) = \{w/v_a(\xi)w(\xi) \in H^2(R)\}$$

and 0 is a simple eigenvalue of A.

Applying the theory of analytic semigroup to the fully nonlinear equation (3.4), we can prove the exponential stability of travelling waves obtained in Theorem 1 in the weighted space D(A).

Theorem 2. *For* $0 < u_1 < 2$ *and any fixed* $0 < a < u_1/4$, *there exists small* $\delta > 0$ *such that for* $\|v_a(\xi)(u_0(\xi) - U(\xi))\|_{H^2(R)} \leq \delta$,

$$\|v_a(\xi)(u_(t,\xi) - U(\xi + \gamma))\|_{H^2(R)} \leq C \exp(-\sigma t),$$

with $\sigma > 0, \gamma$ *and C depending only on* a, u_1 *and* u_0.

Remark. Similar results about the existence and stability of travelling waves of equation (2.1) can be obtained for more general function $f(u)$, and other types of $Q(u_x)$, e.g. $Q(u_x) = \frac{u_x}{\sqrt{1+u_x^2}}$.

Acknowledgments

References

1. A. Kurganov, D. Levy and P.Rosenau, On Burgers-type equations with non-monotonic dissipative fluxes,*Commun. Pure Appl. Math.*51,443-473(1998).
2. A. Kurganov and P.Rosenau, Effects of a saturating dissipation in burgers-type equations,*Commun. Pure Appl. Math.*50,753-771(1997).

3. A.M.Il'in and O.A.Oleinik, Behavior of the solutions of the Cauchy problem for certain quasilinear equations for unbounded increase of time, AMS Translations,**42**,19-23(1964).

4. D.H. Sattinger, On the stability of waves of nonlinear parabolic systems, Advances in Math.**22**,312-355(1976).

5. J.Goodman, Nonlinear asymptotic stability of viscous profiles for conservation laws,Arch. Rat. Mech. Anal.**95**,325-344(1986).

6. C.K.R.T. Jones, Stability of travelling waves for non-convex scalar viscous conservation laws,*Commun. Pure Appl. Math.***46**,505-526(1993).

7. T.P. Liu, Pointwise convergence to shock waves for viscous conservation laws,*Commun. Pure Appl. Math.***50**,1113-1182(1997).

8. T.Kapitula, On the stability of travelling waves in weigted L^∞ spaces, J. Differential Equations.**112**,179-215(1994).

GLOBAL ATTRACTOR OF REACTION-DIFFUSION EQUATIONS WITH DELAYS[j]

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The object of this paper is to consider the asymptotic behavior of solutions of the partial functional differential equations. A new approach is developed to study the invariant sets and global attractor of semilinear reaction-diffusion equations with time delays. The results obtained are new even in the context of reaction-diffusion equations whose the reaction term do not contain delays.

1 Preliminary

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. $C(X,Y)$ denotes the class of continuous mapping from a Banach space X to a Banach space Y, specially, $C = C([-r,0] \times \Omega, R^m)$. $L^2(\Omega)$ is the space of real Lebesgue measurable functions on Ω. It is a Banach space for the norm

$$\|u\| = [\int_\Omega |u(x)|^2 dx]^{1/2},$$

where $|u|$ denotes the Euclid norm of a vector $u \in R^p$ for any integer p.

Let $H^1(\Omega) = \{u \in L^2(\Omega), \ \bigtriangledown u \in L^2(\Omega), \ \bigtriangledown \ \text{is the gradient operator}\}$, H_0^1 is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, where $C_0^\infty(\Omega)$ is the space of real C^∞ functions on Ω with a compact support in Ω. H_0^1 is also a Banach space for the norm

$$|||u||| = [\int_\Omega | \bigtriangledown u(x)|^2 dx]^{1/2}.$$

Let $u_t(x) = (u_t^1(x), \cdots, u_t^m(x))^T \triangleq \text{Col}\{u_t^i(x)\} \in C$, for any $t \geq t_0$ each u_t^i is defined by $u_t^i(s, x) = u^i(t+s, x)$ with $x \in \Omega$ and $s \in [-r, 0]$. For $u \in R^m$, we define $[u]^+ = \text{Col}\{|u^i|\}$, and for $u \in C$, we define $[u_t]_r^+ = \text{Col}\{|u_t^i|_r\}$, where $|u_t^i(x)|_r = \sup_{t-r \leq s \leq t} |u^i(t+s, x)|$.

In the similar manner, we may define

$$[u(t)]^L = \text{Col}\{\|u^i(t)\|\}, \quad [u_t]_r^L = \text{Col}\{\|u_t^i\|_r\}, \quad \|u_t^i\|_r = \sup_{t-r \leq s \leq t} \|u^i(t+s)\|,$$

$$[u(t)]^\bigtriangledown = \text{Col}\{|||u_i(t)|||\}, \quad [u_t]_r^\bigtriangledown = \text{Col}\{|||u_t^i|||_r\}, \quad |||u_t^i|||_r = \sup_{t-r \leq s \leq t} |||u^i(t+s)|||.$$

Then $C^L \triangleq C([-r, 0], L^2(\Omega)^m)$ and $C^\bigtriangledown \triangleq C([-r, 0], H_0^1(\Omega)^m)$ are also Banach space for the norms $\|u\|_r = \{([u]^L)^T [u]^L\}^{1/2}$ and $|||u|||_r = \{([u]^\bigtriangledown)^T [u]^\bigtriangledown\}^{1/2}$, respectively.

$A \geq B$ $(A < B)$ means that each pair of corresponding elements of A and B satisfies the inequality " $\geq$ " (" $<$ "), especially, A is called a nonnegative matrix if $A \geq 0$.

Lemma 1 [3]. *If $M \geq 0$ and $\rho(M) < 1$ then $(I - M)^{-1} \geq 0$.*

The symbol $\rho(M)$ denotes the spectral radius of a square matrix M.

Lemma 2 [6]. *We assume that H is a metric space and the semigroup $T(t)$ is continuous and uniformly compact for t large. We also assume that there exists a bounded set $\mathcal{B}$ such that $\mathcal{B}$ is absorbing in H. Then there is a global attractor $\mathcal{R}$ for the semigroup $T(t)$. Furthermore, if H is a Banach space, then $\mathcal{R}$ is connected too.*

2 Main Result

Consider semilinear reaction diffusion equations with delay, for $i = 1, \cdots, m$,

$$\begin{cases} \frac{\partial u^i(t,x)}{\partial t} = a_i \Delta u^i(t, x) - b_i u^i(t, x) + f_i(u_t(x)), \ t \geq t_0, \ x \in \Omega, \\ u^i(t, x) = 0, \quad t > t_0, \ x \in \partial\Omega, \ \forall i, \\ u^i(x, t_0 + s) = \phi_i(s, x), \ -r \leq s \leq 0, \ x \in \Omega, \end{cases} \quad (2.1)$$

where $a_i > 0$, $b_i \geq 0$ are constants and if $b_i = 0$ we agree that no boundary condition applies to u^i, Δ is the Laplace operator, the initial data $\phi = \mathrm{Col}\{\phi_i\} \in C$ is a given function. $f = \mathrm{Col}\{f_i\} \in C(C, R^m)$ is globally Lipschitz uniformly in $x \in \Omega$.

Under the above conditions, the assumptions of existence and uniqueness results [5] are then satisfied. This is however sufficient to define the semigroup $T(t)$: we set

$$T(t) : \phi \in C^L \rightarrow u_t \in C^L.$$

In the following, we always suppose that:

(H_1) $[f(u_t(x)]^+ \leq B[u_t]_r^+ + P$, $B = (b_{ij})_{m \times m}$, $P = (p_1, \cdots, p_m)^T$,

(H_2) $\rho(M) < 1$, $M = (m_{ij})_{m \times m}$, $m_{ij} = \frac{b_{ij}}{\eta_i}$, $\eta_i = \frac{a_i}{\gamma_i} + b_i$, where $\gamma_i = \gamma_i(\Omega)$ is a constant determined by Poincare inequality. $\gamma_i = h/\sqrt{m}$ if $\Omega = \{x \in R^m \| |x_i| < h\}$.

(H_3) $\rho(N) < 1$, $N = (n_{ij})_{m \times m}$, $n_{ij} = \frac{m \gamma_i^2 b_{ij}^2}{2 a_i b_i}$. Especially, $n_{ij} = \frac{h^2 b_{ij}^2}{2 a_i b_i}$ if $\Omega = \{x \in R^m \| |x_i| < h\}$.

Theorem. *We assume that the hypotheses (H_1), (H_2) and (H_3) are satisfied. Then the semigroup $T(t)$ associated to the system (2.1) possesses a global attractor $\mathcal{A}$ which is bounded in C^∇, compact and connected in C^L; $\mathcal{A}$ attracts the bounded sets of C^L.*

Remark 1. Temam [6] studied Equation (2.1) without delay (i.e. $r = 0$). Under the conditions those (2.1) has an invariant region and f is bounded (i.e., $B = 0$ in the condition (H_1)), Temam[6](Th.1.3,p96) proved that (2.1) possesses a local attractor. Our result leaves out the above Temam's conditions and gives more exact region of positively invariant and attracting set by a pseudo-rectangle instead of pseudo-ball.

3. Proof of theorem

In order to prove our theorem, we first show the following propositions 1-4.

Proposition 1. *Let $\mathcal{M}(\Omega)$ is the measure of Ω and $Q = [q_1, \cdots, q_m]^T$, $q_i = p_i \sqrt{\mathcal{M}(\Omega)}/\eta_i$. If (H_1) and (H_2) hold, then the set $S_\alpha = \{\phi \in C^L | [\phi]_r^L \leq \alpha K = \alpha(I - M)^{-1} Q, \alpha \geq 1\}$, which is called a pseudo-rectangle, is a positively invariant set for the semigroup $T(t)$ associated to the system (2.1).*

Proof After multiplying (1) by $u^i(t, x)$, by integration of (1), and using the Green formula, the Poincare inequality and Holder inequality, we have

$$\frac{d}{dt}[\|u^i\|]^2 \leq -\frac{a_i}{\gamma_i}\|u^i\|^2 - b_i\|u^i\|^2 + \|u^i\|[\sum_{j=1}^{m} b_{ij}\|u_t^j\|_r + p_i\sqrt{\mathcal{M}(\Omega)}]. \quad (2.2)$$

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From $\frac{a_i}{\gamma_i} + b_i = \eta_i$, we obtain

$$\|u^i\|^2 \le e^{-\eta_i(t-t_0)}\|\phi_i\|_r^2 + \int_{t_0}^t e^{-\eta_i(t-s)}\eta_i\|u^i\|[\sum_{j=1}^m m_{ij}\|u_s^j\|_r + q_i]ds. \quad (2.3)$$

Without loss of generality, we assume that $P > 0$, i.e., $Q > 0$. Since $\rho(M) < 1$, from Lemma 1, $(I - M)^{-1} \ge 0$ and $K = (I - M)^{-1}Q > 0$. We now prove that, when $[\phi]_r^L < \alpha K$

$$[u(t)]^L < \alpha K \quad \text{for} \quad t \ge t_0. \quad (2.4)$$

If (2.4) is not so, there must be some i, and $t_1 > t_0$, such that

$$|u^i(t_1)| = \alpha k_i, \quad |u^i(t)| < \alpha k_i, \quad \text{for} \quad t < t_1, \quad (2.5)$$
$$[u(t)]^L \le \alpha K, \quad \text{for} \quad t_0 \le t \le t_1, \quad (2.6)$$

where k_i is the i th component of vector K. Then

$$|u^i(t_1)|^2 < e^{-\eta_i(t_1-t_0)}\alpha^2 k_i^2 + \int_{t_0}^{t_1} e^{-\eta_i(t_1-s)}\eta_i\alpha k_i[\sum_{j=1}^m m_{ij}\alpha k_j + q_i]ds.$$

Let diag$\{k_i\}$ (or diag$K$) is a diagonal matrix with diagonal entries k_i. Noting that $K = (I - M)^{-1}Q$, i.e., $MK + Q = K$, we have

$$\begin{aligned}
\text{Col}\{|u^i(t_1)|^2\} &< \alpha^2\text{diag}\{e^{-\eta_i(t_1-t_0)}k_i\}(MK + Q) \\
&\quad + (I - \text{diag}\{e^{-\eta_i(t_1-t_0)}\})\alpha\text{diag}\{k_i\}[\alpha MK + Q] \\
&\le \alpha(\alpha-1)\text{diag}\{k_i\}Q + \alpha\text{diag}\{k_i\}[\alpha MK + Q] \\
&= \alpha^2\text{diag}\{k_i\}K = \alpha^2\text{Col}\{k_i^2\}. \quad (2.7)
\end{aligned}$$

This implies that $|u^i(t_1)| < \alpha k_i$, which contradicts the equality in (2.5), and so (2.4) holds.

Proposition 2. *If (H_1) and (H_2) hold, then the pseudo-rectangle $S = \{\phi \in C^L | [\phi]_r^L \le K = (I - M)^{-1}Q\}$ is a global attracting set for the semigroup $T(t)$ associated to the system (2.1).*

Proof. For any $\phi \in C^L$, there is a pseudo-rectangle S_α such that $\phi \in S_\alpha$. From Proposition 1, the solution $u(t,x)$ satisfies $[u(t)]^L \le \alpha K$, and there exist a constant vector $\sigma \ge 0$ such that $\lim_{t \to +\infty} \sup[u(t)]^L = \sigma$. Then, for sufficient small positive constant ε, there is $t_2 > t_0$, such that, for any $t \ge t_2$,

$$[u_t]_r^L \le (\sigma + E\varepsilon), \quad E = [1, \cdots, 1]^T. \quad (2.8)$$

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Since $\eta_i > 0$, for the above ε and K, there must be $T > 0$ such that for $t \geq T$,

$$\alpha^2 \text{diag}\{e^{-\eta_i(t-t_0)}k_i^2\} + \int_T^\infty \text{diag}\{e^{-\eta_i s}\eta_i\}\text{diag}\{\alpha K\}(M(\alpha K) + Q)ds \leq E\varepsilon.$$

So, when $t \geq t_2 + T$,

$$\begin{aligned}
\text{Col}\{|u^i|^2\} &\leq \text{Col}\{e^{-\eta_i(t-t_0)}|\phi_i|_r^2\} \\
&\quad + \{\int_{t_0}^{t-T} + \int_{t-T}^t\}\text{diag}\{e^{-\eta_i(t-s)}\eta_i\}\text{diag}\{|u^i|\}(M[u(s)]_r^L + Q)ds \\
&\quad + E\varepsilon + \int_{t-T}^t \text{diag}\{e^{-\eta_i(t-s)}\eta_i\}\text{diag}\{\sigma + E\varepsilon\}(M(\sigma + E\varepsilon) + Q)ds \\
&\leq E\varepsilon + \text{diag}\{\sigma + E\varepsilon\}(M(\sigma + E\varepsilon) + Q).
\end{aligned}$$

Letting $\varepsilon \to 0$ and using (2.13), we obtain

$$\text{diag}\{\sigma\}\sigma \leq \text{diag}\{\sigma\}(Q + M\sigma),$$

that is, $\sigma \leq M\sigma + Q$ or $\sigma \leq (I - M)^{-1}Q$. Hence (3) holds, and the proof is completed.

We now prove the existence of an absorbing set in $H_0^1(\Omega)$.

Proposition 3. *Let $\hat{P} = [\hat{p}_1, \cdots, \hat{p}_m]^T$, $\hat{p}_i = \frac{p_i^2 \mathcal{M}(\Omega)}{2a_i b_i}$. If (H_1) and (H_3) hold, we have the pseudo-rectangle $\Gamma_\alpha = \{\phi \in C^\nabla | \text{Col}\{\|\|\phi_i\|\|_r^2\} \leq \alpha \hat{K} = \alpha(I - N)^{-1}\hat{P}, \alpha \geq 1\}$, is a positively invariant for the semigroup $S(t)$ associated to the system (2.1).*

Proof. Multiplying equation (1) by $-\Delta u^i$ and integrate over Ω, using the boundary condition of (1) and the Green formula, we have

$$\frac{d}{dt}\|\|u^i\|\|^2 \leq -2a_i\|\Delta u^i\|^2 - 2b_i\|\|u^i\|\|^2 + 2\int_\Omega |\Delta u^i||f_i(u_t(x))|dx, \quad (2.9)$$

By using Hölder inequality and the Poincare inequality we obtain

$$\frac{d}{dt}\|\|u^i\|\|^2 \leq -2b_i\|\|u^i\|\|^2 + \sum_{j=1}^m \frac{m\gamma_i^2}{a_i}b_{ij}^2\|\|u_t^j\|\|^2 + \frac{p_i^2\mathcal{M}(\Omega)}{a_i}. \quad (2.10)$$

We now prove that, when $\text{Col}\{\|\|\phi_i\|\|_r^2\} < \alpha\hat{K}$

$$\text{Col}\{\|\|u^i(t)\|\|\} < \alpha\hat{K} \quad \text{for} \quad t \geq t_0. \quad (2.11)$$

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If (2.11) is not so, there must be some i, and $t_1 > t_0$, such that

$$|||u^i(t_1)|||^2 = \alpha \hat{k}_i, \quad |||u^i(t)|||^2 < \alpha \hat{k}_i, \quad \text{for } t < t_1, \tag{2.12}$$

$$\text{Col}\{|||u^i(t)|||^2\} \leq \alpha \hat{K}, \quad \text{for } t_0 \leq t \leq t_1, \tag{2.13}$$

where k_i is the i th component of vector $\hat{K}$. From (2.10)

$$|||u^i(t)|||^2 \leq e^{-2b_i(t-t_0)}|||\phi_i|||_r^2 + \int_{t_0}^t e^{-2b_i(t-s)} 2b_i \sum_{j=1}^m [n_{ij}|||u_s^j|||^2 + \hat{p}_i] ds.$$

Letting $b = \text{diag}\{b_i\}$ and noting that $\hat{K} = (I - N)^{-1}\hat{P}$, i.e., $N\hat{K} + \hat{P} = \hat{K}$, we have

$$\text{Col}\{|||u^i|||^2\} < e^{-2b(t-t_0)}\alpha\hat{K} + (I - e^{-2b(t-t_0)})[N\alpha\hat{K} + \hat{P}]$$
$$\leq (\alpha - 1)\hat{P} + N\alpha\hat{K} + \hat{P} = \alpha(N\hat{K} + \hat{P}) = \alpha\hat{K}. \tag{2.14}$$

This implies that $\|u^i(t)\|^2 < \alpha \hat{k}_i$, which contradicts the equality in (2.12), and so (2.11) holds.

Proposition 4. *If (H_1) and (H_3) hold. Then the pseudo-rectangle $\Gamma = \{\phi \in C^\nabla | \text{Col}\{|||\phi_i|||_r^2\} \leq \hat{K} = (I-N)^{-1}\hat{P}\}$, attracts each bounded set $\mathcal{B} \subset C^\nabla$ under $T(t)$. the semigroup $T(t)$ associated to the system (2.1).*

Proof. For any bounded set $\mathcal{B} \subset C^\nabla$, there is an $\alpha \geq 1$ such that $\mathcal{B} \subset \Gamma_\alpha$. From Proposition 3, for any $\phi \in \mathcal{B}$, the solution $u(t, x)$ satisfies $\|u^i(t)\|^2 \leq \alpha\hat{k}_i$. For any given $\beta > 0$, we choose T_1 such that for $t \geq T_1$

$$e^{-2b(t-t_0)}\text{Col}\{\|\phi_i\|_r^2\} + e^{-2bT_1}(\alpha N\hat{K} + \hat{P}) \leq \beta\text{Col}\{1\}, \tag{2.15}$$

where $|(\cdot)|_E$ denotes the Euclid norm of matrix.

Then, from (2.10), we have

$$\|u^i(t)\|^2 \leq \int_{t-T_1}^t e^{-2b_i(t-s)} 2b_i [\sum_{j=1}^m n_{ij}\|u_s^i\|_r^2 + \hat{p}_i] ds + \beta. \tag{2.16}$$

Let $\lim_{t\to+\infty} \sup |||u^i(t)||| = \sigma_i$, $\sigma = [\sigma_1 \cdots, \sigma_m]^T$, and following the remainder of the proof of Proposition 2,

$$\sigma_i^2 \leq \sum_{j=1}^m n_{ij}\sigma_i^2 + \hat{p}_i + \beta. \tag{2.17}$$

Letting $\beta \to 0$, we have

$$\sigma \leq \hat{K} = (I - N)^{-1}\hat{P}$$

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This implies that Γ attracts each bounded set $\mathcal{B} \subset C^{\nabla}$ under $T(t)$.

Proof of Theorem. We shall prove that the operators $S(t)$ are uniformly compact.

For any given $\phi \in C^L$, by Proposition 1, there is an α such that $\phi \in S_\alpha$, i.e., $\|u^i(t)\| \leq \alpha k_i$ for all $t \geq t_0$. Then from (??), we obtain

$$\frac{d}{dt}\|u^i\|^2 \leq -a_i\|\|u^i\|\|^2 - b_i\|u^i\|^2 + \|u^i\|\sum_{j=1}^m [b_{ij}\|u_t^j\|_r + p_i\sqrt{\mathcal{M}(\Omega)}]$$

$$\leq -a_i\|\|u^i\|\|^2 + \delta \quad \forall t \geq t_0, \tag{2.18}$$

where $\delta = \delta(\alpha K) = \alpha k_i \sum_{j=1}^m [b_{ij}\alpha k_j + p_i\sqrt{\mathcal{M}(\Omega)}]$.

For $h > r$ fixed, we integrate (2.18) between t and $t + h$ and obtain

$$\int_t^{t+h} \|u^i(s)\|^2 ds \leq \frac{h\delta}{a_i} + \frac{1}{a_i}|u^i(t)|^2 \leq \frac{h\delta}{a_i} + \frac{\alpha^2 k_i^2}{a_i}. \tag{2.19}$$

On the other hand, from Proposition 1 and (2.9), we have

$$\frac{d}{dt}\|u^i(t)\|^2 \leq \sum_{j=1}^m \frac{m}{a_i}b_{ij}^2\alpha^2 k_j^2 + \frac{p_i^2\mathcal{M}(\Omega)}{a_i} \triangleq \xi_i. \tag{2.20}$$

We multiply (2.20) by t and obtain

$$\frac{d}{dt}(t\|u^i\|^2) \leq \xi_i t + \|u^i\|^2. \tag{2.21}$$

By integration between 0 and $h > r$ and using (2.19)

$$\|u^i(h)\|^2 \leq \frac{1}{2}\xi_i h + \frac{1}{h}\int_0^h \|u^i\|ds \leq \frac{1}{2}\xi_i h + \frac{1}{h}[\frac{h\delta}{a_i} + \frac{\alpha^2 k_i^2}{a_i}] \triangleq \zeta_i. \tag{2.22}$$

For the above $\zeta = \mathrm{Col}\{\zeta_i\}$, there must be α' such that $\zeta \leq \alpha' K$. If $\phi \in S_\alpha \subset C^L$, then $u_h = T(h)\phi \in \Gamma_{\alpha'} \subset C^{\nabla}$. It is easy to deduce from Proposition 3 and 4 that the pseudo-rectangle $\Gamma_{\alpha'} = \{\phi \in C^{\nabla} | \mathrm{Col}\{\|\phi_i\|_r^2\} \leq \hat{K}, \alpha' \geq 1\}$ is positively invariant and uniformly absorbing in C^{∇}. If $\mathcal{B}$ is any bounded set of C^L included in S_α, after a certain time $t_1 = t_1(\mathcal{B}, \alpha')$, we find that $u(t)$ belongs to the absorbing set $\Gamma_{\alpha'}$. This shows that

$$T(t)\mathcal{B} \subset \Gamma_{\alpha'}, \quad \forall t \geq t_1.$$

The embedding $H^1(\Omega) \subset L^2(\Omega)$ is compact [9], so is $\Gamma_{\alpha'} \subset C^L$.

From the above proof and Proposition 1-4, all the assumptions of Lemma 2 are now satisfied with $H = C^L$ and the proof of Theorem is completed.

References

1. J. K. Hale, (Providence: Amer. Math. Soc., 1988).
2. P.W. Bates, K. Lu and Z. Zeng, Memoirs Amer. Math. Soc.**645**,(1998).
3. Lasalle, J. P., (Philadelphia: SIAM, 1976).
4. D. Y. Xu, S. Li and Z. Pu, Computers and Mathematics with Applications**38**,155-162(1999)
5. R. H. Martin and H. L. Smith, Trans. Amer. Math. Soc.**321**,1-44(1990).
6. R. Temam, (Springer-Verlag, New York, 1988).
7. J. Wu, (Springer-Verlag, New York, 1996).
8. R. A. Horn and Johnson, C. R., (Cambridge University Press, London, 1990).
9. R. S. Adams, (Academic Press, New York, 1975).

ON NON-EXISTENCE OF POSITIVE SOLUTIONS OF NONLINEAR POLYHARMONIC EQUATIONS [k]

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Abstract

One will show that if $f(t)$ is a nonnegative function such that $f(t) \geq t^q$ with $q \geq 1$ for $t \geq 0$, then there is no C^{2p} positive solutions to the equation $(-\Delta)^p u = f(u)$ on R^n provided $n < 2p$.

1 Introduction

Classification for positive solutions of the equation

$$\Delta u + u^p = 0, \quad \text{in} \quad R^n \tag{1.1}$$

has played a very important rule in geometric related problem.

Since the elegant result of Gidas, Ni and Nirenberg[1] has surfaced, there are much work on this type of equations. However the geometric correspondence to such equations has been known earlier. One refers to Obata's work.

From geometric point of view, higher order elliptic differential equation will be much harder since the geometric meaning of them are not clear. The finding of Paneitz's operator can be considered as the breakthrough. It leads one to

completely understand the corresponding biharmonic equations even though it has been widely studied previously. The interesting reader can consult the recent papers [2], [4] and [5].

Once one understands the biharmonic semilinear equations, one can go one step further to study semilinear polyharmonic equations [3]. The nonexistence of the entire (positve) solutions for certain semilinear polyharmonic equations frequently appeared in the mathematical literatures. However it seems that systematically understood such problem has not been appeared yet. Based on our knowledge for biharmonic equations, in this paper we will show the following main result:

Theorem 1.1 *If $n > 2p$, $1 \leq q < (n)/(n - 2p)$ and $f(t) \geq t^q$ for $t \geq 0$, the only C^{2p} nonnegative entire solutions of*

$$(-\Delta)^p u = f(u) \quad in \quad R^n \tag{1.2}$$

are the nonnegative algebraic solutions of $f(t) = 0$.

Theorem 1.2 *There is no C^{2p} entire positive solutions to (1.2) if $n < 2p$ and $f(t) \geq t^q$ for $t \geq 0$ and $q \geq 1$.*

The organization of the paper is as follows: In section 2, one will prove several elementary properties of the entire solutions to (1.2). Proof of theorems 1.1 and 1.2 will be given in Section 3 based on Section 2.

2 Prelimaries

In this section, we should prove the following:

Lemma 2.1 *Let u be a positive solution of*

$$(-\Delta)^p u = f(u) \quad in \quad R^n \tag{2.3}$$

with $f(t) \geq t^q$ for $t \geq 0$ and $q \geq 1$, then we have

$$(-\Delta)^i u > 0, \quad i = 1, 2, \cdots, p - 1. \tag{2.4}$$

Proof: This is known. We refer the reader to [3] for the detail of the proof. The only thing we need to point out is that at there the proof is given for the case $f(t) = t^q$. However, the argument goes through without any difficulty for present case.

3 Proof of Theorems 1.1 and 1.2

A lemma goes first.

Lemma 3.1 *Suppose $w = w(r) > 0$ satisfies*

$$w''(r) + \frac{n-1}{r}w'(r) + \varphi(r) \leq 0, \quad r > 0$$

with φ nonnegative and nonincreasing, and w' is bounded for r near 0. Then

$$w(r) \geq cr^2\varphi(r), \tag{3.5}$$

where $c = c(n)$ is a constant depending only on the dimension n.

Proof: It is clear that the given differential inequality implies, by multiplying r^{n-1}, the following

$$-w'(r) \geq r^{1-n} \int_0^r s^{n-1}\varphi(s)ds$$

$$\geq r^{1-n} \int_0^{r/2} s^{n-1}\varphi(s)ds$$

$$\geq \frac{r}{2^n n}\varphi(r/2).$$

Integrate this from r to $2r$ then to yield

$$w(r) \geq w(2r) + \frac{1}{2^n n} \int_r^{2r} s\varphi(s/2)ds \geq \frac{r^2}{2^n n}\varphi(r)$$

which is as required.

Proof of Theorem 1.1: Some notations first. Let us denote $(-\Delta)^i u = u_i$ with $i = 0, 1, 2, \cdots, p-1$ where $(-\Delta)^0 u = u$. Thus clearly we have

$$\Delta u_i + u_{i+1} = 0 \quad \text{for } i = 0, 1 \cdots p-2 \tag{3.6}$$

and

$$\Delta u_{p-1} + f(u) = 0. \tag{3.7}$$

Denote the sphere in R^n of radius r and center at 0 by ∂B_r and its included solid ball in R^n by B_r respectively. ω_n is equal to its sphere area in R^n. And also let us denote the integral $\{1/(n\omega_n r^{n-1})\} \int_{\partial B_r} u_i d\sigma$ by $\bar{u}_i$ for $i = 0, 1, 2, \cdots, p-1$. By taking spherical average of above equations (3.6) and (3.7), we get, by venture of Jenson's inequality,

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$$\Delta \bar{u}_i + \bar{u}_{i+1} = 0, \quad \text{for} \quad i = 0, 1, \cdots, p-2 \tag{3.8}$$

and

$$\Delta \bar{u}_{p-1} + \bar{u}_0^q \leq 0. \tag{3.9}$$

Now Lemma 2.1 implies that for each i, u_i is nonnegative and so is its spherical average $\bar{u}_i$. Since u is regular, for each i, $\bar{u}_i'(r)$ is bounded near $r = 0$ and it follows from above equations (3.8) and (3.9) that for each i, $\bar{u}_i(r)$ is nonincreasing function of r. Thus by applying Lemma 3.1, one gets

$$\bar{u}_i(r) \geq cr^2 \bar{u}_{i+1}(r)$$

for $i = 0, 1, 2, \cdots, p-2$ and

$$\bar{u}_{p-1} \geq cr^2 \bar{u}_0^q.$$

If $q = 1$, then claerly we have

$$\bar{u}_0(r) \geq cr^2 \bar{u}_1(r) \geq \cdots \geq c^p r^{2p} \bar{u}_0$$

for all r which implies that $\bar{u}_0 \equiv 0$. This implies that u is identically zero.

If $q > 1$, then we get

$$\bar{u}_0(r) \leq c^{-p/(q-1)} r^{-2p/(q-1)} \tag{3.10}$$

and

$$\bar{u}_{p-1}(r) \leq c^{-(p-1+p/(q-1))} r^{-(2(p-1)+2p/(q-1))}. \tag{3.11}$$

Thus we have, for any $R > 0$,

$$\begin{aligned}
\int_{B_R} u^q dx &\leq \int_{B_R} f(u) dx \\
&\leq CR^{n-2} \int_R^{2R} r^{1-n} \int_{B_r} f(u) dx dr \\
&= -CR^{n-2} \int_R^{2R} r^{1-n} \int_{B_r} \Delta u_{p-1} dx dr \\
&= -CR^{n-2} \int_R^{2R} \bar{u}_{p-1}'(r) dr \\
&\leq CR^{n-2} \bar{u}_{p-1}(R) \\
&\leq C_1 R^{n-2-(2(p-1)+2p/(q-1))}. \tag{3.12}
\end{aligned}$$

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If $q \leq \frac{2p}{n-2p}$, then it is not hard to check that

$$n - 2 - (2(p-1) + \frac{2p}{q-1}) < 0.$$

By letting R goes to infinity in (3.12), we find that

$$\int_{R^n} u^q dx \leq 0.$$

Thus u is identically zero.

Proof of Theorem 1.2: It follows from equation (3.12) that if $n \leq 2p$ and $q > 1$, then clearly

$$n - 2 - (2(p-1) + \frac{2p}{q-1}) < 0.$$

Thus same as above, $u \equiv 0$. For $q = 1$, we can again argue as in the proof of Theorem 1.1 to conclude that u must be zero.

Thus we finish the proof.

1. B. Gidas, N. Ni and L. Nirenberg; Symmetry and related properties via the maximum principle, *Comm. Math. Phys.*, 34(1981), 525-598.
2. C. S. Lin; A classification of solutions of a conformally invariant fourth order equation in R^n, *Comment. Math. Helv.*, 73(1998),206-231.
3. J. Wei and X. Xu; Classification of solutions of higher order conformally invariant equation, *Math. Ann.*, 313(1999), 207-228.
4. X. Xu; Classification of solutions of certain fourth order nonlinear elliptic equation in R^4, preprint, 1996.
5. X. Xu; Uniqueness theorem for the entire positive solutions of biharmonic equations in R^n, *The Royal Society of Edinburgh, Proceedings A(Mathematics)*, to appear.

MULTIPLE PERIODIC SOLUTIONS FOR A CLASS OF RETARDED NONAUTONOMOUS WAVE EQUATIONS

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In this paper, by means of a newly developed Z_p index theory, we discuss a class of retarded nonlinear nonautonomous wave equations and obtained multiplicity results to its periodic solutions.

1   Introduction and Main Results

For nonlinear nonautonomous differential systems, many authors have studied the existence of multiple periodic solutions (cf. [1,2] and [5] etc.). However, it should be noted that almost every results on this topic have been established for normal variable systems, and little has been done for delayed systems. Especially, for nonautonomous partial differential equations with deviating arguments, we have not yet found any progress in this field.

In this paper, we discuss the following equation:

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = g(t, x, u(t - \tau_1, x), u(t - \tau_2, x), \cdots, u(t - \tau_s, x)) \\
u(t, 0) - u(t, \frac{T}{2}) = 0 \qquad t \in (0, T) \\
u(t, x) = u(t + T, x) \qquad (t, x) \in (0, T) \times (0, \frac{T}{2})
\end{cases}
\tag{1.1}
$$

where $T > 0$ is constant, $g(t + \frac{T}{p}, x, u(t - \tau_1, x), u(t - \tau_2, x), \cdots, u(t - \tau_s, x)) = g(t, x, u(t - \tau_1, x), u(t - \tau_2, x), \cdots, u(t - \tau_s, x))$, $\forall (t, x, u) \in (0, T) \times (0, \frac{T}{2}) \times R$, $\tau_1, \tau_2, \cdots, \tau_s \in R$ satisfying $\tau_j = r_j T$, r_j integer, $j = 1, 2, \cdots, s$, and there exists a function $f(t, x, u) \in C^1((0, T) \times (0, \frac{T}{2}) \times R, R)$ such that

$$
f(t, x, u) = g(t, x, u, \cdots, u), \forall (t, x, u) \in (0, T) \times (0, \frac{T}{2}) \times R.
$$

Set $Q = (0, T) \times (0, \frac{T}{2})$, $\square = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$, with domain $\mathcal{D}(\square) = \{u \in C^2(Q) | u$ is $T-$ periodic in t and $u(t, 0) = u(t, \frac{T}{2}) = 0\}$. Let H be the self-adjoint extension of $\square$ on Hilbert space $L^2(Q)$. We know the spectrum of H is

$$
\sigma(H) = \{\lambda | \lambda = (\frac{2\pi}{T})^2 (j^2 - k^2) | (j, k) \in N \times Z\}.
\tag{1.2}
$$

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Now, we assume that

(f_1) There exist constants $\alpha < \beta, \alpha\beta > 0$, such that

$$\alpha \le f'_u(t,x,u) \le \beta, \qquad \forall(t,x,u) \in Q \times R. \tag{1.3}$$

Without loss of generality, assume $\alpha, \beta \notin \sigma(H)$. Set

$$\sigma(H) \cap [\alpha,\beta] = \{\lambda_1 < \lambda_2 < \cdots < \lambda_l\},$$

$$B[\alpha,\beta] = \{k \in Z | \exists j \in N \text{ such that } (\frac{2\pi}{T})^2(j^2 - k^2) \in [\alpha,\beta]\}. \tag{1.4}$$

Assume $0 \notin B[\alpha,\beta]$, then $B[\alpha,\beta]$ is a finite set. We denote it by

$$B[\alpha,\beta] = \{-k_a < -k_{a-1} < \cdots < -k_1 < k_1 < \cdots < k_a\}. \tag{1.4}'$$

For any $r, 1 \le r \le l$, we write d_r as the multiplicity of $\lambda_r \in \sigma(H) \cap [\alpha,\beta]$. Let

$$n = \sum_{r=1}^{l} d_r. \tag{1.5}$$

Denote by $< k, l >$ and $[k, l]$ the greatest common divisor and smallest common multiple of positive integer k, l, respectively.

Now, our main results are the following theorem.

Theorem 1.1 Assume that $f(t,x,u)$ satisfies (f_1) and (f_2)

$$[k_1, k_2, \cdots, k_a]^{\frac{n}{2}} \ne 0 (\text{mod } p). \tag{1.6}$$

(f_3) There exist $\varepsilon > 0, \rho > 0$ and $b \in Z$ such that

$$\lambda_b + \varepsilon \le \frac{f(t,x,u)}{u} \le \lambda_{b+1} - \varepsilon, \forall(t,x) \in Q, |u| \ge \rho. \tag{1.7}$$

(f_4) $f(t,x,0) = 0$ and there exists $\lambda_c \in \sigma(H)$ satisfying $\lambda_c \le \lambda_b$ (or $\lambda_c \ge \lambda_{b+1}$) such that $f'_u(t,x,0) \le \lambda_c - \varepsilon$ (or $f'_u(t,x,0) \ge \lambda_c + \varepsilon$), $\forall(t,x) \in Q$.

Then, equation (1.1) has at least $\sum_{r=c}^{b} d_r$ (or $\sum_{r=b}^{c} d_r$) geometrically distinct periodic solutions.

2 Preliminaries

For a given positive integer $p, p > 1$, we need a Z_p index theory defined in [6].

Let Y be a Banach space and μ be a linear isometric action of Z_p on Y. A subset A of Y is called μ-invariant if $\mu(A) \subset A$. A continuous map $\varphi : A \to Y$ is called μ-equivariant if $\varphi(\mu y) = \mu\varphi(y), \forall y \in A$.

Set

$$\Sigma = \{A \subset Y \text{ is closed and } \mu\text{-invariant}\}, \tag{2.1}$$

$$W = \{z \in C | \arg z = \frac{2\pi r}{p}, r = 0, 1, \cdots, p-1\}. \tag{2.2}$$

According to [6], $\forall A \in \Sigma$, a Z_p geometrical index $i_m(A)$ can be defined as follows

$$i_m(A) = \min\{k \in N | \text{there exists a } (\mu, E_m)^k\text{-type map } \varphi : A \to W^k \backslash \{\theta\}\}. \tag{2.3}$$

If no such map exists, we define $i_m(A) = +\infty$ and set $i_m(\emptyset) = 0$, where the definition of $(\mu, E_m)^k$-type map can be found in [6].

Lemma 2.1 Let $J \in C^1(Y, R)$ be a μ-invariant functional satisfying the (P. S.) condition. For integer j define

$$c_j = \inf_{\substack{i_m(A) \geq j \\ A \in \Sigma}} \sup_{u \in A} J(u). \tag{2.4}$$

Assume $-\infty < c_j < +\infty$, then c_j is a critical value of J. And if $c = c_j = c_{j+1} = \cdots = c_{j+l-1}$, then $i_m(K_c) \geq l$, where $K_c = \{u \in X | J(u) = c, \theta \in \partial_Y J(u)\}$.

3 The Proof of the Main Results

In this section, we fix m, n as follows

$$\begin{aligned} m &= [k_1, k_2, \cdots, k_a] \\ n &= \sum_{r=1}^{l} d_r \end{aligned} \tag{3.1}$$

where $k_r \in B[\alpha, \beta], r = 1, 2, \cdots, a$, $d_r, r = 1, 2, \cdots, l$ is the same as those introduced in Section 1.

Proof of Theorem 1.1 For simplicity, we set $T = 2\pi$. Assume that (f_1)-(f_4) hold. For each $r, r = 1, 2, \cdots, l$, we denoted by G_r the subspace of $L^2(Q)$ which is spanned by the following set

$$\{\sin(j_r x)e^{i(k_r t)}, \sin(j_r x)e^{-i(k_r t)} | (j_r, k_r) \in N \times Z \text{ such that } \lambda_r = (j_r^2 - k_r^2)\},$$

i.e. G_r is the eigenspace corresponding to the eigenvalue λ_r.

Set

$$L_n = G_1 \oplus G_2 \oplus \cdots \oplus G_l. \tag{3.2}$$

We define a Z_p action μ on $L^2(Q)$ as follows

$$\mu(u(t,x)) = u(t + \frac{2\pi}{p}, x).$$

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On the subspace G_r,

$$\mu(\sin(j_r x)e^{i(k_r t)}) = e^{i\frac{2\pi k_r}{p}}\sin(j_r x)e^{i(k_r t)},$$

$$\mu(\sin(j_r x)e^{-i(k_r t)}) = e^{-i\frac{2\pi k_r}{p}}\sin(j_r x)e^{-i(k_r t)},$$

i.e., $\mu(u(t,x)) = e^{i\frac{2\pi k_r}{p}}u(t,x)$ or $\mu(u(t,x)) = e^{-i\frac{2\pi k_r}{p}}u(t,x)$.

We can easily check that $\dim L_n = n$, where n is even, defined by equation (3.1). So, when identifying L_n with $C^{\frac{n}{2}}$, we can apply our Z_p index theory to a corresponding functional on L_n.

By similar discussions as in [1,2] etc, we have:

Lemma 3.1 Under assumptions (f_1)-(f_4), there exists a μ-invariant functional $J \in C^2(L_n, R)$ satisfying

(i) The critical points of J are the solutions of equation (1.1).

(ii) $J(\theta) = 0$, $J'(\theta) = \theta$ and J satisfies the (P.S.) condition.

(iii) There exists a $\delta > 0$, such that

$$\begin{aligned}\frac{1}{2} < (H - \lambda_{b+1} + \frac{\varepsilon}{2})u, u > -\delta \leq J(u) \\ \leq \frac{1}{2} < (H - \lambda_b - \frac{\varepsilon}{2})u, u > +\delta, \forall u \in L_n\end{aligned} \qquad (3.3)$$

(iv) If $f_u'(t,x,0) \leq \lambda_c - \varepsilon$ (or $f_u'(t,x,0) \geq \lambda_c + \varepsilon$), then

$$J(u) \geq \frac{1}{2} < (H - \lambda_c + \varepsilon)u, u > +O(\|u\|)^2 \text{ as } u \to \theta, \qquad (3.4)$$

$$(\text{or } J(u) \leq \frac{1}{2} < (H - \lambda_c - \varepsilon)u, u > +O(\|u\|)^2)) \text{ as } u \to \theta. \qquad (3.4)'$$

This conclusion reduces the problem to finding the critical points of functional J.

For each $j \in N$, define

$$c_j = \inf_{\substack{i_m(A)\geq j \\ A\in\Sigma}} \sup_{u\in A}[-J(u)].$$

Obviously,

$$-\infty \leq c_1 \leq c_2 \leq \cdots \leq +\infty$$

Set

$$G_c = \text{span}\{\sin jx e^{ikt}|j^2 - k^2 \in \sigma(H) \cap [\lambda_c, \beta]\}.$$

From (iv) of Lemma 3.1, for $u \in G_c, u \to 0$,

$$-J(u) \leq -\frac{1}{2} < (H - \lambda_c + \varepsilon)u, u > -O(\|u\|)^2 \leq -\frac{\varepsilon}{2} < u, u > -O(\|u\|)^2$$

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When we take $\varepsilon > 0$ small enough, set

$$S_\varepsilon = \{u \in G| \ \|u\| = \varepsilon\}.$$

Then,

$$-J(u) < 0, \forall u \in S_\varepsilon.$$

By the conclusion of [6], we know that

$$i_m(S_\varepsilon) = \sum_{r=c}^{l} d_r = \dim(G_c).$$

From the definition of c_j, we have

$$-\infty \le c_1 \le c_2 \le \cdots \le c_{\dim(G_c)} \le 0.$$

To complete the proof of Theorem 1.1, we also need the following lemma.

Lemma 3.2 Assume that $J \in C^1(L_n, R)$ is an μ- invariant functional satisfying the (P.S) condition and $J(\theta) = 0$. Then J possesses at least $n_2 - n_1$ critical points under the following assumptions:

(i) There exists a μ-invariant subspace $Y_1 \subset L_n$ with dimension n_1 and a constant γ such that $J(u) \ge \gamma, \forall u \in Y_1^\perp$, where n_1 is even.

(ii) There exists a μ-invariant subset $A \subset L_n$ satisfying $i_m(A) \ge n_2 > n_1$ such that $\sup_{u \in A} J(u) < 0$.

The proof of this lemma is similar to Theorem 2.3 in [5] with slight modification and Lemma 3.2 is a Z_p version of the result concerning even functionals in [4].

We now apply Lemma 3.1 to functional $-J$.

Set

$$G_{b+1} = \text{span}\{\sin jx e^{ikt}|j^2 - k^2 \in \sigma(H) \cap [\lambda_{b+1}, \beta]\}.$$

By Lemma 3.1 (iv), $\forall u \in G_{b+1}^\perp$, we have

$$-J(u) \ge -\frac{1}{2} < (H - \lambda_b - \frac{\varepsilon}{2})u, u > -\delta \ge \frac{\varepsilon}{4} < u, u > -\delta \ge -\delta.$$

Note that $\dim(G_{b+1}) = \sum_{r=b+1}^{l} d_r$, By Lemma 3.2, there exist at least $\dim(G_c) - \dim(G_{b+1}) = \sum_{r=c}^{b} d_r$ critical points for $-J$. And this complete the proof of Theorem 1.1.

Finally, we give a example to illustrate the applications of our results. Consider the following equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \frac{1}{32}[11u(t-2\pi, x) - 8\sin u(t-4\pi, x)][17 + \sin 5t\varphi(x)], \quad (3.5)$$

where $\varphi(x) \in C(R, R)$ is any given bounded function satisfying $|\varphi(x)| \le 1$.

We can easily check that equation (3.5) satisfies all the assumptions of Theorem 1.1. According to the expression of Theorem 1.1, $b = 2, c = 1$ and $d_1 = d_2 = 2$. By Theorem 1.1, there exist at least 4 periodic solutions with period 2π to equation (3.5).

References

1. H. Amann & E. Zehnder, Houston J. Math. **7**, 147-174(1981).
2. K. C. Chang, Comm. Pure & Applied Math. **34**, 693-712 (1981).
3. K. C. Chang, Critical Point Theory and its Applications, Science and Tech. Press, Shanghai, China, 1986.
4. P. H. Rabinowitz, Nonlinear Analysis (T. M. A.), 161-177(1978).
5. Z. Q. Wang, Acta Math. Sinica ,New Series **5**, 197-213(1989).
6. Z. M. Guo, Y. T. Xu & Z. P. Lin, A new Z_p index theory and its applications to functional differential equations (to appear).

EXISTENCE OF STABLE SUBHARMONIC SOLUTIONS FOR REACTION-DIFFUSION EQUATIONS

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The semilinear parabolic problem

$$(P) \quad \begin{cases} u_t = \Delta u + f(u, t), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

is studied, where $u = u(x, t) \in R$, Ω is a smooth bounded domain in R^N, and $f : R^2 \to R$ is a smooth function that is periodic in t with period $\tau > 0$. A solution of (P) is said to be subharmonic if it is periodic in t with the minimal period $k\tau$ for some integer $k > 1$. It is shown that for any $N \ge 2$ and $k \ge 2$ there exist Ω and f such that (P) has a linearly stable subharmonic solution with the minimal period $k\tau$.

1 Introduction

This article is based on a joint work with Peter Poláčik[17]. We consider the parabolic problem

$$(P) \quad \begin{aligned} u_t &= \Delta u + f(u,t), & x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, & x \in \partial\Omega, \end{aligned}$$

where $u = u(x,t) \in R$, Ω is a smooth bounded domain in R^N and $f : R^2 \to R$ is a smooth function that is periodic in t with period $\tau > 0$. We say that a solution $\phi(x,t)$ is subharmonic if it is periodic in t with the minimal period $k\tau$ for some integer $k > 1$. The subharmonic solution $\phi(x,t)$ is said to be linearly stable if the period map of the linearized problem

$$(L) \quad \begin{aligned} v_t &= \Delta v + f_u(\phi(x,t),t)v, & x \in \Omega, \\ \frac{\partial v}{\partial \nu} &= 0, & x \in \partial\Omega, \end{aligned}$$

has all eigenvalues inside the unit circle in the complex plane.

Let us take

$$X = C(\overline{\Omega})$$

as the state space for (P). For any $u_0 \in X$, let $u(\cdot, t; u_0)$ denote the solution of (P). If $t \mapsto \|u(\cdot, t)\|_X$ is bounded, then this solution exists globally in time and approaches, as $t \to \infty$, its ω-limit set $\omega(u_0)$. In general, the structure of $\omega(u_0)$ is not known and can probably be rather complicated, however, a description is available for typical trajectories. More specifically, there is a residual set $G \subset X$ with the following property: if $u_0 \in G$ and $u(\cdot, t; u_0)$ is bounded then there is an integer $k \geq 1$ and a $k\tau$-periodic solution ϕ of (P) such that

$$\|u(\cdot, t; u_0) - \phi(\cdot, t)\|_X \to 0 \quad \text{as } t \to \infty$$

and ϕ is at least linearly neutrally stable, that is, the spectrum of the period map of the linearized problem (L) is contained inside or on the unit circle. See [15,?] for more general results. See also [19] for a sharper abstract result and [11] for an additional information on periods of stable subharmonic solutions; for a general background on monotone dynamical systems see the monographs [10] and references therein.

Thus a typical bounded trajectory of (P) is asymptotically periodic, and the minimal asymptotic period can be larger than τ if (P) has a stable subharmonic solution. In [18,5], Takáč, and Dancer and Hess independently proved that linearly stable subharmonic solutions do occur for some reaction-diffusion

equations, provided the nonlinearity is allowed to depend on x explicitly: $f = f(u, x, t)$.

In this article, we consider the spatial homogeneous problem (P). In this situation, the problem of existence of stable subharmonic solutions becomes more difficult and more interesting for, first, one has less freedom in varying the nonlinearity around and, second, the effect of the domain shape becomes important. In fact, on some domains there are no stable subharmonic solutions, no matter how the nonlinearity is chosen.

When Ω is convex, by a result of Hess [9], any periodic solution of (P) which is at least linearly neutrally stable is spatially homogeneous (similar results on autonomous equations appeared earlier in [2, 13]). Now, for spatially homogeneous solutions, (P) is simplified to

$$u_t = f(u, t).$$

An elementary inspection shows that there are no subharmonic solutions of this ODE, thus any stable periodic solution of the original PDE is necessarily τ-periodic. Thus there is an interesting interplay between spatial and temporal behavior of solutions of (P). In particular, stable oscillations of higher period must always be accompanied by nontrivial spatial patterns.

We remark that if $N = 1$ then no subharmonic solutions, stable or not, can occur (see [3,7]).

Our main result shows that spatially homogeneous equations with linearly stable subharmonic solutions do exist on some domains.

Theorem 1.1 *For any integers $N \geq 2$ and $k \geq 2$ there exist a domain $\Omega \subset R^N$ with smooth boundary and a smooth function $f = f(u, t)$, τ-periodic in t, such that (P) has a linearly stable subharmonic solution of minimal period $k\tau$.*

Our strategy to prove Theorem 1.1 is as follows. First, we construct a stable subharmonic solution for a one dimensional equation with variable diffusion coefficient,

$$v_t = \frac{1}{d(\theta)}(d(\theta)v_\theta)_\theta + f(v, t), \quad \theta \in S^1. \tag{1.1}$$

The construction involves an analysis of solutions with two sharp transition layers, that is, solutions that stay close to ± 1 everywhere except for two small space intervals, depending on t, in which transition layers occur. The motion of the layers is governed by ordinary differential equations on S^1. Finding subharmonic solutions of these ODEs (Section 2) and using the monotonicity method along the lines of [6], we obtain a stable subharmonic solution of (1.1) (see Section 3).

Next we consider the higher dimensional problem (P) on a thin domain around S^1. More specifically, we assume that the domain $\Omega = \Omega(\mu)$ is a

tubular neighborhood of S^1 in R^N that varies with a small parameter μ and and shrinks to S^1 when $\mu \to 0$. Under certain conditions, one can show that (1.1) serves as a "limit equation" for this family of problems (see[8] for a general discussion on thin domain problems and their limit equations). We show that the stable subharmonic solution of (1.1) is approximated by a subharmonic solution of the equation on $\Omega(\mu)$, for any sufficiently small $\mu > 0$ (see Section 4). Here we use a comparison method in a way similar to that in[20]. This gives a subharmonic solution that is at least neutrally linearly stable.

Finally, we perturb the nonlinearity so that the subharmonic solution becomes linearly stable (Section 5).

2 ODE

In this section, we study ordinary differential equations that will later be used to describe the motion of transition layers of a solution to a reaction-diffusion problem.

Consider the equations

$$\frac{d}{dt}p(t) = g(p(t)) + a(t), \tag{2.2}$$

$$\frac{d}{dt}q(t) = g(q(t)) - a(t), \tag{2.3}$$

where $g(\theta)$ is smooth and 2π-periodic in θ and $a(t)$ is smooth and τ-periodic in t. We impose the following condition on g:

$$\int_0^{2\pi} g(\theta)d\theta = 0. \tag{2.4}$$

We first prove the following result.

Proposition 2.1 *For any integer $k \geq 2$, there exist $g(\theta)$ and $a(t)$ such that (2.2) and (2.3) have linearly stable solutions satisfying $p(t + k\tau) \equiv p(t) + 2\pi$, $q(t + k\tau) \equiv q(t) + 2\pi$ and $q(t) < p(t) < q(t) + 2\pi$ for all t. Moreover, these solutions are such that $k\tau$ is the minimal period of $p(t)$, $q(t)$ mod 2π.*

Note that the linear stability of $p(t)$ requires that the solutions of

$$\frac{d\overline{p}}{dt} = g'(p(t))\overline{p}$$

decay to 0 exponentially as $t \to \infty$. This is equivalent to

$$\int_0^{k\tau} g'(p(t))dt < 0.$$

A similar remark applies to $q(t)$.

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3 One-dimensional problem

Let S^1 be the unit circle parameterized by θ, and consider the equation

$$v_t = \frac{1}{d(\theta)}(d(\theta)v_\theta)_\theta + f(v,t), \quad \theta \in S^1. \tag{3.5}$$

We assume that $d(\theta)$ is smooth and 2π-periodic in θ, and f is smooth in (v,t) and τ-periodic in t. In this section, we show the existence of a stable subharmonic solution of (3.5). As usual, we identify $\theta, \theta \pm 2\pi, \theta \pm 4\pi, \cdots$.

Let $\alpha \in (-1,1)$ and β be parameters, and consider the autonomous reaction-diffusion equation

$$v_t = v_{xx} + \hat{f}(v;\alpha) + \beta, \quad x \in R, \tag{3.6}$$

where $\hat{f}(v;\alpha) := (v - \alpha)(1 - v^2)$. For each α, if $|\beta|$ is small, the function $\hat{f}(v;\alpha) + \beta$ has exactly three zeros, say $b_1(\alpha,\beta) < b_0(\alpha,\beta) < b_2(\alpha,\beta)$. It is known that (3.6) has a traveling front solution $v = \Phi(z;\alpha,\beta)$, $z = x - ct$, which satisfies

$$\begin{aligned} &\Phi_{zz} + c\Phi_z + \hat{f}(\Phi;\alpha) + \beta = 0, \\ &\Phi(-\infty;\alpha,\beta) = b_2(\alpha,\beta), \quad \Phi(+\infty;\alpha,\beta) = b_1(\alpha,\beta). \end{aligned} \tag{3.7}$$

We impose an additional condition $\Phi(0;\alpha,\beta) = 0$ in order to fix the phase of the traveling solution. We note that by $\hat{f}_v(b_2,\alpha) < 0$ and $\hat{f}_v(b_1,\alpha) < 0$, the convergence of Φ to $b_1(\alpha,\beta)$ and $b_2(\alpha,\beta)$ must be exponential.

The following result is well-known (see, e.g.,[7]).

Lemma 3.1 *If $\hat{f}(v;\alpha) + \beta$ has three zeros, then (3.7) has a unique solution $(\Phi,c) = (\Phi(z;\alpha,\beta), c(\alpha,\beta))$ that depends on (α,β) smoothly. Moreover, $c(0,0) = 0$ and $\dfrac{\partial c}{\partial \alpha}(0,0) > 0$.*

Let $g(\theta)$ and $a(t)$ be as in Proposition 2.1, and $\varepsilon > 0$ be a small parameter. By Lemma 3.1 (ii), we can find τ-periodic function $\hat{\alpha}(t) = O(\varepsilon)$ such that

$$c(\hat{\alpha}(t), 0) = \varepsilon a(t). \tag{3.8}$$

Now we define $f(v,t)$ by

$$f(v,t) := \varepsilon^{-2}\hat{f}(v;\hat{\alpha}(t)),$$

and $d(\theta)$ by

$$d(\theta) := \exp(-\int g(\theta)d\theta),$$

or equivalently,

$$g(\theta) := -d'(\theta)/d(\theta).$$

We note that $f(v,t)$ is τ-periodic in t and $d(\theta)$ is 2π-periodic in θ by (2.4).

We have the following result concerning the existence of a stable $k\tau$-periodic solution of (3.5).

Proposition 3.1 *Let $d(\theta)$ and $f(v,t)$ be given as above. If $\varepsilon > 0$ is sufficiently small, then (3.5) has a stable solution that is periodic in t with the minimal period $k\tau$.*

Let us briefly explain the mechanism for the existence of a stable subharmonic solution. Suppose that initial data has two transition layers at $\theta = p(0)$ and $q(0)$, and is close to -1 for $\theta \in (q(0), p(0))$ and close to $+1$ for $\theta \in (p(0), q(0))$. Since the nonlinearity f given as above is strongly bistable, the solution of (3.5) is mostly close to the stable state $+1$ or -1, and there appear very thin transition layers between these two stable states. Therefore, roughly speaking, the dynamics of solutions of (3.5) can be reduced to that for the layers which depends on the nonlinearity, spatial inhomogeneity, and interaction between two layers. We can show by a formal approximation (see Section 3 of.[6]) that the asymmetry of f and the spatial homogeneity yield the driving force $\pm a(t)$ and $g(\theta)$, respectively, but the interaction between two layers is very small and negligible. Thus the positions $\theta = p(t)$ and $\theta = q(t)$ of the layers are described by (2.2) and (2.3), respectively. Then, by Proposition 2.1, we see that the layers rotate around S^1 with period $k\tau$, and the linear stability of $p(t)$ and $q(t)$ together with the smoothing effect by diffusion ensures the existence of a stable $k\tau$-periodic solution of (3.5).

We can verify the above intuitive consideration in a mathematically rigorous manner by a comparison method.

4 Thin domain

In this section, we consider the existence of a stable subharmonic solution of the equation

$$
\begin{aligned}
u_t &= \Delta u + f(u,t), & x \in \Omega, \\
\frac{\partial u}{\partial \nu} &= 0, & x \in \partial\Omega,
\end{aligned}
\tag{4.9}
$$

where Ω is a bounded thin domain in R^N ($N \geq 2$) and f, d are as in Proposition 3.1. We define the domain Ω as follows. Let

$$\xi(\theta) = (r\cos\theta, r\sin\theta, 0, \cdots, 0)^t \in R^N,$$

and let S^1 be a unit circle embedded in R^N given by

$$S^1 = \{\xi(\theta) \; ; \; \theta \in [0, 2\pi)\}.$$

We note that $\eta(\theta) := (d/d\theta)\xi(\theta)$ is a unit tangent vector of S^1. Let $D(\theta, \mu)$ be an $(N-1)$-dimensional disc given by

$$D(\theta, \mu) = \{x \in R^N \; ; \; (x - \xi(\theta)) \cdot \eta(\theta) = 0 \text{ and } |x - \xi(\theta)| < \mu d(\theta)^{1/(N-1)}\},$$

where $\mu > 0$ is a small parameter and $d(\theta)$ is as in Proposition 3.1. Then we define $\Omega = \Omega(\mu)$ by

$$\Omega(\mu) := \{x \in R^N \; ; \; x \in D(\theta, \mu), \; \theta \in [0, 2\pi)\}.$$

We have the following result.

Proposition 4.1 *Let $f(u, t)$ and Ω be given as above. If $\mu > 0$ is sufficiently small, (4.9) has a stable solution that is periodic in t with the minimal period $k\tau$.*

Our proof of this proposition is based on the comparison method. We can construct a super-solution of (4.9) by modifying the method of [20].

5 Linear stability

By the comparison method we cannot show the linear stability of subharmonic solutions. In this section, we refine Propositions 3.1 and 4.1 by perturbing the nonlinearity.

Let us first recall that the linear stability of a $k\tau$-periodic solution u means that the period-$k\tau$ map of the linear variational equation

$$(\text{L}) \quad \begin{aligned} v_t &= \Delta v + f_u(u, t)v, & x \in \Omega, \\ \frac{\partial v}{\partial \nu} &= 0, & x \in \partial\Omega, \end{aligned}$$

has all eigenvalues inside the unit circle. By the Krein-Rutman theorem, this is true if and only if the principal eigenvalue λ of the period map, which is a positive algebraically simple eigenvalue, satisfies $\lambda < 1$.

Consider the following problem

$$\begin{aligned} u_t &= \Delta u + f(u, t) + \varepsilon h(t), & x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, & x \in \partial\Omega. \end{aligned} \tag{5.10}$$

where ε is a small parameter, $h(t)$ is a smooth function that is positive and τ-periodic and Ω, f are as in Section 4. Specifically, the following properties of f and Ω are assumed:

Both f and Ω are smooth and f is τ-periodic in t and real analytic in u. Problem (4.9) has a $k\tau$-periodic supersolution u^+ and a $k\tau$-periodic subsolution u^- such that

$$u^-(x,t) < u^+(x,t) \quad (x \in \overline{\Omega},\ t \in [0, k\tau],)$$

and if $t \mapsto u(\cdot, t)$ is $k\tau$-periodic function satisfying

$$u^-(x,t) < u(x,t) < u^+(x,t) \quad (x \in \overline{\Omega},\ t \in [0, k\tau]) \tag{5.11}$$

then $k\tau$ is its minimal period.

If ε is sufficiently small, then u^- and u^+ are a subsolution and supersolution of (5.10) as well. For such ε we denote by u^ε the $k\tau$-periodic solution of (5.10) satisfying (5.11) that is maximal in the sense that any other $k\tau$-periodic solution $\widetilde{u}$ of (5.10) that satisfies (5.11) also satisfies

$$\widetilde{u}(x,t) < u^\varepsilon(x,t) \quad (x \in \overline{\Omega},\ t \in [0, k\tau]).$$

See[10] for the proof of the maximal solution. This solution is stable from above, in a appropriate sense, and, in particular, it is either linearly stable of linearly neutrally stable:

$$\lambda^\varepsilon \leq 1.$$

Here λ^ε is the principal eigenvalue of the period-$k\tau$ map of the linearization

$$\begin{aligned}
v_t &= \Delta v + f_u(u^\varepsilon, t)v, & x \in \Omega, \\
\frac{\partial v}{\partial \nu} &= 0, & x \in \partial\Omega.
\end{aligned} \tag{5.12}$$

Our aim is to prove that for some ε one has $\lambda^\varepsilon \neq 1$, which implies that u^ε is linearly stable. There is nothing to prove if $\lambda^0 \neq 1$, so we assume below that

$$\lambda^0 = 1.$$

The remaining part of the proof of the linear stability is carried out in the following lemmas.

Lemma 5.1 *There is an interval I containing 0 such that the function*

$$\varepsilon \mapsto u^\varepsilon \in C^{2,1}(\overline{\Omega} \times [0, k\tau])$$

is continuous on I and of class C^1 on $I \setminus \{0\}$.

Lemma 5.2 *Let I be as in Lemma 5.1. Then for any $\varepsilon \in I \setminus \{0\}$ one has $\lambda^\varepsilon \neq 1$, hence u^ε is linearly stable.*

For details, see[17].

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References

1. P. Brunovský, P. Poláčik and B. Sandstede, Convergence in general periodic parabolic equations in one space dimension, *Nonlinear Analysis TMA* **18** (1992), 209–215.

2. R. C. Casten and C. J. Holland, Instability results for reaction diffusion equations with Neumann boundary conditions, *J. Differential Equations* **27** (1978), 266–273.

3. X.-Y. Chen and H. Matano, Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations, *J. Differential Equations* **78** (1989), 160–190.

4. X.-Y. Chen and P. Poláčik, Asymptotic periodicity of positive solutions of reaction diffusion equations on a ball, *J. Reine Angew. Math.* **472** (1996), 17–51.

5. E. N. Dancer and P. Hess, Stable subharmonic solutions in periodic reaction-diffusion equations, *J. Differential Equations* **108** (1994), 190–200.

6. M. Ei, S.-I. Iida and E. Yanagida, Dynamics of interfaces in a scalar parabolic equation with variable diffusion, *Japan J. Indust. Appl. Math.* **14** (1997), 1–23.

7. P. C. Fife and J. B. McLeod, The approach of solutions of nonlinear diffusion equations to traveling front solutions, *Arch. Ration. Mech. Anal.* **65** (1977), 335–361.

8. J. K. Hale and G. Raugel, Reaction-diffusion equations on thin domains, *J. Math. Pures Appl.* **71** (1992), 33–95.

9. P. Hess, Spatial homogeneity of stable solutions of some periodic-parabolic problems with Neumann boundary conditions, *J. Differential Equations* **68** (1987), 320–331.

10. P. Hess, *Periodic-parabolic boundary value problems and positivity*, Longman Scientific & Technical, Harlow, 1991.

11. P. Hess and P. Poláčik, Boundedness of prime periods of stable cycles and convergence to fixed points in discrete monotone dynamical systems, *SIAM J. Math. Anal.* **24** (1993), 1312–1330.

12. A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Berlin, 1995.

13. H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, *Publ. Res. Inst. Math. Sci.* **15** (1979), 401–458.

14. X. Mora, Semilinear parabolic problems define semiflows on C^k spaces, *Trans. Amer. Math. Soc.* **278** (1983), 21–55.

15. P. Poláčik and I. Tereščák, Convergence to cycles as a typical asymptotic

behavior in smooth strongly monotone discrete-time dynamical systems, *Arch. Rational Mech. Anal.* **116** (1992), 339–360.

16. P. Poláčik and I. Tereščák, Exponential separation and invariant bundles for maps in ordered Banach spaces with applications to parabolic equations, *J. Dynamics Differential Equations* **5** (1993), 279–303.

17. P. Poláčik and E. Yanagida, Existence of stable subharmonic solutions for reaction-diffusion equations, J. Diff. Eqs., to appear.

18. P. Takáč, A construction of stable subharmonic orbits in monotone time-periodic dynamical systems, *Monatsh. Math.* **115** (1993), 215–244.

19. I. Tereščák, Dynamics of C^1 smooth strongly monotone discrete-time dynamical systems, *preprint*.

20. E. Yanagida, Existence of stable stationary solutions of scalar reaction-diffusion equations in thin tubular domains, *Applicable Anal.* **36** (1990), 171–188.

A NOTE ON THE LIFESPAN OF SMOOTH SOLUTIONS TO THE THREE DIMENSIONAL COMPRESSIBLE ISENTROPIC EULER EQUATIONS

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In this note we consider the three-dimensional isentropic Euler equations

$$\partial_t + \nabla \cdot \rho u = 0,$$
$$\rho(\partial_t u + (u \cdot \nabla)u) + \nabla \rho^\gamma = 0$$

with the initial values which are a small perturbation of a constant state

$$\rho(0, x) = \bar{\rho} + \epsilon \rho_0^\epsilon, \qquad u(0, x) = \epsilon u_o^\epsilon,$$

where $\epsilon > 0$ is a small parameter, $\bar{\rho} > 0$ is a constant, and prove the classical solutions to the above Cauchy problem satisfies

$$e^{\frac{c}{\epsilon}} \leq T(\epsilon) \leq e^{\frac{C}{\epsilon}},$$

where c and C are positive constants independent of ϵ, $T(\epsilon)$ denotes the lifespan of the classical solutions.

1 Introduction

In this note we will obtain the sharp result for the lifespan of classical solutions to the three-dimensional compressible isentropic Euler equations

$$\partial_t + \nabla \cdot \rho u = 0, \tag{1.1}$$

$$\rho(\partial_t u + (u \cdot \nabla)u) + \nabla \rho^\gamma = 0 \tag{1.2}$$

with initial data of the form

$$\rho(0, x) = \bar{\rho} + \epsilon \rho_0^\epsilon, \qquad u(0, x) = \epsilon u_0^\epsilon, \tag{1.3}$$

representing a perturbation of order ϵ from the constant background state $(\rho, u) = (\bar{\rho}, 0)$. For cosmetic reasons, we will omit the dependence of the solution on the parameter ϵ, here, $\bar{\rho}$ is a positive constant, $\gamma \geq 1$, and the function ρ_0^ϵ and u_0^ϵ are uniformly bounded in the Schwartz space $S(R^3)$, for all $\epsilon \geq 0$. If u_0^ϵ is irrotational, then we will show that the lifespan $T(\epsilon)$ of the classical solutions to (1.1)-(1.3) satisfies

$$e^{\frac{c}{\epsilon}} \leq T(\epsilon) \leq e^{\frac{C}{\epsilon}} \tag{1.4}$$

for small ϵ, where c and C are positive constants independent of ϵ, extending the general results [4]

$$\frac{c}{\epsilon} \leq T(\epsilon), \tag{1.5}$$

which holds quite generally for symmetric hyperbolic systems in any number of space dimensions.

It was also shown by Sideris [2, 3] that the lifespan satisfies

$$e^{\frac{c}{\epsilon}} \leq T(\epsilon) \leq e^{\frac{C}{\epsilon^2}} \tag{1.6}$$

under mild conditions on the initial data ρ_0^ϵ, u_0^ϵ in (1.3). Here, we can see the upper and lower bound of lifespan in (1.6) have different order on ϵ, comparing this with (1.4) and obtain the sharp result.

2 Main theorem

Following Sideris [3], we rewrite equations (1.1), (1.2) in symmetric hyperbolic form. If (ρ, u) is a smooth solution of (1.1), (1.2), define the new variables

$$\xi(t, x) = \frac{2}{\gamma - 2} \left[\left(\frac{\rho(\frac{t}{\epsilon}, x)}{\bar{\rho}} \right)^{\frac{\gamma-1}{2}} - 1 \right], \qquad v(t, x) = \frac{u(\frac{t}{\epsilon}, x)}{\bar{c}},$$

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where $\bar{c} = \sqrt{\gamma}\rho^{\frac{\gamma-1}{2}}$. The equation for (ξ, v) now take the form

$$\partial_t \xi + \nabla \cdot v + v \cdot \nabla \xi + \frac{\gamma-1}{2}\xi \nabla \cdot v = 0, \tag{2.1}$$

$$\partial_t v + \nabla \xi + (v \cdot \nabla)v + \frac{\gamma-1}{2}\xi \nabla \xi = 0, \tag{2.2}$$

with initial conditions of order ϵ

$$\xi(0, x) = \epsilon \xi_0^\epsilon(x), \qquad v(0, x) = \epsilon v_0^\epsilon(x), \tag{2.3}$$

where

$$\xi_0^\epsilon(x) = \frac{1}{\epsilon}\frac{2}{\gamma-1}[(1 + \frac{\epsilon \rho_0^\epsilon(x)}{\bar{\rho}})^{\frac{\gamma-1}{2}} - 1], \quad v_0^\epsilon(x) = \frac{u_0^\epsilon(x)}{\bar{c}}.$$

Once again, ξ_0^ϵ and v_0^ϵ are uniformly bounded in $S(R^3)$. Introducing a series partial differential operators as in Klainerman [1], Sideris [4] has obtained the following result by using generalized Sobolev inequality.

Theorem 1 *There exists a positive constant c such that if v_0 is irrotational, the initial value problem (2.1)-(2.3) has a unique C^1 solutions on $[0, T(\epsilon)) \times R^3$, where the lifespan satisfies the lower bound*

$$T(\epsilon) \geq e^{\frac{c}{\epsilon}}. \tag{2.4}$$

Now, we state our result.

Theorem 2 *Suppose that the initial data $(\epsilon \xi_0, \epsilon v_0)$ is support in $\{|x| \leq R\}$ and satisfies $\xi_0 > 0$, $x \cdot v_0(x) > 0$ on the same annulus $\{R_1 < |x| < R\}$. Then there is fixed constant C such that the C^1 local solution (ξ, v) cannot be extended to the region*

$$\{(t, x) \in R^4 : \qquad R_1 + t < |x| < R + t, \qquad 0 \leq t \leq T(\epsilon)\}$$

for $T(\epsilon) > e^{\frac{C}{\epsilon}}$.

Combining Theorem 1 and Theorem 2, we end the proof of (1.4).

The proof of Theorem 2 follows directly from the following Lemma 1 and Lemma 2. Now set

$$F(t) = \int_0^t \int_{R_1+t}^{R+t} r^{-1} \int_{|x|>r} \frac{(|x| - r)^2 \xi(x)}{|x|} dx dr d\tau.$$

Then for $F(t)$, we have from Sideris [2]

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Lemma 1 $F(t)$ *is a* C^2 *function with* $F(0) = 0$ *and* $F'(0) = 0$ *and also satisfies*

$$F''(t) \geq C\epsilon(t + R)^{-1}; \qquad 0 \leq t \leq T$$

and

$$F''(t) \geq C[(R + t)^3 \log(\frac{t + R}{R})]^{-1} F^2(t); \qquad 0 < R_2 \leq t < T$$

where C*, and* R_2 *are positive constants.*

To finish the proof of Theorem 2, we only need to prove

Lemma 2 $H(t)$ *is a* C^2 *function with* $H(0) = 0$ *and* $H'(0) = 0$ *and also satisfies*

$$H''(t) \geq cA(T + R)^{-1}; \qquad 0 \leq t \leq T$$

and

$$H''(t) \geq c[(R + t)^3 \log(\frac{t + R}{R})]^{-1} H^2(t); \qquad 0 < R_2 \leq t < T,$$

where c*,* A*,* R_2 *are positive constants. Then for sufficiently small* A*, we have* $T < \infty$*. More precisely, we have*

$$T \leq e^{\frac{c}{A}} \tag{2.5}$$

for some positive constant c *independent of* A*.*

Proof The proof of this lemma one can see Yang [5].

Reference

1. Klainerman S., Remarks onthe global Sobolev inequalities in the Minkowski space R^n, Comm. Pure Appl. Math., 38, 631-641(1985)
2. Sideris T. C., Formation of singularities in the three dimensional compressible fluids, Comm. Math. Phys., 101, 475-485(1985).
3. Sideris T. C., THe lifespan of $3D$ compressible flow, Seminaire Equations aux Dérivées Partielles, 1991-1992.
4. John F., Nonlinear wave equations, formation of singularties, AMS, 1990.
5. Yang H., On the lifespan of smooth solutions to the three dimensional isentropic flow, J. Fudan Univ., 37(1999) no.5.

GLOBAL STABILITY OF THE SECOND ORDER EQUATION
$$\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0$$

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This paper investigates the qualitative behavior of the equation $\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0$. The new sufficient conditions for globally asymptotically stability are established. It improved and generalized the results of Antosiewicz, Opial and Jiang's. In particular, in this paper, we also give necessary and sufficient conditions on the global stability for the system $\ddot{x} + [f_1(x) + f_2(x)\dot{x}]\dot{x} + g(x) = 0$.

1 Introduction

This paper is devoted to the investigation of the global stability of the second order equation $\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0$ or its equivalent system

$$\dot{x} = y \qquad \dot{y} = -f(x,y)y - g(x) \tag{1}$$

which has been widely studied on global stability. See [1]−[6] and cited therein. Antosiewicz[1] considered the boundedness of all solution of the following system $\ddot{x} + [f_1(x) + f_2(x)\dot{x}]\dot{x} + g(x) = 0$, which is equivalent to

$$\dot{x} = y \qquad \dot{y} = -[f_1(x) + f_2(x)y]y - g(x), \tag{2}$$

under the assumption

$$(A_1) \quad f(x,y) \geq 0 \quad \text{for} \quad x^2 + y^2 \geq 0.$$

Opial[2] generalized the results in [1] and proved that the origin of (1) is globally stable if there exists a continuous function $f_2(x)$ on $\mathbf{R}$ such that (A_2) and suitable additional conditions hold, where

$$(A_2) \quad f(x,y) - f_2(x)y > 0 \quad \text{for} \quad x^2 + y^2 > 0.$$

The results in [1,2] have been generalized by [3] et.al. . In [3], Jiang studied the qualitative behavior such as the global stability for (1) under the more general assumption:

(H_1) there are two continuous functions $f_1(x)$ and $f_2(x)$ on R such that $f_1(x) = inf\{f(x,y) - f_2(x)y : y \in R\}$

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However, in [1] − [6], it plays important role in the following assumptions:

(A_3) $F(G^{-1}(-z)) \leq F(G(t)^{-1}))$ for $0 < z < min(G(-\infty), G(+\infty))$

(A_4) $F(G^{-1}(-z)) \neq F(G^{-1}(z))$ for $0 < z << 1$,

where $a(x) = exp(\int_0^x f_2(s)ds), F(x) = \int_0^x a(s)f_1(s)ds$ and $G^{-1}(z)$ is inverse function of $z = G(x)sgnx = \int_0^x a^2(s)g(s)ds \cdot sgnx$.

Our aim in this paper is to obtain some sufficient conditions for the origin of (1) and necessary and sufficient conditions for the origin of (2) to be globally stable respectively. Our results are different from those of [1] − [6] in some sense. In particular, all our results also allow to avoid the classical assumptions $(A_1)-(A_4)$.

2   Main Results

In this section, we first introduce basic notations and hypotheses, then state our main results for global stability. Throughout this paper, we assume that

(H) The origin is global stable.

(H_0) $f(x,y)$ and $g(x)$ are locally Lipschitz continuous with $xg(x) > 0$ for $x \neq 0$.

From any point $p \in R^2$, we denote by $O^+(P)(O^-(p))$ the positive (negative) semiorbit of (1) , (2) or system

$$(E)   \dot{x} = u - F(x)   \dot{u} = -a^2(x)g(x),$$

where $a(x) = exp(\int_0^x f_2(s)ds), F(x) = \int_0^x a(s)f(s)ds$.

For the sake of convenience, let

$$C^+ = \{(x,u) \mid x \geq 0, u = F(x)\}   C^- = \{(x,u) \mid x \leq 0, u = F(x)\}$$

$$D_1 = \{(x,u) \mid x \geq 0, u > F(x)\}   D_2 = \{(x,u) \mid x > 0, u < F(x)\}$$

$$D_3 = \{(x,u) \mid x \leq 0, u < F(x)\}   D_4 = \{(x,u) \mid x < 0, u > F(x)\}$$

Now, we introduce some definitions which concepts are stated in [4].

Definition 2.1 System (E) has property (X^+) in right half–plane (resp. left half–plane) if, for every point $p \in D_1(resp., D_3)$, the positive semitrajectory $O^+(p)$ crosses the curve $C^+(C^-)$.

Definition 2.2 System (E) has property $(Z_1^+)(resp.(Z_3^+))$ if there exists a point $p \in C^+(C^-)$ such that the positive semitrajectory $O^+(p)$ approaches the origin through only the first (third) quadrant.

definition 2.3 System (E) has property $(Z_2^-)(resp.(Z_4^-))$ if these exists a $p \in C^-(C^+)$ such that the negative semitrajectory $O^-(p)$ approaches the origin through the second (forth) quadrant.

We are in the position to state our main results for system (1) or (2).

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Theorem 17 *Assume that system (2) satisfies the following conditions:*

$(H_1)_1$ $f_1(x), f_2(x)$ *and* $g(x)$ *are continuous functions;*

(H_2) *there exists function* $\psi(x) \in C^0$ *together with constants* $b \geq 0$ *and* $c \in R$ *such that* $\psi(b) + c > 0$ *and* $\psi(x) < F(x)$, $\int_b^x fraca^2(s)g(s)F(s) - \psi(s)ds < \psi(x) + c$ *for* $x > b$;

(H_3) *Let* $S = \{x \mid x < 0, F(x) > 0\}$, *if* $S \neq \phi$, *let* $\alpha = -supS$, *and otherwise let* $\alpha = +\infty$. $F(x) \leq 0$ *when* $-\alpha \leq x \leq 0$; *moreover, there exists a non-negative constant* β *such that* $G(\beta) \leq G(-\alpha)$ *and* $F(x) > 0$ *when* $0 < x < \beta$;

(H_4) $\frac{c^2}{2} + G(b) \leq G(\beta)$ *where* $G(x) = \int_0^x a^2(x)g(s)ds$,
then (H) *holds if and only if* (Y_0) $xg(x) > 0$ *for* $x \neq 0$ *and*
(Y_1) *System* (E) *has property* (X^+) *in the right and left-plane.*

Furthermore, we have

Theorem 18 *Suppose that assumptions* $(H_0) - (H_4)$ *and* (Y_1) *are satisfied, then system (1) has property* (H).

Our next theorem is on global stable for system (2) under assumption (A_3). The proof of the case is similar to Theorem 3.1 in[4].

Theorem 19 *Assume that hypotheses* $(H_1)_1$ *and* (A_3) *hold, then system (2) has property* (H) *if and only if* $(Y_0), (Y_1)$ *and* (Y_2) *system* (E) *has neither property* (Z_2^-) *nor property* (Z_4^-); (Y_3) *there has sequence* $\{z_n\}$ *with* $z_n \to 0^+$ *such that* $F(G^{-1}(-z_n)) < F(G^{-1}(z_n))$.

From Theorem 2.3, we may prove that the following theorem is true by the method of theorem 6.4 in[3].

Theorem 20 *Suppose that system (1) satisfied the assumptions* (H_0), (H_1), (A_3) *and* $(Y_1) - (Y_3)$, *then property* (H) *holds.*

The proof of Theorem 2.1 consists of four main steps based of that the qualitative behaviour of (2) is just as the same as that of system (E).

(a) According to the proof of theorem 1 in[5] and definition 2.1. the necessity may be obtained.

(b) Sufficiency. First, we shall proved that system (2) has no periodic solution and no homoclinic orbit under $(H_1)_1$ and $(H_2) - (H_4)$.

(c) All point in R^2 can be divided into two classes: $S_1 = \{p \in R^2 \mid$ there is $t_0 > 0$ such that $O^+(p) \in D_2 \bigcup D_4$ for $t > t_0\}$ and $S_2 = \{p \in R^2 \mid O^+(p)$ spirals around the origin$\}$. Moreover, $R^2 = S_1 \bigcup S_2$. From this, we may prove that all positive semiorbits converge to the origin.

(d) By (H_3), it can be proved that the origin is (locally) stable. The proof of Theorem 2.2 is based on the comparison approach. The method was in theorem 6.4 in[3].

Acknowledgements

References

1. H. A. Antosiewicz, *J. London Math. Soc.*, **30**, 64(1955).
2. Z. Opial, *Ann. Polon. Math.*, **8**, 65(1960).
3. J. F. Jiang, *Math. Proc. Camb. Phil. Soc.*, **122**, 122(1997).
4. J. Sugie, Chen D. and H. Matsunaga, *J. Math. Anal. Appl.*, **219**, 140(1998).
5. C. Qian, *Nonl. Anal.*, **22**, 823(1994).
6. Jiang Jifa, *Nonl. Anal.* , **28**, 855(1997).
7. J. Sugie, T. Yoneyama, *Math. Proc. Camb. Phil. Soc.*, **113**, 413(1993).

LIMIT CYCLES OF KUKLES SYSTEM WITH TWO FINE FOCI

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The so-called Kukles system is a cubic system in the form of

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = Q(x,y), \tag{0.1}$$

where $Q(x,y)$ is a polynomial of degree 3. It is well-known that the first investigating the centre problem of such a system is due to I.S.Kukles[1]. Nowadays the main problem of Kukles system is to study the number of its limit cycles. The modern approach of investigating this problem is based on bifurcation theory—closed orbit bifurcation, homoclinic bifurcation and Hopf bifurcation. Perturbating the system such that the stability of the critical point (generally focus) is changed, then by Hopf bifurcation theory, there will generate at least one small amplitude limit cycle surrounding that critical point. So the problem of generating limit cycles can be reduced to the so-called cyclicity problem of fine focus. And at last it has to depend on the calculation of the focal quantities.

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There were many papers concerned with the calculation of focal quantities to special Kukles system [2-7]. In this paper we shall study the limit cycles to Kukles system with two fine foci.

Without loss of generality, we may assume that system (1) has two fine foci at $O(0,0)$ and $B(a,0)$, where $a > 1$, and $A(1,0)$ is a saddle. Then system (1) can be reduced to

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x + (1 + \frac{1}{a})x^2 - \frac{1}{a}x^3 - aa_5xy + a_3y^2 + a_5x^2y + a_6xy^2 + a_7y^3. \end{cases} \tag{0.2}$$

We first consider the case of the two fine foci having the same order. By means of computer, we can find that the highest order of these two fine foci are two, and system (2) can be reduced into

$$\begin{cases} \frac{dx}{dt} = y = P(x,y), \\ \frac{dy}{dt} = -x + (1 + \frac{1}{a})x^2 - \frac{1}{a}x^3 + \frac{2-a+a_6-aa_6}{a-1}y^2 + a_6xy^2 - aa_5xy + \\ \qquad a_5x^2y - \frac{aa_5[1-a_6(a-1)]}{3(a-1)}y^3 = Q(x,y). \end{cases} \tag{0.3}$$

where $\tilde{h} = \text{arctg}\frac{Q}{P}$, so system (3) forms a rotated vector field in some certain region of the (x,y)-plane. The curve $C : (x - \frac{a}{2})^2 + \frac{a[1-a_6(a-1)]}{3(a-1)}y^2 = \frac{a^2}{4}$ depends on parameter $a_6 : C$ is elliptic, if $a_6 < \frac{1}{a-1}, C$ consists of two straight lines $x = 0$ and $x = a$, if $a_6 = \frac{1}{a-1}, C$ is hyperbolic, if $a_6 > \frac{1}{a-1}$. System (3) can be regarded as a perturbed system of system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x + (1 + \frac{1}{a})x^2 - \frac{1}{a}x^3 + \frac{2-a+a_6-aa_6}{a-1}y^2 + a_6xy^2, \end{cases} \tag{0.4}$$

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by perturbing parameter a_5. The limit cycles of system (3) are closely relative to the global phase portraits of system (4).

By analyzing the topological structure of the trajectories to system (4) in the neighbourhoods of critical points at infinity, the global phase portraits of system (4) can be shown in Fig. 1-3.

1^0 $a_6 < 0$.

Since

$$\frac{dy}{dx}\bigg|_{(3)} - \frac{dy}{dx}\bigg|_{(4)} = a_5 \left\lceil (x - \frac{a}{2})^2 + \frac{a[1 - (a-1)a_6]}{3(a-1)}y^2 - \frac{a^2}{4} \right\rceil ,$$

from the phase portrait of system (4) (Fig.1), we can choose a closed orbit L of system (4) which contains the ellipse $(x - \frac{a}{2})^2 + \frac{a[1-(a-1)a_6]}{3(a-1)}y^2 = \frac{a^2}{4}$. Thus when $a_5 > 0$ along the curve L we have $\frac{dy}{dx}|_{(3)} > \frac{dy}{dx}|_{(4)}$, and the two fine foci are stable, so there is at least one limit cycle in int L (Fig.4(a)). Similarly, when $a_5 < 0$ the assertion is still valid, it merely has a different stability of the limit cycle (Fig.4(b)). By perturbing the system twice, the changes of the stability of the foci yields 4 limit cycles. Hence system (1) has at least 5 limit cycles, among which one simultaneously surrounds critical points $O(0,0), A(1,0)$ and $B(a,0)$; two surround critical point $O(0,0)$ and the other two surround critical point $B(a,0)$.

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2^0 $a_6 = 0$

In case of Fig.2(2), the generation of limit cycles is the same as that of Fig.1. In cases of Fig.2(1) and (3), by the continuity of the vector field with respect to a, we can also choose a closed orbit of system (4) which contains the ellipse. And then the process of generating 5 limit cycles is the same as that in case of $a_6 < 0$.

3^0 $0 < a_6 < \frac{1}{a-1}$.

It is sufficient to consider the case of $a = 2$. In this case, after translating the origin to critical point $A(1,0)$, system (4) becomes

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = \frac{x}{2}(1-x)(1+x) + a_6 x y^2, \end{cases} \tag{0.5}$$

which is an integrable system, and its integral curve which passes through critical points $(\pm 1, \frac{1}{\sqrt{2a_6}}, 0)$ and $(\pm 1, -\frac{1}{\sqrt{2a_6}}, 0)$ is $y^2 = -\frac{1}{2a_6}(1-x^2) + \frac{1}{a_6^2}$. It is not difficult to prove that this integral curve lies in the exterior of ellipse $x^2 + \frac{2}{3}(1 - a_6)y^2 = 1$, and in those regions $\frac{\partial \theta}{\partial a_5} > 0(< 0)$ as $a_5 < 0(< 0)$. So from Fig.3 we know that the relative position of separatrices to system (3) which leave critical point $(\pm 1, \frac{1}{\sqrt{2a_6}}, 0)$ and enter critical point $(\pm 1, -\frac{1}{\sqrt{2a_6}}, 0)$ are shown as Fig.5. In case of $a_5 > 0(< 0)$, the fine foci are stable (unstable), thus system (3) has at least one limit cycle surrounding simultaneously three critical points, and by further perturbation, system (1) has at least 5 limit cycles.

Sum up the above analysis we have

Theorem 1 *Kukles system (1) has at least 5 limit cycles if its two fine foci have the same order.*

Now we consider the case when the two fine foci of system (2) have different orders. By means of computer, we can find that if the highest order of the focal quantity to one of the critical points, say $O(0,0)$ is one, then the highest order of the focal quantity to the other, say $B(a,0)$, is five. Hence by Hopf bifurcation we have.

Theorem 2 *Kukles system (1) has at least 6 limit cycles if its two fine foci have different orders. In this case, $O(0,0)$ is at least surrounded by one limit cycles; and $B(a,0)$ is at least surrounded by 5 limit cycles.*

At last we conclude our discussion with the following conjecture.

Conjecture Kukles system with two fine foci could at least have 7 limit cycles if the highest order of the two fine foci are different.

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Acknowledgments

References

1. I.S.Kukles, *Dokl. Akad. Nauk USSR* **42**, 160-163 (1994).
2. X.F.Jin & D. M. Wang, *Bull. of The London Math. Soc.*, **94**, 1-4 (1990).
3. C.J.Christopher & N.G.Lloyd, *Bull. of The London Math. Soc.*, **94**, 5-22 (1990).
4. N.G.Lloyd & J.M.Pearson, *Lect. Notes in Math.* **1445**, 230-242 (1989).
5. N.G.Lloyd & J.M.Pearson, *J. Computational and Appl. Math.*, **2**, 323-336 (1992).
6. Y. S. Huang & R. Liu, *Ann. of Diff. Eqs.* **3**, 249-259(1999).

SOME MATHEMATICAL PROBLEMS IN SYNCHRONIZATION THEORY

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This paper review and propose some mathematical problem in synchronization theory of dynamical systems.

1 Introduction

Synchronization is a ubiquitous phenomenon in natural systems and has received intensive investigations for many decades due to its fundamental importance for many physical, biological and technical systems. Roughly speaking, if two systems have something in common, then a synchronization-like phenomenon will occur between them when they interact or couple each other in a way. The so-called frequency locking between two clocks is a well known synchronization-like phenomenon.

In the last decades, much attention has been paid to synchronization and a considerable research [1-10] has been carried out on the mechanism of this universal phenomenon which makes its show in various forms in nature. Much

work has been done, for example, on drive-response type synchronization of chaos in dynamical systems in view of its potential applications to secure communication and synchronization in two systems coupled via feedback from the physical standpoint. From biological point of view, the synchronization of biological oscillators has been studied by several authors.

Without knowing the rapid development of the theory of synchronization in dynamical systems, some mathematicians[14,15] also consider a similar problem concerning synchronization between two dynamical systems in terms of mutual stability, but in a unpractical context.

Although synchronization theory is of considerable significance to many areas of science and engineering and it is a very active branch of nonlinear science currently, few people from mathematical community paid their attention to this problem to the knowledge of the author. However, a lot of mathematical problems (which are usually not easy to be coped with) arose with the progress of the synchronization theory, and much activity of mathematicians is apparently in need.

2 Review of Some Notions of Synchronization

Consider the coupled systems

$$\dot{x} = f(x, y) \tag{2.1}$$

and

$$\dot{y} = g(x, y), \tag{2.2}$$

where $x \in R^n$, $y \in R^n$.

In the literature, the so-called (identical) synchronization [1,2,6,7] between two dynamical systems is as simple as follows:

Definition 1 Let $x(t, x_0)$ and $y(t, y_0)$ be the solutions to (1) and (2), respectively. If $\lim_{t \to \infty} ||x(t, x_0) - y(t, y_0)|| = 0$, provided $||x_0 - y_0|| < \delta$ for some δ. Then (1) and (2) are said to be in synchronization. **Remark** It is easy to see from the above definition that a necessary condition for identical synchronization is that $f(x, x) = g(x, x)$ on the omega limit set of $x(t, x_0)$.

Because the boundedness of trajectories is neglected in the above definition, and it is physically meaningful to have the hypothesis of the boundedness, it is natural to adopt the concept of *synchronizor* which is formulated as follows.

Definition 2 [5,10] Suppose that A is an attractor of coupled systems (1) and (2), A is said to be a synchronizor, if A satisfies

$$A \subset \{(x, y) : x = y, (x, y) \in R^n \times R^n\}.$$

In case that A is asymptotically stable in Liapunov's sense, the two systems (1) and (2) are said to be in synchronization.

Remark It is easy to see from the above definition that a necessary condition for identical synchronization is that $f(x,x) = g(x,x)$ on A.

Definition 3 Suppose $y \in R^m$ in system (2), and let P_A be the image of the natural projection of A onto R^n. If the above attractor can be expressed as

$$A = \{(x,y) : y = H(x), x \in P_A, H \text{ is a transformation from } P_A \text{ to } R^m\},$$

then A is called a generalized synchronizor. As in the previous definition, if A is asymptotically stable in Liapunov's sense, then the systems (1) and (2) are said to be in generalized synchronization. If the transformation H is continuous, then A is called a continuous synchronizor.

Remark It is easy to see from the above definition that a necessary condition for generalized synchronization is that $f(x,H(x)) = g(x,H(x))$ on A.

3 Some Problems

The fundamental problem is:

Problem A Under what conditions two coupled dynamical systems are in synchronization or in generalized synchronization?

A related problem is

Problem B If the (generalized) synchronizor exists, then whether it is smooth? If not, what conditions can guarantee the smoothness of synchronizor?

A basic treatment of the problem A is by virtue of Liapunov stability theory. For details, the readers are referred to [2,7] and references therein.

Recently the problem A has received a lot of investigations in the case of drive and response type dynamical systems.

Consider the following coupled systems:

$$\dot{x} = f(x) \quad \text{(drive system)} \tag{3.3}$$

and

$$\dot{y} = g(x,y), \quad \text{(response system)} \tag{3.4}$$

where $x \in R^n$, $y \in R^m$.

Under the hypothesis that (3) has an attractor, the existence of generalized synchronizor or the exhibition of generalized synchronization between (3) and (4) has been discussed by many authors from areas other than mathematical sciences, for example see [1,2,6]. And some *theorems* were incorrectly proved. In view of such situation, [10] presented a mathematical theory for the problem in question.

Theorem 4 *Consider the following coupled systems*

$$\dot{x} = f(x) \tag{3.5}$$

$$\dot{y} = By + h(x) \tag{3.6}$$

Suppose that the system $\dot{x} = f(x)$ possesses an attractor A, which is a bounded set, and the system $\dot{y} = By$ is asymptotically stable, where B is a constant matrix, and $h(x)$ is continuous. Then (6) has a continuous synchronizor S_A.

For more general nonlinear case, the following statement is obtained.

Consider the following unidirectional coupled systems

$$\dot{x} = f(x) \tag{3.7}$$

$$\dot{y} = g(y, h(x)) \tag{3.8}$$

where $x \in R^n$, $y \in R^m$, and $f \in c^1[R^d, R^d]$, $g \in c^1[R^d \times R^k, R^d]$, $h \in c^1[R^d, R^k]$.

Theorem 5 *Suppose that the drive system (7) and response system (8) satisfy the following conditions;*
a) (7) has an attractor A;
b) $\max_{x \in A} \|g(0, h(x))\| \leq C < \infty$;
c) $\lambda(y, h(x)) \leq -k < 0$.
Here $\lambda(y, h(x))$ denotes the maximal eigenvalue of the matrix

$$M(y, h(x)) = \frac{1}{2}[\partial_y g(y, h(x) + (\partial_y g(y, h(x)))T].$$

Then systems (7) and (8) possess a synchronizor S_A.

An open question. In [10] the author failed to prove the continuity of S_A, thus *whether S_A is continuous is still an open question.*

Generally one is interested in the following problem.

A conjecture *If $g(0, h(x))$ is bounded on a neighborhood B of a compact attractor of (7) and (8) is asymptotically stable for every solution $x(t, p)$ in B, then (7) and (8) possess a generalized synchronizor.*

An considerably significant aspect of synchronization between two dynamical systems is that the delay of effect is inevitable in couplings. Therefore the following question is more practical.

Consider the coupled systems

$$\dot{x} = f(x, y(t - \tau)) \tag{3.9}$$

and

$$\dot{y} = g(x(t - \lambda), y), \tag{3.10}$$

where $x \in R^n$, $y \in R^m$, τ and λ are time delays, respectively.

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Problem C *Under what conditions (9) and (10) are in identical (generalized) synchronization?*

It is expected that the theory of functional differential equations [11, 12] should be useful in coping with this problem

In this section we just touch upon the synchronization theory for case of continuous dynamical systems. For the difference systems, the readers are referred to [13]

4 Comments on the Current Mutual Stability in terms of Synchronization

Mutual stability between dynamical systems is a useful concept, because it describes a kind of cooperation behavior in dynamical systems. Recently several authors [14, 15] have dwelt upon this topic in context of difference systems. However, as commented in [3], from the scientific point of view, the current treatment [14,15] on this concept neglects the interaction or coupling between the mutual stable systems, and this is very important because it is less meaningful to study the cooperative behavior in dynamical systems without considering their mutual interaction or coupling. Now let us recall the notions of mutual stability.

Consider difference systems

$$x(n+1) = f(x(n)) \tag{4.11}$$

and

$$y(n+1) = g(y(n)) \tag{4.12}$$

Recall that a solution or orbit $x(n, x_0)$ to (1.1) with initial condition x_0 is defined to be $x(n) = f^n(x_0)$, $n = 0, 1, \dots$.

According to [14, 15], a kind of mutual stability is defined as follows:

Definition 6 System (11) and system (12) are said to be mutually stable if for any $\epsilon > 0$, there exists a continuous positive function $\delta = \delta(\epsilon)$ such that any two solutions $x(n, x_0)$ and $y(n, y_0)$ of (11) and (12), respectively, satisfy

$$\|x(n, x_0) - y(n, y_0)\| < \epsilon, \quad \text{for} \quad n \geq 0,$$

Provided $\|x_0 - y_0\| < \delta$.

Definition 7 System (11) and system (12) are said to be mutually attractive if for any two positive numbers ϵ and η. There exists a nonnegative integer N such that any two solutions x and y of (11) and (12), respectively, satisfy

$$\|x(n, x_0) - y(n.y_0)\| < \epsilon, \quad \text{for} \quad n \geq N,$$

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provided $||x_0 - y_0|| < \eta$.

Remarks Definition 7 clearly implies that for the initial conditions satisfying $||x_0 - y_0|| < \eta$ the expression $||x(n, x_0) - y(n.y_0)|| \to 0$ as $n \to \infty$ holds. In fact one can chose a sequence $\epsilon \to 0$ to verify this observation.

Note that the mutual attraction does not imply the asymptotic mutual stability, thus we restate the definition of asymptotically mutual stability of two dynamical systems as follows.

Definition 8 System (11) and system (12) are said to be asymptotically mutually stable, if for any positive number ϵ, There exists a continuous positive function $\delta(\epsilon)$, such that any two solutions $x(n, x_0)$ and $y(n, y_0)$ of (11) and (12), respectively, satisfy

$$||x(n, x_0) - y(n, y_0)|| < \epsilon, \quad \text{for} \quad n \geq 0,$$

and

$$||x(n, x_0) - y(n.y_0)|| \to 0, \quad \text{as} \quad n \to \infty$$

provided $||x_0 - y_0|| < \eta$.

Remark Clearly the so-called mutual stability is just a special case of synchronization in the current literature.

Recently, it has been shown that the mutual stability between two dynamical systems is also less of interests from mathematical point of view.

Theorem 9 [16] *Suppose that system (11) and system (12) are asymptotically mutually stable on a compact invariant set A in terms of Definition 8, then (11) and (12) possess a common asymptotically stable equilibrium point in A with A being its basin of attraction.*

The same is true of the continuous system, i.e., the systems described by autonomous ODE [16].

This theorem assert that the dynamic of two asymptotically mutually stable autonomous dynamical systems is trivial.

References

1. A. Kittel et L., *Physica D*, **112**, 459(1998).
2. L. Kocarev and U. Parlitz, *Phys. Rev. Lett.*, **76**, 1816(1996).
3. Xiao-song Yang, Concepts of synchronization in dynamical systems, *Pyhs. Lett. A.*, 260(1999)340-344.
4. Xiao-song Yang, Synchronization and Synchronizor, submitted to *Phys. Lett. A.*.
5. U. Parlitz et al., *Phys. Rev. Lett.*, **79**, 3158(1997).
6. S. S. Yang and C. K. Duan, *J. Phys. A: Math. Gen.*, **30**, 3273(1997).
7. C. K. Duan and S. S. Yang, *Phys. Lett. A*, **229**, 151(1997).

8. S. S. Yang and C. K. Duan, *Chaos, Solitons and Fractals*, **9**, 1703(1998).

9. X. S. Yang(S.S. Yang), Existence of synchronizor in coupled nonlinear systems, (submitted).

10. D. Y. Xu, *Nonlinear Anal.*, **27**, 413(1996).

11. Xinzhi Liu and Dao-Yi Xu, *J. Math. Anal. Appl.*, **216**, 625(1997).

12. K. M. Cambell and M. E. Davies, *Nonlinearity*, **9**, 801(1996).

13. P.Y.H. Pang and R.P. Agarwal, *Computer, Math. Appl.*, **36**, 423(1998).

14. X. S. Yang, Dynamics of mutual stable autonomous dynamical systems without interactions, (submitted).

THE STRUCTURE OF NON-WANDERING SETS OF INTERVAL MAPS

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The structure of non-wandering sets of continuous maps of an interval has been studied by many authors. In this note by providing several examples we show that some situations in the theory really occur.

1 Introduction

Let X be a topological space and $f : X \longrightarrow X$ be continuous. $x \in X$ is non-wandering if for each neighborhood U of x there is $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. The collection of all non-wandering points of f is denoted by $\Omega(f)$. When X is a closed interval the structure of $\Omega(f)$ is quite well understood. The best way to describe the results is to look at the complement of the closure of the periodic points of f. Let $P(f)$ be the set of periodic points of f, $\omega(x, f)$ be the ω-limit set of x, and $\Lambda(f) = \cup_{x \in X}\omega(x, f)$. For an interval map, i.e. a continuous map from a closed interval I into itself, let C be a connected component of $I \setminus \overline{P(f)}$. We have

(1) $\Omega(f) \setminus \overline{P(f)}$ is countable and no-where dense [3].

(2) Isolated periodic point is also isolated in $\Omega(f)$ [4].

(3) The derived set of $\Omega(f)$ is contained in $\Lambda(f)$ [1].

(4) C contains at most one point from $\Omega(f)$ with infinite orbit [3], and if $x \in C \cap \Lambda(f)$ then $\omega(x,f)$ is a non-trivial minimal set [2]. Moreover, if $x \in C \cap \Omega(f)$ has infinite orbit then there is an end point of C with infinite orbit [3].

(5) There are interval maps f and g [2] such that

(a) $C \cap \Lambda(f) \neq \emptyset$. (b) there is $x \in \Omega(g) \setminus \Lambda(g)$ with infinite orbit.

In this note we will show

Theorem 21 *There are interval maps f, g and h with*

(1) there is a component C of $I \setminus \overline{P(f)}$ such that $\{x_i\} \subset C \cap \Omega(f)$ is infinite and $\lim x_i = x \in C$.

(2) there is a component C of $I \setminus \overline{P(g)}$ such that $\{x_i\} \subset C \cap \Omega(g)$ is infinite, $\lim x_i = x \in \overline{C} \setminus C$ and x has infinite orbit.

(3) there is a component C of $I \setminus \overline{P(h)}$ such that $\{x_i\} \subset C \cap \Omega(f)$ is infinite, $\lim x_i = x \in \overline{C} \setminus C$ and x has finite orbit.

2   Proof of (1) and (2) of Theorem 1.1

In this section we shall show that there are examples which satisfy (1) and (2) of Theorem 1.1. The method we use to construct the example can be explained as follows. First we define a subset $\mathcal{A}$ of $C(I,I)$ (the set of all continuous maps of I with $d(f,g) = \sup_{x \in I} |f(x) - g(x)|$). We then define an operator from $\mathcal{A}$ into itself and show that the operator is a contraction. Finally we prove that the unique fixed point provided by the operator satisfies (1) of Theorem 1.1. Modifying the construction, we get an example which satisfies (2) of Theorem 1.1. We start with the construction of (1) of Theorem 1.1.

Choose real numbers a_i^1, $1 \leq i \leq 10$ such that

$$-1 = a_1^1 < a_2^1 < a_3^1 < -\frac{1}{2} < 0 < a_4^1 < a_5^1 < a_6^1 < a_7^1 < a_8^1 < a_9^1 < a_{10}^1 = 2.$$

Let $\mathcal{A}$ be the set of all continuous maps from $[-1,2]$ to itself with $f(-1) = 2 = f(2)$ and $f([-1,2]) \subset [a_2^1, 2]$.

Let h_1 be an orientation preserving homeomorphism from $[a_1^1, a_{10}^1]$ to $[a_3^1, a_4^1]$ such that $h_1|_{[-\frac{1}{2},0]} = id$, $h_1|_{[a_1^1, -\frac{1}{2}]}$ and $h_1|_{[0, a_{10}^1]}$ are linear. Let h_2 be an orientation preserving linear homeomorphism from $[a_1^1, a_{10}^1]$ to $[a_5^1, a_6^1]$ and $h_3 = h_1 \circ h_2^{-1} : [a_5^1, a_6^1] \longrightarrow [a_3^1, a_4^1]$.

For $f \in \mathcal{A}$ define a continuous map $\tilde{f} \in \mathcal{A}$ such that

1 $\tilde{f}(a_1^1) = a_{10}^1$, $\tilde{f}(a_2^1) = a_7^1$, $\tilde{f}(a_3^1) = a_6^1$, $\tilde{f}(a_5^1) = a_3^1$, $\tilde{f}(a_6^1) = a_4^1$, $\tilde{f}(a_7^1) = a_7^1$, $\tilde{f}(a_8^1) = a_{10}^1$, $\tilde{f}(a_9^1) = a_2^1$, $\tilde{f}(a_{10}^1) = a_{10}^1$;

2 $\tilde{f}|_{[a_i^1, a_{i+1}^1]}$ is linear for $i \neq 3, 5$ and $1 \leq i \leq 9$;

3 $\tilde{f}|_{[a_3^1, a_4^1]} = h_2 \circ f \circ h_1^{-1}$ and $\tilde{f}|_{[a_5^1, a_6^1]} = h_1 \circ h_2^{-1}$.

To see the construction clearly, the reader may draw a picture.

For each $x \in \Omega(f)$ we can associate a subset of I defined by: $P_f(x) = \bigcap_U \overline{(\cup_{n=1}^\infty f^n(U))}$, where U runs over all neighborhood of x. Note that $P_f(x)$ describe the behavior of points near x.

Lemma 10 $\tilde{f}$ *has the following properties:*

A. $\tilde{f}[a_3^1, a_4^1] \subset [a_5^1, a_6^1]$, $\tilde{f}[a_5^1, a_6^1] = [a_3^1, a_4^1]$ and $\tilde{f}^2|_{[a_3^1, a_4^1]} = h_1 \circ f \circ h_1^{-1}$.

B. $a_3^1 \notin \Omega(\tilde{f})$, $a_4^1 \in P(\tilde{f})$ and $\Omega(\tilde{f}) \cap [a_3^1, a_4^1] = h_1(\Omega(f))$.

C. $\Omega(\tilde{f}) \cap [a_1^1, a_3^1) = \{a_2^1\}$ and $\Omega(\tilde{f}) \cap [a_1^1, a_4^1] = \{a_2^1\} \cup h_1(\Omega(f))$.

D. *if* $d(f, g) < \epsilon$, *then* $d(\tilde{f}, \tilde{g}) < \frac{a_6^1 - a_5^1}{3} \epsilon$.

E. $P_{\tilde{f}}(a_2^1) = [a_2^1, a_{10}^1]$.

Proof: **A** follows by the definition of $\tilde{f}$.

As $f([-1, 2]) \subset [a_2^1, 2]$, we know that there is $\epsilon > 0$ such that $\cup_{n=1}^\infty \tilde{f}^n((a_3^1 - \epsilon, a_3^1 + \epsilon)) \subset [a_3^1 + \epsilon, a_6^1]$. This implies $a_3^1 \notin \Omega(\tilde{f})$. It is easy to see that a_4^1 is a periodic point of $\tilde{f}$. We now show that $\Omega(\tilde{f}) \cap [a_3^1, a_4^1] = h_1(\Omega(f))$.

In fact, since $f([a_3^1, a_4^1]) \subset [a_5^1, a_6^1]$ and $f([a_5^1, a_6^1]) \subset [a_3^1, a_4^1]$ we have that $\Omega(\tilde{f}) \cap (a_3^1, a_4^1) = \Omega(\tilde{f}^2) \cap (a_3^1, a_4^1)$. Thus

$$\Omega(\tilde{f}) \cap (a_3^1, a_4^1) = \Omega(h_1 \circ f \circ h_1^{-1}) \cap (a_3^1, a_4^1) = h_1(\Omega(f)).$$

Combining the other results we just proved we finish the proof of **B**.

To prove **C** note that for each $x \in [-1, a_2^1)$ there is $n \in \mathbb{N}$ such that $\tilde{f}^n(x) \in [a_2^1, 2]$, and for each $x \in (a_2^1, a_3^1)$ there is $n \in \mathbb{N}$ such that $\tilde{f}^n(x) \in [a_3^1, a_6^1]$. As $a_3^1 \notin \Omega(\tilde{f})$, we have that $\Omega(\tilde{f}) \cap [a_1^1, a_3^1) = \{a_2^1\}$. By **B**, $\Omega(\tilde{f}) \cap [a_3^1, a_4^1] = h_1(\Omega(f))$, thus we get $\Omega(\tilde{f}) \cap [a_1^1, a_4^1] = \{a_2^1\} \cup h_1(\Omega(f))$. This proves **C**.

It is easy to see that **D** follows from the definition of $\tilde{f}$ and **E** can be checked easily. This ends the proof of Lemma 2.1.

Let $\mathcal{O} : \mathcal{A} \longrightarrow \mathcal{A}$ such that $\mathcal{O}(f) = \tilde{f}$ for each $f \in \mathcal{A}$. Since $\mathcal{A}$ is a Banach space, by Lemma 2.1(D), there is a unique fixed point F for $\mathcal{O}$ and for each

$f \in \mathcal{A}$, $F = \lim \mathcal{O}^n(f)$. For a fixed $f \in \mathcal{A}$ let $f_n = \mathcal{O}^n(f)$ for each $n \in \{0\} \cup \mathbb{N}$ (we set $f_0 = f$).

Let $a_i^n = h_1^{n-1}(a_i^1)$, $1 \leq i \leq 10$, $n \in \mathbb{N}$. Then

$$a_1^1 < a_2^1 < a_3^1 = a_1^2 < a_2^2 < a_3^2 = a_1^3 < \ldots < -\frac{1}{2} < 0 < \ldots < a_4^3 < a_4^2 < a_4^1$$

and $\lim_n a_1^n = -\frac{1}{2}$, $\lim_n a_4^n = 0$. Note that for each $i \in \mathbb{N}$, $h_1^i : [a_1^1, a_{10}^1] \longrightarrow [a_3^i, a_4^i]$.

Lemma 11 *We have the following observation:*

(1) *for each $n \in \mathbb{N}$ and $1 \leq i \leq n$, $f_n^{2^i}|_{[a_3^i, a_4^i]} = h_1^i \circ f_{n-i} \circ h_1^{-i}$.*

(2) *for each $n \in \mathbb{N}$, $\Omega(f_n) \cap [a_1^1, a_3^n] = \{a_2^1, a_2^2 \ldots, a_2^n\}$.*

(3) *for each $n \in \mathbb{N}$, $d(f_{n+1}, f_n) < 3(\frac{a_6^1 - a_5^1}{3})^n$.*

(4) *for each $n \in \mathbb{N}$, $P_{f_n}(a_2^n) = \{f_n^i([a_2^n, a_{10}^n]) : 0 \leq i \leq 2^{n-1} - 1\}$ which is the disjoint union of 2^{n-1} intervals.*

Proof: (1) follows from A of Lemma 2.2.

To see (2) we use induction. First we have $\Omega(f_1) \cap [a_1^1, a_3^1] = \{a_2^1\}$. Assume that $\Omega(f_n) \cap [a_1^1, a_3^n] = \{a_2^1, a_2^2 \ldots, a_2^n\}$. Then

$$
\begin{aligned}
\Omega(f_{n+1}) \cap [a_1^1, a_3^{n+1}] &= \Omega(f_{n+1}) \cap ([a_1^1, a_3^1] \cup [a_3^1, a_3^{n+1}]) \\
&= \{a_2^1\} \cup (\Omega(f_{n+1}) \cap [a_3^1, a_3^{n+1}]) \\
&= \{a_2^1\} \cup (h_1(\Omega(f_n)) \cap h_1[a_1^1, a_3^n]) \\
&= \{a_2^1\} \cup (h_1(\Omega(f_n) \cap h_1[a_1^1, a_3^n])) \\
&= \{a_2^1\} \cup (h_1\{a_2^1 \ldots, a_2^n\}) \\
&= \{a_2^1, \ldots, a_2^{n+1}\}.
\end{aligned}
$$

(3) and (4) can be checked directly. This ends the proof of Lemma 2.2.

According to Lemma 2.2 we have

Theorem 22 *The unique fixed point F of $\mathcal{O}$ has the following properties:*

1. $F|_{[-\frac{1}{2}, 0]}$ *is constant.*

2. $\Omega(F) \cap [a_1^1, 0) = \{a_2^1, a_2^2, \ldots\} \cup \{-\frac{1}{2}\}$ *and* $P(F) \cap [a_1^1, 0) = \emptyset$;

3. $P_F(-\frac{1}{2}) = \cap_{n=1}^{\infty} P_F(a_2^n)$ *which is the union of a Cantor set with countably many non-degenerate connected components.*

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Proof: (1) Let $F_n = \mathcal{O}^n(F) = F$ for each $n \in \mathbb{N}$. Since $F_1|_{[-\frac{1}{2},0]} = h_2 \circ F \circ h_1^{-1}|_{[-\frac{1}{2},0]}$, and since $h_1^{-1}|_{[-\frac{1}{2},0]} = id$ and $F_1 = F$, we have that $F|_{[-\frac{1}{2},0]} = h_2 \circ F|_{[-\frac{1}{2},0]}$. As h_2 has a unique fixed point a, we get that $F([-\frac{1}{2},0]) = \{a\}$.

(2) As $\Omega(F) \cap (-\frac{1}{2},0) = \emptyset$ (by 1), $\Omega(F) \cap [a_1^1, a_3^n] = \{a_2^1, a_2^2 \ldots, a_2^n\}$, for each $n \in \mathbb{N}$ (by (2) of Lemma 2.2), and $\lim_n a_3^n = -\frac{1}{2}$, we have that

$$\Omega(F) \cap [a_1^1, 0) = \bigcup_{n=1}^{\infty} (\Omega(F) \cap [a_1^1, a_3^n]) \cup \{-\frac{1}{2}\} = \{a_2^1, a_2^2, \ldots\} \cup \{-\frac{1}{2}\}.$$

It is easy to see that $P(F) \cap [a_1^1, 0) = \emptyset$.

(3) It is obvious that $P_F(-\frac{1}{2}) = \cap_{n=1}^{\infty} P_F(a_2^n)$. This set contains $[-\frac{1}{2}, 0]$ and is the union of a Cantor set with countably many non-degenerate connected components. This ends the proof of Theorem 2.1.

Proof of (1) and (2) of Theorem 1.1: The F constructed in Theorem 2.1 satisfies (1) of Theorem 1.1. To Prove (2) of Theorem 1.1 we just need to modify the construction. Namely, we take $h_1 : [a_1^1, a_{10}^1] \longrightarrow [a_3^1, a_4^1]$ to be an orientation preserving homeomorphism with $h_1(-\frac{1}{2}) = -\frac{1}{2}$, and $h_1|_{[a_1^1, -\frac{1}{2}]}$ and $h_1|_{[-\frac{1}{2}, a_{10}^1]}$ being linear. Set $a_i^n = h_1^{n-1}(a_i^1)$, $1 \le i \le 10$. Then $\lim_n a_1^n = \lim_n a_4^n = -\frac{1}{2}$. With this modification we get a fixed point of $\mathcal{O}$ which satisfies (2) of Theorem 1.1.

3 Proof of (3) of Theorem 1.1

In this section we give the proof of (3) of Theorem 1.1.

Let b_1^i, $1 \le i \le 8$ with

$$-1 = b_1^1 < b_2^1 < b_3^1 = 0 < b_4^1 = 1 < b_5^1 < b_6^1 < b_7^1 < b_8^1 = 2$$

and $\mathcal{B}$ be the collection of continuous maps from $[-1, 2]$ into itself with $f(-1) = f(2)$ and $f([-1,2]) \subset [b_2^1, b_8^1]$. For $f \in \mathcal{B}$ define $\tilde{f}$ such that

(1) $\tilde{f}(b_1^1) = b_8^1$, $\tilde{f}(b_2^1) = b_5^1$, $\tilde{f}(b_3^1) = b_4^1 = \tilde{f}(b_4^1)$, $\tilde{f}(b_5^1) = b_5^1$, $\tilde{f}(b_6^1) = b_8^1$, $\tilde{f}(b_7^1) = b_2^1$, $\tilde{f}(b_8^1) = b_8^1$.

(2) $\tilde{f}|_{[b_i^1, b_{i+1}^1]}$ is linear for $i \ne 3$ and $1 \le i \le 7$.

(3) $\tilde{f}|_{[b_3^1, b_4^1]} = h_1 \circ f \circ h_1^{-1}$, where $h_1 : [b_1^1, b_8^1] \longrightarrow [b_3^1, b_4^1]$ is an orientation preserving homeomorphism. Note that h_1 has a unique fixed point $\frac{1}{2}$. We have the following observation

Lemma 12 *(a).* $b_3^1 \in \Omega(\tilde{f})$ *iff* $-1 \in \Omega(f)$, b_4^1 *is a fixed point of* $\tilde{f}$ *and* $\Omega(\tilde{f}) \cap [b_3^1, \frac{1}{2}] = h_1(\Omega(f) \cap [b_1^1, \frac{1}{2}])$.

(b). $\Omega(\tilde{f}) \cap [b_1^1, b_3^1) = \{b_2^1\}$ *and* $\Omega(\tilde{f}) \cap [b_1^1, \frac{1}{2}] = \{b_2^1\} \cup h_1(\Omega(f) \cap [b_1^1, \frac{1}{2}])$.

(c). If $d(f,g) < \epsilon$, then $d(\tilde{f}, \tilde{g}) < \frac{1}{3}\epsilon$.

Proof: Similar to the proof of Lemma 2.1.

Let $\mathcal{O} : \mathcal{B} \longrightarrow \mathcal{B}$ with $\mathcal{O}(f) = \tilde{f}$. Then we see that $\mathcal{O}$ has a unique fixed point F. Moreover, for each $f \in \mathcal{B}$, $F = \lim \mathcal{O}^n(f)$. Fixed $f \in \mathcal{B}$, let $f_n = \mathcal{B}^n(f)$ with $f_0 = f$ and $b_i^n = h_1^{n-1}(b_i^1)$, $n \in \mathbb{N}, 1 \leq i \leq 8$. It is clear that

$$\lim b_1^n = \lim b_2^n = \lim b_3^n = \lim b_4^n = \frac{1}{2}.$$

Lemma 13 *1. For each $n \in \mathbb{N}$, $\Omega(f_n) \cap [b_1^1, b_3^n] = \{b_2^1, b_2^2, b_2^3, \ldots\}$.*

2. For each $n \in \mathbb{N}$, $d(f_n, f_{n+1}) < (\frac{1}{3})^n$.

3. b_4^n is a fixed point of f_n.

Finally we have

Theorem 23 *(1). $\Omega(F) \cap [b_1^1, \frac{1}{2}] = \{b_2^1, b_2^2, b_2^3, \ldots\}$ and there is no periodic point of F in $[b_1^1, \frac{1}{2})$.*

(2). $\frac{1}{2}$ is a fixed point of F.

Proof of (3) of Theorem 1.1: The function F we obtained in this section satisfies (3) of Theorem 1.1.

Acknowledgments

Reference

1. L. Block and W.A. Coppel, Dynamics in one dimension, *Lect. Notes in Math.* **1513**, Springer-Verlag, 1991.
2. L. Block and E. Coven, ω-limit sets for maps of the interval, *Ergod. Th. and Dynam. Sys.* **6** (1986), 335-344.
3. J. Xiong, Nonwandering sets of continuous interval self-maps, *Kexue Tongbao*(English ED.) **29** (1984), 1431-1433.
4. J. Xiong, The topological structure of nonwandering sets of self-maps of an interval (chinese), *Acta Math. Sinica* **29** (1986), 691-696.

SOME PROMISING DIRECTIONS OF RESEARCH IN THE FIELD OF PLANAR AUTONOMOUS DIFFERENTIAL SYSTEMS[1]

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Topics and problems on Poincare index theorem as well as distribution of critical points, limit cycles and Hilbert's 16th problem of planar polynomial differential systems are presented.

I. Generalizations and applications of the Poincare index theorem.

1. Poincare index theorem. G: simply-connected plane region with smooth boundary line L. With respect to a vector field F, there are σ (ν) inner (outer) contact points but no critical points of F on L. Then the sum of indices of critical points of F within G is [1]:

$$\sum = \sum_i Ind\ O_i = 1 + \frac{1}{2}(\sigma - \nu) \qquad (0.1)$$

2. Generalizations

a. To a multiply-connected plane region and in general, to any two-dimensional orientable or non-orientable closed surface S with boundaries [2,3,4,5]. We have:

$$\sum = \sum_i Ind\ O_i = \chi(S) + \frac{1}{2}(\sigma - \nu) \qquad (0.2)$$

where $\chi(T_{g,b}) = 2 - 2g - b, \chi(P_{g,b}) = 2 - g - b$ are Euler-Poincare characteristics, b & g are number of boundary circles and genus of S.

Remark: The proof in [5] is more elementary and having less assumptions than those in [2] and [3].

b. To a multiply-connected plane region with elementary critical points and segments of trajectories (finite or at infinity) on its boundaries L_j (piecewise smooth), see [6] & [7].

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c. Problem 1. When there non-elementary critical points of F on the boundaries, how to calculate $\sum$? (See [11], Chap.1,after Fig.1.30).

Problem 2. From (2) and the well-known Gauss-Bonnet formula in differential geometry, we have now:

$$\int_C k_g ds + \int_S K dA + \sum_i (\pi - \alpha_i) = 2\pi \sum_i Ind\ O_i + (\nu - \sigma)\pi.$$

Is there any use of this formula?

3. Applications.

a. Some new theorems on the existence or non-existence of limit cycles see [8,9,10].

b. The estimation of passages passing through a multiply-connected region, see [8,9].

c. The distribution of critical points of n-degree polynomial differential system, see [7].

d. Problem: Find other applications of the generalized Poincare index theorem, especially, for plane autonomous differential systems.

Remark: Most of the above results can be found in my Monograph [11].

II. Distributions of critical points of an n-degree polynomial system (especially, cubic system):

$$\begin{cases} \dot{x} = P_n(x,y) \\ \dot{y} = Q_n(x,y) \end{cases} \tag{0.3}$$

1. Known results.

a. There exists a system (3) with $P_n(x,y) = 0$ a bundle of straight lines having center A, and $Q_n(x,y) = 0$ another bundle of straight lines having center $B \neq A$, such that (3) has n^2 elementary critical points, their distribution has the form:

$$(2n - 1) - (2n - 3) + (2n - 5) - \cdots + (-1)^{n-1} \tag{0.4}$$

Among these critical points

$$(2n - 1) + (2n - 5) + (2n - 9) + \cdots = \frac{n}{2}(n + 1) \tag{0.5}$$

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have index +1,

$$(2n-3)+(2n-7)+(2n-11)+\cdots=\frac{n}{2}(n-1) \qquad (0.6)$$

have index -1, so the difference of numbers of anti-saddles and saddles attains its maximum:

$$\frac{n}{2}(n+1)-\frac{n}{2}(n-1)=n.$$

see [12,13].

b. If the 2n-1 outmost critical points of index +1 form a convex polygon, then among them there is at least one node [7].

c. For cubic system having 6 critical points with index +1 and 3 critical points with index -1, the distributions:

$$5-3+1, \qquad 6-3, \qquad 4-3+2$$

can all be realized by concrete cubic systems, see [14,15]. But the distribution 3-3+3 is proved to be impossible [11, Chap.13]

d. For cubic system having 5 critical points with index +1 and 4 critical points with index -1, the distribution 4-4+1 is realized by

$$\dot{x}=y(x^2-y^2+1), \qquad \dot{y}=x(x^2-2y^2-3).$$

We conjecture that the distribution 5-4 is also possible, but we are still unable to realize it by a concrete cubic system.

e. There exist other kinds of distribution for the cubic system:

$$\begin{cases} \dot{x}=-y+\delta x+lx^2+mxy+ny^2 \\ \dot{y}=x(1+ax+qx^2+by) \end{cases} \qquad (0.7)$$

which has one higher order critical point at infinity, and at most 6 finite elementary critical points , see [16].

2. Problems.

a. Are there any other distribution of critical points for cubic systems having 9 finite critical points? (See [11], Chap. 3, before Theorem 3.18)

b. What is the relation between distributions of critical points and intersection properties of $P_n(x,y)=0$ and $Q_n(x,y)=0$ when $n>2$? For example, in the simplest case: a cubic system. When $n=2$ this is clear.

c. If in 1, b) the 2n-1 outmost critical points of index +1 form a concave polygon, is the assertion there still true?

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d. Find a cubic system (7) such that it has 3 foci , their relative position is Fig.3(3) in the paper [16].

e. How about the property of the 2n-5 anti-saddles (forming a 3rd polygon) in (4), the 2n-9 anti-saddles (forming a fifth polygon) in (4), etc.?

III. The highest order of a fine focus and the distributions of foci as well as their order for n-degree polynomial systems.

As is well-known, these problems have not been solved even for cubic systems. Up to now, there appear papers studying many special cubic systems, we mention here only some remarkable results obtained in recent years.

1. In 1993 V .G.Romanovsky [17] discovered an inductive method for the calculation of the focus quantities of polynomial systems. By using this method Cherkas, Romanovsky and Zoladek solved completely the center conditions for the 8-parameter system [18]:

$$\dot{z} = iz + Az^2 + Dz^3 + Ez^2\bar{z} + Fz\bar{z}^2 \tag{0.8}$$

where $z = x + iy \in C = R^2, A, D, E, F \in C$. There are other papers (by Romanovsky himself or with his colleagues) using this method, too.

2. In a preprint of the Univ. Autonoma de Barcelona Gasull & Torregrosa[19] studied the relation between the maximum number H of small amplitude limit cycles generated by Hopf bifurcation and the maximum number P of the big limit cycles (around the same focus) generated by Poincare bifurcation. They conjectured that P can be greater than H. Among others they studied also the order of fine focus of the cubic system:

$$\begin{cases} \dot{x} = -y + a_2x^2 + a_3x^3 \\ \dot{y} = x + b_2y^2 + b_3y^3 \end{cases} \tag{0.9}$$

They found that the order of O(0,0) is 4, but after perturbation it can only generate 3 limit cycles. The reason is $v_7v_9 \geq 0$. Actually, such phenomenon has already appeared in some quadratic systems.

3. In a recent paper by Y. R. Liu[20] the cubic system:

$$\begin{cases} \dot{x} = (-y + \delta x)(x^2 + y^2) + X_2(x, y) \\ \dot{y} = (x + \delta y)(x^2 + y^2) + Y_2(x, y) \end{cases} \tag{0.10}$$

$(X_2, Y_2$: homogeneous of degree 2) was studied. The line at infinity is a trajectory of (10) passing no critical points. After the transformation $x = \dfrac{\cos\theta}{r}, y = \dfrac{\sin\theta}{r}$, it becomes:

$$\frac{dr}{d\theta} = -r\frac{\delta + r[\cos\theta\, X_2(\cos\theta, \sin\theta) + \sin\theta\, Y_2(\cos\theta, \sin\theta)]}{1 + r[\cos\theta\, Y_2(\cos\theta, \sin\theta) - \sin\theta\, X_2(\cos\theta, \sin\theta)]} \tag{0.11}$$

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and the line at infinity becomes the origin: r=0.

The right hand side of (11) differs from that of the polar coordinates of the qudratic system:

$$\dot{x} = -y + \delta x + X_2(x,y), \dot{y} = x + \delta y + Y_2(x,y)$$

only a minus sign. But after perturbation there can appear 4 limit cycles around the origin r=0 of (11).

4. V. A. Gaiko [21] proved that a quadratic system cannot have a swallow-tail bifurcation surface of multiplicity-four limit cycles. This agree with our conjecture that (4,x) distribution of limit cycles is impossible for any quadratic system.

5. Problems.

a. Prove or disprove the conjecture of Gasull & Torregrosa.

b.Find the order of fine focus and conditions for center of the critical point $O(0,0)$ of system $(7)_{\delta=0}$ in II.

6. Special cubic or quartic systems studied for global analysis, order of fine focus or center conditions appeared in recent years are [22-34], etc...

IV. The number of sudden appearance of semi-stable limit cycles (LC, for short) in a polynomial system containing a varying parameter which rotates the vector field, and its relation with the Hilbert's 16th problem.

1. Proposition A. (to be proved) For the system:

$$\begin{cases} \dot{x} = -y + \delta x + lx^2 + ny^2 - m(1 + ax - y) \\ \dot{y} = x(1 + ax - y) \end{cases} \tag{0.12}$$

as m or δ varies monotonously the sudden appearance of semi-stable LC can happen at most once.

2. Proposition A was stated by me early in 1993 (see [11], Chap.14). The reason for me to state this Proposition is:

When a cubic curve C (it must be a solution curve of a certain quadratic system) moves upward (see Fig.1) or downward (see Fig.2), it can contact a given horizontal line L at most once (e.g., at the point P). Now, if we replace L by the real plane $\mathcal{L}$: $x_2 = y_2 = 0$ in $C^2(x_1 + ix_2, y_1 + iy_2)$, and C by a special solution surface $S \in C^2 \cong R^4$ of the complex quadratic system, which intersects $\mathcal{L}$ at a closed curve Γ_1, it is a LC in $\mathcal{L}$. Certainly, S will move as a parameter in the system varies monotonously. We conjecture that " the sudden appearance of a contact point P in Fig.1 or 2" would correspond to: "the sudden appearance of a semi-stable LC Γ_2 on $\mathcal{L} \cap S$", because P will

then become two points of intersection of C and L, just as the newly appeared semi-stable LC Γ_2 will split into a stable LC Γ_3 and an unstable LC Γ_4.

3. An example of the intersection of a two dimensional surface in $C^2(x_1 + ix_2, y_1 + iy_2) \cong R^4(x_1, x_2, y_1, y_2)$ with the real plane $\mathcal{L}$: $x_2 = y_2 = 0$ as the parameter varies.

Let

$$F(x, y, \epsilon) = (1 + \epsilon)(x^2 + y^2)^2 - 2(1 + \epsilon)(x^2 + y^2) + 1 + \epsilon - \epsilon^3 = 0 \quad (0.13)$$

be the equation of the surface S in R^4, where $x = x_1 + ix_2, y = y_1 + iy_2$, and ϵ is a parameter. Then the equation of $S \cap \mathcal{L}$ in $\mathcal{L}$ is $F(x_1, y_1, \epsilon) = 0$. The discriminant of $F(x_1, y_1, \epsilon) = 0$ as a quadratic equation of $(x_1^2 + y_1^2)$ is:

$$\sharp = (1 + \epsilon)^2 - (1 + \epsilon)(1 + \epsilon - \epsilon^3) = (1 + \epsilon) \epsilon^3.$$

a. When $0 < \epsilon < \epsilon_1 \cong 1.32472, \sharp > 0, S \cap \mathcal{L}$ are two single circles.

b. When $\epsilon = 0, \sharp = 0, S \cap \mathcal{L}$ is a double circle $[(x_1^2 + y_1^2) - 1]^2 = 0$.

c. When $-1 < \epsilon < 0, \sharp < 0, S \cap \mathcal{L} = \emptyset$.

d. When $\epsilon = -1, F(x_1, y_1, -1) = 1, S \cap \mathcal{L} = \emptyset$.

e. When $\epsilon = \epsilon_1, \sharp > 0, S \cap \mathcal{L}$ are a circle $x_1^2 + y_1^2 = 2$ and a point (0.0).

f. When $\epsilon > \epsilon_1, \sharp > 0, S \cap \mathcal{L}$ is a single circle:

$$x_1^2 + y_1^2 = \frac{1 + \epsilon + \sqrt{(1 + \epsilon)\epsilon^3}}{1 + \epsilon} > 2.$$

g. When $\epsilon < -1, S \cap \mathcal{L}$ is a single circle.

Fig.1   Fig.2

4. It is clear that Proposition A is not true for $n \geq 3$ degree polynomial systems as is easily seen from the facts:

a. The number of small-amplitude LC of a cubic system around a focus can be equal to 11 [35], say : $L_1 \supset L_2 \supset \cdots \supset L_{11} \supset O$.

b. By the rotated vector field theory (with parameter α) LC will expand or shrink according to the stability & orientation as the vector field rotates. So, as the field rotates in a suitable direction say, from α_1 to α_2, L_i and $L_{i+1}(i = 1, 3, 5, 7, 9)$ will coincide and then disappear for certain values of α . Now, let α varies from α_2 to α_1, then semi-stable LC will appear many times , as is easily seen.

Therefore, this Proposition must be a special property only for quadratic systems.

5. In a recent paper [36] I proved that as δ increases the quadratic system:

$$\begin{cases} \dot{x} = -y + \delta x + \dfrac{7x^2}{6} + 3xy + \dfrac{3y^2}{4} \\ \dot{y} = x(1 - \dfrac{x}{2} - y) \end{cases} \tag{0.14}$$

can appear a LC Γ_1 around N$(0, \frac{4}{3})$ when δ passes through -4; then a LC Γ_2 around S$_2$(12.9281, -5.4628) when passes through -0.8453; finally, a LC Γ_3 will shrink to O(0,0) when δ passes 0. I call this property : The ergodicity of LC. This work was extended by W. Y. Ye [37] and Zhang Xiang [38] later on.

6. Problems

a. Prove or disprove Proposition A.

b. Try to explain the ergodic phnomenon in the complex space $\mathcal{C}^2$.

V. The maximum number of integral lines of n-degree polynomial systems.

Let $\alpha(n)$ be the maximum number of integral lines of n-degree polynomial systems, $\beta(n)$ be the maximum number of slopes that these integral lines can have (assume $\alpha(n)\&\beta(n)$ are finite). Then:

1. $\alpha(n) \leq 3n - 1$ (H. Zoladek, see [43])

2. $\beta(n) = \alpha(n-1) + 1$ (J.C.Artes & J.Llibre [39])

3. $\alpha(2) = 5 = 3 \cdot 2 - 1$, trivial.

4. $\alpha(3) = 8 = 3 \cdot 3 - 1$, $\alpha(4) = 9 < 3 \cdot 4 - 1$ ([40-42], etc..)

5. $\alpha(5) = 14 = 3 \cdot 5 - 1$ (J.C.Artes,B.Grunbaum & J.Llibre [43])

Problem: Does there exist a general formula for $\alpha(n)$?

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Notice that algebraic solutions of polynomial systems and their relation with integrability as well as the existence of LC is also an interesting topic deserves to be studied (see [11] Chap.17).

VI. The behavior of LC generating from bifurcation method when the parameter becomes larger.

So far as we know, most papers on the bifurcation (both Hopf bif. and Poincare bif.) of polynomial systems only considered one independent parameter, and only the behavior of the perturbed system when this parameter is small. But in some papers by me, D.J.Luo, W.Y.Ye and Zhang Xiang, we have considered two independent parameters; moreover, when these parameters become larger. Nevertheless, in order to get satisfactory results we must ask for help from Proposition A in IV. After all, in order to solve completely the qualitative problem of any polynomial system in which LC can appear, one must consider the problem of "the sudden appearance of semi-stable LC" when the varying parameter becomes larger. So, it seems to me that this problem is crucial in the ultimate solution of the Hilbert's 16th problem.

There are three steps in our approach for quadratic systems:

1. To prove the system:

$$\begin{cases} \dot{x} = -y + lx^2 + ny^2 \\ \dot{y} = x(1 + ax - y) \end{cases} \tag{0.15}$$

(assume $0 < n < 1, a < 0$, without loss of generality) has two foci: O(0,0) (below $1 + ax - y = 0$) and N$(0,\frac{1}{n})$ (above $1 + ax - y = 0$) with different stability when $l \neq \frac{1}{2}$ (by using the Dulac function $(1-y)^{2l-1}$); but O and N are both centers when $l = \frac{1}{2}, (P_x + Q_y \equiv 0)$.

2. To prove the distribution of LC of

$$\begin{cases} \dot{x} = -y + \delta x + lx^2 + ny^2 \\ \dot{y} = x(1 + ax - y) \end{cases} \qquad (0 < n < 1, a < 0) \tag{0.16}$$

are (0,0), (0,1) & (1,0) (see [11,Chap.20], there is still a gap in our proof).

Notice that when δ varies (16) forms half-plane rotated vector fields on each side of the line $1 + ax - y = 0$ with opposite orientation, and we assume unwelcome semi-stable LC will not appear suddenly around O and around N. Actually, if this happens, then in the next step we will meet a contradiction to Proposition A.

3. To get all possible distributions of LC we have [52], in which by using Proposition A in IV W.Y.Ye proved that all possible distributions of LC for quadratic systems are:

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(3,1),(1,3),(3,0),(0,3),(2,1), (1,2),(2,0),(0,2),(1,1),(1,0),(0,1),(0,0).

We mention here also some results obtained by three mathematicians, because the way of their approach is different from that of other mathematicians.

a. In a series of papers by P.G.Zhang [44-50] in 1995-1999 he has succeeded in proving:

If a quadratic system has LC around both two foci, then at least around one focus the LC must be unique.

So, for quadratic systems (2,2) distribution of LC is impossible, this gives a remedy of our imperfect proof in [11,Chap.20].

b. Recently R.E.Kooij & A.Zegeling also got beautiful results in this respect, see [33] & [51].

Problem 1. How to solve the Hilbert's 16th problem for the general cubic systems or for a special class of cubic systems by using a method similar to the above steps for quadratic systems?

Problem 2. What will be the Proposition used therein, similar to Proposition A?

Problem 3. Is it possible to give an example showing that (16) can have two LC around O(0,0)?

VII. Index-inverse systems.

See my recent paper [53].

References

1. P.Hartman,Ordinary Differential Equations,Chap.VII (Birkhauser, Boston, 1964).
2. C.C.Pugh, Topology**7**,217-226(1968).
3. K.R.Mayer,Proc.Amer.Math. Soc.**760-763**,1968(.)
4. Ye Yanqian, Ann.of Diff. Eqs.**6**,241-262(1990).
5. J.Llibre & Y.Q.Ye, *On the dynamic of surface vector fields and homeomorphisms* (Preprint,1992, to appear).
6. Y.Q.Ye & W.Y.Ye, Ann. of Diff. Eqs.**7**,362-383(1991).
7. Y.Q.Ye, Chin. Ann. of Math.**16(B)**,453-458(1995).
8. Y.Q.Ye, Chin.Ann.of Math.**v.13(B)**,257-265(1992).
9. Y.Q.Ye, Chin.Ann. of Math.**15(B)**,489-492(1994).
10. Zhang Xiang, Chin.Ann. of Math.**20(B)**,169-172(1999).
11. Ye Yanqian, *Qualitative Theory of Polynomial Differential Systems, (Shanghai Sci. & Tech. Publ., 1995).*
12. *Ye Weiyin, Ann.of Diff. Eqs.**v.5(3)**,335-340(1989).*

13. Sun Jitao, *J.of Math. & Phys.***15**,*319-322(1995)*.

14. Ye Weiyin, *Ann.of Diff. Eqs.***4(4)**,*503-509(1988)*.

15. R.E.Kooij, *Some properties of cubic systems,* Report 90-62, Delft Univ. of Technology.

16. Y.Q.Ye, *A special cubic system close related to the general quadratic system (*to appear in Ann.of Diff.Eqs., v.15 or 16 (1999 or 2000).

17. V.G.Romanovsky, Diff. Urav.**29**,910-912(1993).

18. L.A.Cherkas,V.G.Romanovsky & H.Zoladek, Diff.Eqs. & Dyn.Syst.**5**,299-302(.)

19. A.Gasull & J.Torregrosa, *Center problem for several differential equations via Cherkas method. (*Preprint of the Univ. Autonoma de Barcelona, Num.6, 1999).

20. Y.R.Liu, *Limit cycle bifurcation of a class of higher order critical point and point at infinity. (*Preprint, 1999).

21. V.A.Gaiko, *On global bifurcation theory of polynomial differential systems and its applications (*to appear).

22. N.G.Lloyd & L.M.Pearson, Publ. Matem.**41**,199-208(1997).

23. A.Gasull & R.Prohens, J.Math.Anal. & Appl.**198**,25-34(1996).

24. Wang Xian, Ann. of Diff. Eqs.**12**,475-482(1996).

25. A.Gasull, A.Guillamon & V.Manosa, J.Math. Anal. & Appl.**211**,190-212(1997).

26. A.Gasull, A.Guillamon & V.Manosa, *Center and isochronicity conditions for systems with homogeneous non-linearities,* Proc.of 2nd Catalan days in appl. math. (to appear).

27. H.Zoladek, J.Diff.Eqs.**137**,94-118(1997).

28. F.Dumortier & C.Z.Li, J.Diff.Eqs**139**,41-59(1997).

29. F.Dumortier,R.Roussarie & C.Rousseau, J.Diff.Eqs.**110**,86-113(1994).

30. C.Christopher, J.Math.Anal. & Appl.**229**,319-329(1999).

31. L.A.Cherkas & S.I.Dovnar, *Some unexpected properties of limit cycles of quadratic systems in the plane.,* CRM Preprint, num.402, Gener, 1999.

32. J.Chavarriga & J.Gene, Publ. Matem.**40**,21-39(1996); andand,41(335-356)1997.

33. R.E.Kooij & A.Zegeling, Kyungpook Math. J.**38**,323-340(1998).

34. C.Z.Li, W.G.Li,J.Llibre & Z.F.Zhang, *Bifurcation of limit cycles from quadratic isochronous centers,* CRM Preprint,num.413,Maig.(1999).

35. H.Zoladek, Nonlinearity8,843-860(1995).

36. Ye Yanqian, Ann. of Diff. Eqs.**14**,392-401(1998).

37. Ye Weiyin, Ann. of Diff. Eqs.**14**,384-391(1998).

38. Zhang Xiang, *Ergodicity of limit cycles for the quadratic differential systems. (*to appear).

39. J.C.Artes & J.Llibre, J. Nanjing Univ., Math. Biquarterly**13**,143-149(1996).

40. G.R.Dai, Acta Math. Scientia**16**,232-240(1996).

41. J.Sokulski, Nonlinearity**9** ,479-485(1996).

42. Zhang Xiang & Y.Q.Ye, Proc. AMS**126**,2249-2265(1998).

43. J.C.Artes, B.Grunbaum & J.Llibre, Diff. Eqs. & Dyn. Syst.**5**,317-327(1997).

44. P.G.Zhang, *Quadratic system with a 3rd-order (or 2nd-order) weak focus* (Preprint, 1998).

45. P.G.Zhang, Appl. Math. JCU**13(A)**,1-12(1998).

46. P.G.Zhang, *Uniqueness of limit cycles around a first order weak focus of the quadratic system (III) n=0,* (Preprint, 1998).

47. P.G.Zhang, Acta Math. Sinica**44**,1-16(1999).

48. P.G.Zhang, Appl. Math., JCU**14(B)**,7-14(1999).

49. P.G.Zhang, Appl. Math., JCU**14(A)**,246-253(1999).

50. P.G.Zhang, *The number and distribution of limit cycles of a quadratic system having two foci.* (Preprint, 1999).

51. A.Zegeling & R.E.Kooij, J.Diff.Eqs.**151**,373-385(1999).

52. Ye Weiyin, Appl.Math., JCU**14(A)**,254-260(1999).

53. Ye Yanqian, Ann. of Diff.Eqs.**15**,211-220(1999).

THE ASYMPTOTIC THEORY OF INITIAL VALUE PROBLEM FOR SEMILINEAR WAVE EQUATIONS IN THREE SPACE DIMENSIONS

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In this paper, we consider the existence, uniqueness and asymptotic theory of semilinear wave equations in three space dimensions.

1 Introduction

The aim of this paper is to establish an asymptotic theory for the following initial value problem for a semilinear perturbed wave equation

$$\begin{cases} u_{tt} - \Delta u = \epsilon G(u, \epsilon), & x \in R^3, \quad t > 0, \\ u(0, x, \epsilon) = u_0(x, \epsilon), \quad u_t(0, x, \epsilon) = u_1(x, \epsilon), & x \in R^3, \end{cases} \tag{1.1}$$

where $u(\epsilon, x, t)$ is a real-valued unknown function, $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$, ϵ is a parameter with $0 < |\epsilon| < \epsilon_0 \ll 1$, $G(u, \epsilon)$, $u_0(x, \epsilon)$ and $u_1(x, \epsilon)$ satisfy some assumptions mentioned in Section 2. In the papers ref.1-ref.4, the asymptotic theory for validation of formal approximations of the solutions of initial-boundary value problems for the second order semilinear wave equations in one space dimension with the best order time function $T = O(|\epsilon|^{-1})$ has been presented. In ref.5, the asymptotic theory of solutions of initial value problems for the equation $u_{tt} - u_{xx} + p^2 u = \epsilon f(t, x, u, \epsilon)$ $(-\infty < x < \infty, p^2 > 0)$ was obtained on the long time scale of order $|\epsilon|^{-1}$ in a suitable Sobolev space. In ref. 6, the asymptotic theory of initial value problems for second order semilinear wave equations is presented on the time scale of order $|\epsilon|^{-1}$ in three space dimensions. In this paper, an interesting result is that the asymptotic theory and validation of formal approximations for the second order semilinear wave equation in three space dimensions on the long time scale including $0 \le t \le T = O(|\epsilon|^{-\sigma})$ $(\sigma > 0, \epsilon \to 0)$ and $0 \le t \le T = \infty$ are established in the classical sense of C^2.

For simplicity, we will denote by C any constants appearing in our paper, which never depends on ϵ.

2 Existence and uniqueness

Suppose that the nonlinear term $G(u, \epsilon)$ and initial value $u_0(x, \epsilon)$, $u_1(x, \epsilon)$ satisfy following assumptions
   (i) $G(u, \epsilon) \in C^2$ with respect to u, $G(0, \epsilon) = G_u(0, \epsilon) = G_{uu}(0, \epsilon) = 0$.
   (ii) If $|u(t, x, \epsilon)| < M$, $|v(t, x, \epsilon)| < M$, there exist constants $p > 3$ and $A > 0$ such that

$$|G(u, \epsilon)| \le A \quad \text{and} \quad |G_{uu}(u, \epsilon) - G_{uu}(v, \epsilon)| \le A|w|^{p-1}|u - v|,$$

where $w = \max\{|u|, |v|\}$, M and A are independent of ϵ.
   (iii) $u_0(x, \epsilon)$ and $u_1(x, \epsilon)$ satisfy

$$|\partial_x^\alpha u_0(x, \epsilon)|, \quad |\partial_x^\beta u_1(x, \epsilon)| \le \frac{G}{(1 + |x|)^{1+k}}, \quad 0 < k < 1,$$

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where multi-integers α and β satisfy $|\alpha| \leq 3$, $|\beta| \leq 2$, G is independent of ϵ.

Let J_k be given by

$$J_k = \{ \begin{array}{lll} (t,x), & (t,x) \in [0,\infty) \times R^3, & k > 2/(p-1), \\ (t,x), & (t,x) \in [0,T] \times R^3, & 0 < k < 2/(p-1). \end{array}$$

We define $C^2(J_k)$ be the space of all real-valued and twice continuously differentiable functions W on J_k with norm $\| \cdot \|_{J_k}$ given by

$$\|W\|_{J_k} = \sup_{(t,x) \in J_k} [(1 + t + |x|)^k \|W(t,x,\epsilon)|\|] < \infty, \tag{2.2}$$

where

$$\|W(t,x,\epsilon)\| = \sum_{0 \leq j + j_1 + j_2 + j_3 \leq 2} |\frac{\partial^{j+j_1+j_2+j_3} W(t,x,\epsilon)}{\partial t^j \partial x_1^{j_1} \partial x_2^{j_2} \partial x_3^{j_3}}|.$$

We know that $C^2(J_k)$ is a Banach space with the norm defined by (2), and for any $u \in C^2(J_k)$, $\|u\|_{J_k}$ is bounded. By the fixed point theorem we prove the existence and uniqueness of solutions to (1) in the space $C^2(J_k)$.

Theorem 1 *Suppose that the nonlinear term $G(u,\epsilon)$, initial value $u_0(x,\epsilon)$ and $u_1(x,\epsilon)$ satisfy assumptions (i)-(iii) with $0 < |\epsilon| \leq \epsilon_0 \ll 1$, then we have*

(1) If $k > 2/(p-1)$ $(p > 3)$, there exists a unique global C^2 solution to problem (1).

(2) If

$$0 < k < \min\{1, 2/(p-1)\}(p > 3), \ 0 \leq t \leq T = O(|\epsilon|^{-1/(2-kp+k)}),$$

there exists a unique solution $u \in C^2(J_k)$ to problem (1).

3 Validation of formal approximations

Suppose that on $J_k \times [-\epsilon_0, \epsilon_0]$, the function $v(t,x,\epsilon)$ satisfies

$$\begin{cases} v_{tt} - \Delta v = \epsilon G(v,\epsilon) + |\epsilon|^m c_1(t,x,\epsilon), & m > 1, \\ v(0,x,\epsilon) = u_0(x,\epsilon) + |\epsilon|^{m-1} c_2(x,\epsilon) = v_0(x,\epsilon), & 0 < |\epsilon| \leq \epsilon_0 \ll 1, \\ v_t(0,x,\epsilon) = u_1(x,\epsilon) + |\epsilon|^{m-1} c_3(x,\epsilon) = v_1(x,\epsilon), & 0 < |\epsilon| \leq \epsilon_0 \ll 1, \end{cases} \tag{3.3}$$

where $G(v,\epsilon)$, $u_0(x,\epsilon)$, $u_1(x,\epsilon)$ satisfy assumptions (i)-(iii). Suppose that $c_1(t,x,\epsilon)$, $c_2(x,\epsilon)$ and $c_3(x,\epsilon)$ satisfy following conditions

$$c_1(t,x,\epsilon) \in C^2(J_k), \text{ and } \|c_1(t,x,\epsilon)\| \leq 1/(1+t+|x|)^{kp}, \tag{3.4}$$

$$|\partial_x^\alpha c_2(x,\epsilon)|, \partial_x^\beta c_3(x,\epsilon)| \leq C/(1+t+|x|)^{k+1}, |\alpha| \leq 3, |\beta| \leq 2, 0 < k < 1. \tag{3.5}$$

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From Theorem 1 it follows that the initial value problem (3) has a unique solution $v(t, x, \epsilon) \in C^2(J_k)$. On the other hand, the problem (3) can be transformed into the following equivalent equation

$$v(t, x, \epsilon) = \frac{\partial}{\partial t} [\frac{t}{4\pi} \int_{|\omega|=1} v_0(x + t\omega, \epsilon) d\sigma_\omega] + \frac{t}{4\pi} \int_{|\omega|=1} v_1(x + t\omega, \epsilon) d\sigma_\omega$$
$$+ \frac{\epsilon}{4\pi} \int_0^t (t - \tau) \int_{|\omega|=1} [G(v, (\tau, x + (t - \tau)\omega, \epsilon), \epsilon)$$
$$+ |\epsilon|^m c_1(\tau, x + (t - \tau)\omega, \epsilon)] d\sigma_\omega d\tau.$$

If $u \in C^2(J_k)$ is the solution of problem (1), then

$$v(t, x, \epsilon) - u(t, x, \epsilon)$$
$$= \frac{\partial}{\partial t} [\frac{t}{4\pi} \int_{|\omega|=1} |\epsilon|^{m-1} c_2(x + t\omega, \epsilon) d\sigma_\omega]$$
$$+ \frac{t}{4\pi} \int_{|\omega|=1} |\epsilon|^{m-1} c_3(x + t\omega, \epsilon) d\sigma_\omega$$
$$+ \frac{\epsilon}{4\pi} \int_0^t (t - \tau) \int_{|\omega|=1} [G(v(\tau, x + (t - \tau)\omega, \epsilon), \epsilon) \qquad (3.6)$$
$$- G(u(\tau, x + (t - \tau)\omega, \epsilon), \epsilon)] d\sigma_\omega d\tau$$
$$+ \frac{\epsilon}{4\pi} \int_0^t (t - \tau) \int_{|\omega|=1} |\epsilon|^m c_1(\tau, x(t - \tau)\xi, \epsilon) d\sigma_\omega d\tau.$$

Thus, we get the following asymptotic approximation theorem

Theorem 2 *Suppose that $v(t, x, \epsilon)$ is the solution of the problem (3), and nonlinear term G, initial data u_0, u_1 satisfy assumptions (i)-(iii). Let $c_1(t, x, \epsilon)$, $c_2(x, \epsilon)$ and $c_3(x, \epsilon)$ satisfy (4) (5). Then for $m > 1$, the formal approximation $v(t, x, \epsilon)$ is an asymptotic approximation (as $\epsilon \to 0$) of the solution $u(t, x, \epsilon)$ of problem (1). Furthermore*

$$\|u - v\|_{J_k} = O(|\epsilon|^{m-1}) \text{ for } (t, x) \in J_k. \qquad (3.7)$$

References

1. W. T. Van Horssen, *SIAM J. Appl. Math.*, **48**, 1227 (1988).
2. W. T. Van Horssen and A. H. P. Van Der Burgh, *SIAM J. Appl. Math.*, **48**, 719 (1988).
3. W. T. Van Horssen, *Nonliear Analysis TMA.*, **19**, 501 (1992).
4. C. J. Blom and A. H. P. Van Der Burgh, *Applicable Analysis*, **52**, 155 (1994).
5. S. Y. Lai, *Appl. Math. Mech.*, **7**, 657 (1997).
6. S. Y. Lai and C. L. Mu, *Appl. Math. JCU(B)*, **12**, 321 (1997).

ON THE GLOBAL STABILITY FOR A CLASS OF DIFFERENCE EQUATIONS WITH DELAY

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G.Ladas presented a conjecture that the positive equilibrium $\bar{x}$ of the following delay difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}, (B \in [0, \infty), \alpha, \beta, A, C \in (0, \infty), x_0, x_{-1} \in (0, \infty))$$

is globally asymptotically stable. Under the additional condition $A > \beta$, the paper [3] has proved the result recently. In this paper, we shall prove the positive equilibrium $\bar{x}$ is globally stable under weaker additional conditions.

1   Introduction

Consider the following difference equation with delay:

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}, n = 1, 2 \cdots \tag{1.1}$$

where

$$B \in [0, \infty), \alpha, \beta, A, C \in (0, \infty), x_0, x_{-1} \in (0, \infty). \tag{1.2}$$

Under condition (2), Equation (1) has a unique positive equilibrium:

$$\bar{x} = \frac{\beta - A + \sqrt{(\beta - A)^2 + 4(B + C)\alpha}}{2(B + C)}.$$

For Equation (1), a solution x_n is called to be permanent if there exist two positive constants M and m $(m \leq M)$, such that for any initial values $x_0, x_1 \in (0, \infty)$, there exists a positive integer N (depends on x_0, x_{-1}), such that $m \leq x_n \leq M$ for $n \geq N$; A solution $\{x_n\}$ is called to be globally attractive about the positive equilibrium $\bar{x}$ if for any initial values $x_0, x_{-1} \in (0, \infty)$, $\lim_{n \to \infty} x_n = \bar{x}$; If the positive equilibrium $\bar{x}$ is locally asymptotically stable and is globally attractive then we say Equation (1) is globally asymptotically stable.

Jarom, Kocid and G.Ladas have proved if one of the conditions (1) $\alpha = 0, \beta > A > 0$; (2) $\alpha > 0, \beta > A > \alpha(B + C)$ is satisfied, then the positive equilibrium $\bar{x}$ of Equation (1) is globally asymptotically stable. G. Ladas presented the conjecture that Equation (1) with condition (2) is globally stable without

any other additional conditions [1]. Recently, the paper [3] has discussed the permanence, the global stability and the global attractivity of Equation (1) with condition (2) and proved if $A > \beta$ then the positive equilibrium $\bar{x}$ is globally attractive about any positive solutions, therefore it is globally stable. In this paper, we shall discuss the global stability of Equation (1) and give weaker sufficient conditions for the global stability of the positive equilibrium $\bar{x}$ of Equation (1).

2   Results

Suppose condition (2) holds henceforth.

Lemma 1 Equation (1) is permanent, i.e. for any solution $\{x_n\}$, there exist positive constants M and $m(m \leq M)$, such that for any initial values $x_0, x_{-1} \in (0, \infty)$, there exists a positive integer N (depends on x_0, x_{-1}), such that if $n \geq N$ then:

$$m \leq x_n \leq M.$$

Lemma 2 The positive equilibrium $\bar{x}$ of Equation (1) is locally asymptotically stable.

For the proofs of the lemmas, see [3].

Lemma 3 If $m > \frac{\beta - A}{B + C}$, then the positive equilibrium $\bar{x}$ of Equation (1) is globally stable.

Proof It suffices to show $\bar{x}$ is globally attractive. Let $\{x_n\}$ be a solution with initial values $x_0, x_{-1} \in (0, \infty)$. We have

$$
\begin{aligned}
x_{n+1} &= \frac{\alpha}{A + Bx_n + Cx_{n-1}} + \frac{\beta x_n}{A + Bx_n + Cx_{n-1}} \\
&= \frac{\alpha}{A + Bx_n + Cx_{n-1}} + \frac{\beta}{A + Bx_n + Cx_{n-1}} \frac{\alpha + \beta x_{n-1}}{A + Bx_{n-1} + Cx_{n-2}} \\
&= \frac{\alpha}{A + Bx_n + Cx_{n-1}} + \frac{\alpha\beta}{(A + Bx_n + Cx_{n-1})(A + Bx_{n-1} + Cx_{n-2})} \\
&\quad + \frac{\beta^2 x_{n-1}}{(A + Bx_n + Cx_{n-1})(A + Bx_{n-1} + Cx_{n-2})} \\
&= \cdots \\
&= \sum_{j=1}^{n} \frac{\alpha\beta^2}{\prod_{j=1}^{I}(A + Bx_{n-j} + Cx_{n-1-j})} \\
&\quad + \frac{\beta^{n+1}x_0}{\prod_{j=0}^{n}(A + Bx_{n-j} + Cx_{n-1-j})}.
\end{aligned}
\tag{2.3}
$$

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As $n \to \infty$, the right side of (3) is a series of positive items. Since Equation (1) is permanent, it is follows that there exist positive constants $M < m$ and an integer N, such that if $n \geq N$ then $m \leq x_n \leq M$. Thus we have

$$\sum_{I=0}^{n} \frac{\alpha \beta^I}{\prod_{j=0}^{I}(A + Bx_{n-j} + Cx_{n-1-j})} + \frac{\beta^{n+1}x_0}{\prod_{j=0}^{n}(A + Bx_{n-j} + Cx_{n-1-j})}$$

$$\leq \sum_{I=0}^{n-N-1} \frac{\alpha \beta^I}{(A + Bm + Cm)^{I+1}} + \frac{\alpha \beta^{n-N}}{(A + Bm + Cm)^{n-N}}$$

$$\times \sum_{I=0}^{N} \frac{\beta^I}{\prod_{j=0}^{I}(A + Bx_{N-j} + Cx_{N-j-1})}$$

$$+ \frac{\beta^{n-N}}{(A + Bm + Cm)^{n-N}} \frac{\beta^{N+1}x_0}{\prod_{j=0}^{N}(A + Bx_{N-j} + Cx_{N-j-1})}. \tag{2.4}$$

Since $(A + Bm + Cm) > \beta$, it is follows that $\sum_{I=0}^{\infty} \frac{\alpha \beta^I}{(A+Bm+Cm)^{I+1}}$ is convergent. It is easy to see that the first part and the second part on the right side in (4) tend to zero as $n \to \infty$, so the right side of (3) is convergent, i.e. $\lim_{n \to \infty} x_n$ exists. Suppose $\lim_{n \to \infty} x_n = \lambda$. Taking the limits on both sides of (1) as $n \to \infty$ yields:

$$\lambda = \frac{\alpha + \beta \lambda}{A + B\lambda + C\lambda}.$$

Since the positive equilibrium of Equation (1) is unique, we have $\lim_{n \to \infty} x_n = \bar{x}$. Therefore, the positive equilibrium $\bar{x}$ of Equation (1) is globally stable. The proof is completed.

¿From the lemma, an immediate consequence follows:

Theorem 1 If $A \geq \beta$, then the equilibrium $\bar{x}$ of Equation (1) is globally stable.

The following theorem give weaker sufficient condition for the global stability of the equilibrium $\bar{x}$ of Equation (1).

Theorem 2 Let $A_1 = \max\{\frac{\alpha}{A}, \frac{\alpha + \beta \bar{x}}{A + B\bar{x}}, 1\}$, $A_2 = \max\{\frac{\frac{\alpha}{\bar{x}} + \beta}{A + B\bar{x}}, 1\}$, $L = \max\{2\bar{x}, A_1, A_2\}$; $B_1 = \min\{\frac{\alpha + \beta \bar{x}}{A + B\bar{x} + CL}, \frac{\alpha + \beta L}{A + BL + CL}, 1\}$, $B_2 = \min\{1, \frac{\frac{\alpha}{\bar{x}} + \beta}{A + B\bar{x} + CL}\}$, $B_3 = \min\{\frac{\bar{x}}{2}, B_1 B_2\}$. If $B_3 > \frac{\beta - A}{B + C}$, then the equilibrium $\bar{x}$ of Equation (1) is globally asymptotically stable.

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Proof According to Lemma 3, we need only to show $m = B_3$. Denote

$$x_n f(x_n, x_{n-1}) = x_n \frac{\dfrac{\alpha}{x_n} + \beta}{A + Bx_n + Cx_{n-1}}.$$

For $0 \leq x \leq \bar{x}$, it is easy to show

$$\max x f(x, 0) = \begin{cases} \frac{\alpha}{A}, & \beta A - \alpha B \leq 0; \\ \frac{\alpha + \beta \bar{x}}{A + B\bar{x}}, & \beta A - \alpha B > 0. \end{cases}$$

Let $A_0 = \max\{\frac{\alpha}{A}, \frac{\alpha + \beta \bar{x}}{A + B\bar{x}}\}, A_1 = \max\{A_0, 1\}, A_2 = \max\{f(\bar{x}, 0), 1\} = \max\{\dfrac{\frac{\alpha}{\bar{x}} + \beta}{A + B\bar{x}}, 1\}, L = \max\{2\bar{x}, A_1 A_2\};$

For $\bar{x} \leq x \leq L$, it is easy to find

$$\min x f(x, L) = \begin{cases} \frac{\alpha + \beta \bar{x}}{A + B\bar{x} + CL}, & \beta A - \beta CL - \alpha B > 0; \\ \frac{\alpha + \beta L}{A + BL + CL}, & \beta A + \beta CL - \alpha B \leq 0. \end{cases}$$

Let $B_0 = \min x f(x, L) = \min\{\frac{\alpha + \beta \bar{x}}{A + B\bar{x} + CL}, \frac{\alpha + \beta L}{A + BL + CL}\}. B_1 = \min\{B_0, 1\},$

$B_2 = \min\{f(\bar{x}, L), 1\} = \min\{\dfrac{\frac{\alpha}{\bar{x}} + \beta}{A + B\bar{x} + CL}, 1\}, B_3 = \min\{\frac{\bar{x}}{2}, B_1 B_2\}.$ ¿From the proof of Theorem 2.1 in [4], we can take :

$$M = L, m = B_3 = \min\{\frac{\bar{x}}{2}, B_1 B_2\}.$$

¿From Lemma 3 , Theorem 2 follows.

3   An example

Consider the global stability of the following difference equation:

$$x_{n+1} = \frac{1 + 3x_n}{2 + 3x_n + 3x_{n-1}} \tag{3.5}$$

In this case $\alpha = 1, A = 2, \beta = B = C = 3$. The theorems in previous papers can't assure the global stability of Equation (5).

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The positive equilibrium of Equation (5) is $\bar{x} = \frac{1}{2}$. It is easy to find:

$$A_0 = \frac{\alpha + \beta\bar{x}}{A + B\bar{x}} = \frac{5}{7},$$

Thus

$$A_1 = \max\{A_0, 1\} = 1, \qquad A_2 = \max\{\frac{\frac{\alpha}{\bar{x}} + \beta}{A + B\bar{x}}, 1\} = \frac{10}{7},$$

$$L = \max\{2\bar{x}, A_1 A_2\} = \frac{10}{7}, \quad B_0 = \frac{\alpha + \beta\bar{x}}{A + B\bar{x} + CL} = \frac{70}{218},$$

Thus

$$B_1 = \min\{B_0, 1\} = \frac{70}{218}, \qquad B_2 = \min\{\frac{\frac{\alpha}{\bar{x}} + \beta}{A + B\bar{x} + CL}, 1\} = \frac{70}{109},$$

$$B_3 = \min\{\frac{\bar{x}}{2}, B_1 B_2\} = \frac{70}{218}\frac{70}{109}.$$

Since $B_3 = \frac{70}{218}\frac{70}{109} > \frac{\beta - A}{B + C} = \frac{1}{6}$, by Theorem 2, the equilibrium $\bar{x} = \frac{1}{2}$ is globally stable.

References

1. G.Ladas, Journal of Difference Equations and Applications **1**, 413-419 (1995).
2. V. L. Kocis and G. Ladas, Global Asymptotic Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer Academic Publishers , Dordrect (1993).
3. Xianyi Li et al, Theory of Differential Equations with Applications , 523-526 (1998).
4. Xialiang Zhou, Differential Equations And Its Applications , 137-141 (1996).

ON THE $\frac{3}{2}$ GLOBAL ATTRACTIVITY AND STABILITY RESULTS OF SOME NONLINEAR DELAY EQUATIONS OF LOGISTIC TYPE [m]

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In this report, we will surrey some results on the $\frac{3}{2}$ global attractivity and uniform stability of the positive equilibriums of the following nonlinear delay differential equations of logistic type

$$N'(t) = r(t)N(t)\left\lceil 1 - \frac{N(t-\tau)}{K}\right\rceil, \quad t \geq 0$$

$$N'(t) = r(t)N(t)\frac{1-N(t-\tau)}{1+s(t)N(t-\tau)}, \quad t \geq 0$$

$$N'(t) = r(t)N(t)\frac{1-N(t-\tau(t))}{1-cN(t-\tau(t))}, \quad t \geq 0$$

where $K, \tau > 0, 0 < c < 1, r \in C([0,\infty),(0,\infty))$ and $s \in C([0,\infty)[0,\infty))$. We will also put forth some interesting open questions.

1   Introduction and Open Questions

Consider the following delay Logistic equations

$$N'(t) = r(t)N(t)\left\lceil 1 - \frac{N(t-\tau)}{K}\right\rceil, \quad t \geq 0, \quad (\text{see}[1]) \tag{1.1}$$

$$N'(t) = r(t)N(t)\frac{1-N(t-\tau(t))}{1-cN(t-\tau(t))}, \quad t \gg 0, \quad (\text{see}[2]) \tag{1.2}$$

and

$$N'(t) = r(t)N(t)\frac{1-N(t-\tau)}{1+s(t)N(t-\tau(t))}, \quad t \gg 0. (\text{see}[3]) \tag{1.3}$$

In 1955, Wright [4] proved that if

$$r(t) \equiv r > 0 \quad \text{and} \quad r\tau \leq \frac{3}{2} \tag{1.4}$$

then all positive solutions of (1) tend to K.

In 1993, Kuang [5] conjectured that if

$$0 < r(t)\tau < \frac{3}{2}, \quad t \geq 0 \tag{1.5}$$

then all positive solutions of (1) tend to K.

In 1995, Chen, Yu, Qian and Wang [6] solved Kuang's conjecture and proved that if

$$0 < r(t)\tau \le \frac{3}{2}, \quad t \ge 0 \tag{1.6}$$

and

$$\int_0^\infty r(s)ds = \infty \tag{1.7}$$

then all positive solutions of (1) tend to K.

At the same year, So and Yu [7] showed that (6) can be replaced by

$$\int_{t-\tau}^t r(s)ds < \frac{3}{2}, \quad \text{for large } t. \tag{1.8}$$

In 1992, Sugie [8] proved that if

$$r(t)\tau \le \alpha_0 < \frac{3}{2}, \quad t \ge 0 \tag{1.9}$$

then the positive equilibrium K of (1) is uniformly stable.

In [6], (9) was improved by

$$\int_{t-\tau}^t r(s)ds \le \alpha_0 < \frac{3}{2}, \quad t \ge \tau. \tag{1.10}$$

The linear equation

$$z'(t) + r(t)z(t - \tau) = 0, \quad t \ge 0 \tag{1.11}$$

—linearized equation associated with (1) at K. (11) has been investigated by many authors, for example see [5,9,10].

In [10], it was proved that if

$$\int_{t-\tau}^t r(s)ds \le \frac{3}{2}, \quad t \ge \tau \tag{1.12}$$

then the zero solution of (11) is uniformly stable.

Open Question 1: Is it true that, if (12) holds, then the equilibrium K of (1) is uniformly stable?

In 1991, Kuang et al., [2] proved that if (7) holds and for some $t_1 > 0$

$$\int_{t-\tau(t)}^t r(s)ds \le \delta < 1 - ce^\delta \quad \text{for} \quad t \ge t_1 \quad \text{and} \quad e^\delta < c^{-1}, \tag{1.13}$$

then every global positive solution of (2) tends to 1.

In 1996, Yu [11] showed that (13) can be improved by

$$\int_{t-\tau(t)}^{t} r(s)ds \le \delta \le \frac{3}{2}(1 - ce^{\delta}) \text{ for } t \ge t_1 \text{ and } e^{\delta} < c^{-1}. \qquad (1.14)$$

In 1996, Yu and Zhang [12] proved that

$$\int_{t-\tau(t)}^{t} r(s)ds \le \delta \le \frac{3}{2}(1 - c), \text{ for } t \ge \bar{t} \text{ and } e^{\delta} < c^{-1}. \qquad (1.15)$$

implies the uniform stability of the equilibrium 1 of (2), where $\bar{t} > 0$ with $\bar{t} = \tau(\bar{t})$.

Open Question 2: Is it true that, if

$$\int_{t-\tau(t)}^{t} r(s)ds \le \frac{3}{2}(1 - c), \text{ for large } t, \qquad (1.16)$$

then every global positive solution of (2) tends to 1?

Open Question 3. Is it true that, if

$$\int_{t-\tau(t)}^{t} r(s)ds \le \frac{3}{2}(1 - c), \text{ for } t \ge \bar{t}, \qquad (1.17)$$

then the equilibrium 1 of (2) is uniformly stable?

In 1998, Gopalsamy, Kulenovic and Ladas [13] introduced the special form of (3)

$$N'(t) = rN(t)\frac{1 - N(t - \tau)}{1 + crN(t - \tau)}, \quad t \ge 0. \qquad (1.18)$$

see also [14]. In [13], it was proved that if

$$r\tau e^{r\tau} < 1 \qquad (1.19)$$

then every positive solution of (18) tends to 1.

In 1993, Groved, Ladas and Qian [3] proved that if (7) holds and

$$\int_{t-\tau}^{t} r(s)ds \le 1 + \varepsilon_0, \text{for } t \ge T \text{ and } \varepsilon_0 = \inf\{s(t) : t \ge 0\}, \qquad (1.20)$$

then every positive solution of (3) tends to 1.

In 1996, Yu [11] showed that if (7) holds and

$$\int_{t-\tau}^{t} r(s)ds \leq \frac{3}{2}, \quad \text{for large } t \tag{1.21}$$

then every positive solution of (3) tends to 1.

In 1995, So and Yu [15] proved that if

$$\int_{t-\tau}^{t} \frac{r(u)}{1+s(u)}du \leq \alpha < \frac{3}{2}, \quad \text{for } t \geq \tau \tag{1.22}$$

then the equilibrium 1 of (3) is uniformly stable.

The linearized equation associated with (3) at 1 is

$$y'(t) + \frac{r(t)}{1+s(t)}y(t-\tau) = 0, t \geq 0. \tag{1.23}$$

Open Question4: Is it true that, if

$$\int_{t-\tau}^{t} \frac{r(u)}{1+s(y)}ds \leq \alpha < \frac{3}{2}, \quad \text{for } t \geq \tau \tag{1.24}$$

then the equilibrium 1 of (3) is uniformly stable?

Open question 5: Is it true that, if

$$\int_{0}^{\infty} \frac{r(u)}{1+s(u)}du = \infty \tag{1.25}$$

and

$$\int_{t-\tau}^{t} \frac{r(u)}{1+s(u)}du \leq \frac{3}{2}, \quad \text{for large } t, \tag{1.26}$$

then every positive solution of (3) goes to 1?

2 Some Results on Open Questions

Result 1 [16]. The answer to open question 1 is positive

Result 2 [17]. Open question 2 is true.

Result 3 [18]. If (7) holds and $s(t) \geq s_0$,

$$\int_{t-\tau}^{t} r(u)du \leq \frac{3}{2}(1+s_0), \quad \text{for large } t \tag{2.27}$$

then every positive solution of (3) goes to 1.

— A partial answer to question 5.

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References

1. G.E.Hutchinson, *Ann.New York Acad.Sci.* **50**, 221 (1948).
2. Y.Kuang, B.G.Zhang and T.Zhao *Tohoku Math.J.* **43**, 509 (1991).
3. E.A.Grove, *Dynamic systems and Appl.* **2**, 243 (1993).
4. E.M.Wright, *J.Reine Angew.Math.* **194**, 66 (1955).
5. Y.Kuang, *Acad.Press,Boston* , (1993).
6. Ming-Po Chen, J.S.Yu, X.Z.Qian and Z.C.Wang *Nonlinear Analysis, TMA.* **25**, 187 (1995).
7. Josepho W.H So and J.S.Yu, *Proc.Amer.Math.So.* **123**, 2687 (1995).
8. J.Sugie, *Proc.Royal Soc.Edinburgh,* **120A**, 179 (1992).
9. J.A.Yorke, *J.Differential Equations,* **7**, 189 (1970).
10. T.Yoneyama, *J.Math.Anal.Appl.,* **165**, 133 (1992).
11. J.S.Yu, *Science in China* **39A**, 225 (1996).
12. J.S.Yu and B.G.Zhang, *Advances in Nonlinear Dynamics, Stability and Control:Theory, Methods and Applications,* **5**, 137 (1996)
13. K.Gopalsamy, M.R.S.Kulenoric and G.Ladas, *Applicable Analysis,* **31**, 225 (1998)
14. K.Gopalsamy, M.R.S. Kulenovic and G.Ladas, *J.Math.Anal. Appl.,* **147**, 545 (1990)
15. Joseph W.H. So and J.S.Yu, *Proc.Royal SoC.Edinburgh,* **125A**, 991 (1995)
16. J.S.Yu and X.H.Tang, to appear
17. J.S.Y, X.H.Tang and Z.C.Wang, to appear
18. X.H.Tang and J.S.Yu, to appear

HOMOCLINIC ORBITS FOR SINGULARLY PERTURBED NONLINEAR SCHÖDINGER EQUATION

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In this work, we study a perturbed nonlinear Schrödinger equation

$$iu_t = u_{xx} + 2(u\bar{u} - \omega^2)u + i\epsilon(u_{xx} - \alpha u - \beta)$$

for u even and periodic in x. The diffusion $i\epsilon u_{xx}$ is an unbounded perturbation. We prove the existence of homoclinic orbits for this diffusively perturbed NLS. The method was based on invariant manifolds, foliations, and Melnikov analysis.

1   Introduction and Main Theorem

We consider the perturbed nonlinear Schrödinger equation

$$iu_t = u_{xx} + 2(u\bar{u} - \omega^2)u + i\epsilon(u_{xx} - \alpha u - \beta), \tag{1.1}$$

where u is 2π periodic and even in x. The parameters ω, α, and β are assumed to satisfy $\omega \in (\frac{1}{2}, 1)$ and $0 < \alpha\omega < \beta$, and $\epsilon > 0$ is a small dissipative perturbation parameter. We shall prove

Theorem 1.1 *For each small $\epsilon > 0$ and $\omega, \alpha > 0$ in certain intervals, there exists $\beta(\epsilon, \alpha)$ such that equation (1.1) has an orbit homoclinic to an equilibrium.*

When $\epsilon = 0$, the unperturbed NLS is a completely integrable Hamiltonian PDE. Starting with the Lax pair for the integrable NLS, an infinite family of commuting constants of motion has been constructed. Thus, One has a rather clear understanding of the dynamics of the integrable NLS. However, these structures are sensitive to small perturbations.

The perturbations in equation (1.1) break these structures. The perturbations contain three parts. The term $-\epsilon\alpha u$ represents damping and ϵu_{xx} diffusion. These dissipative perturbations make the energy of the system decay as time evolves. Under these perturbations, equation (1.1) is in the form of the important complex Ginzburg-Landau equations. It has been proved that there exist a finite dimensional compact attractor and a finite dimensional inertial manifold for the perturbed NLS (1.1)[35]. Though the qualitative properties of the infinite dimensional system (1.1) are contained in the reduced finite dimensional system on the inertial manifold, in general, there is not much

known about the dynamics on the attractors and inertial manifolds of dissipative systems except for some special cases. Furthermore, the dimension of the attractor of equation (1.1) increases to ∞ as $\epsilon \to 0$. Therefore, the existence of the attractor of (1.1) does not help to study homoclinic orbits or other important special solutions. On the contrary, starting with the structure of integrable PDEs, it is natural to study detailed dynamics of weakly dissipative systems, such as (1.1), as perturbations of integrable PDEs. In this direction, recently, G. Cruz-Pacheco, C. D. Levermore, and B. P. Luce also studied Ginzburg-Landau equations as perturbation of NLS which preserve the phase symmetry. Here, the phase symmetry means that if $u_0(t)$ is a solution, then so is $u_0(t)e^{i\theta}$ for any constant θ. In a series[4],[5],[6],[7], they discussed the persistence of special solutions, such as rotating waves, traveling waves, quasi-periodic solutions, and homoclinic orbits, through Melnikov approach. Some necessary conditions are derived by utilizing some invariants of NLS and, in some cases, sufficiency is also studied. The situation is different for equation (1.1) since the phase symmetry is destroyed by the external force $-\epsilon\beta$.

The reason that we choose to study homoclinic orbits is that they are closely related to chaos. In recent years, there has been extensive work done on the existence of chaotic behaviors in dynamical systems. Studies on the chaos for perturbed integrable PDEs have been carried out by D. W. McLaughlin and co-workers.

When the perturbed NLS is studied numerically, one finds solutions, which, at large time $t \gg 1$, consist in very regular spatial patterns which oscillate irregularly, appearing to be chaotic in time. Details can be found in[28]. This chaotic behavior is believed to be closely related to persistent homoclinic structures from the integrable systems. Thus, the proof of the existence of homoclinic orbits for the perturbed NLS is naturally the next step. A lot of work has been done in this direction. Some historical background for can be found in[23].

In finite-dimensional systems there are well developed techniques to show the existence of homoclinic orbits and chaos, see, for example,[11],[31], and[36]. Some techniques developed for finite-dimensional systems have been extended to infinite-dimensional ones, see[15], for example. These methods, which will be described below, are used in our construction.

We begin with a brief description of the unperturbed NLS. It has a spatially uniform periodic solution

$$q(t) = re^{-i(2(r^2-\omega^2)t-\theta)},$$

for any fixed $r \neq \omega$ and θ, whose orbit is the circle

$$C_r = \{u \mid u_x = 0, |u| = r\}.$$

Using Bäklund-Darboux transformations, one can obtain exact solutions homoclinic to $q(t)$,

$$q_h^{\pm}(t) = e^{-2ip} \left(\frac{\cos 2p - i \sin 2p \tanh \tau \pm \frac{\sin p \cos x}{\cosh \tau}}{1 \mp \frac{\sin p \cos x}{\cosh \tau}} \right) q(t), \qquad (1.2)$$

where

$$\tau = \sigma_r(t + t_0), \quad \sigma_r = \sqrt{4r^2 - 1}, \; and \; e^{ip} = \frac{1 + i\sigma_r}{2r}.$$

It is easy to see that $q_h^{\pm}(t)$ are homoclinic to $q(t)$ with a phase shift $-4p$. When $r = \omega$, C_ω is a circle of fixed points and $q_h^{\pm}(t)$ are heteoclinic orbits. These periodic orbits and their homoclinic orbits are special cases of general whiskered tori and 'figure 8' structures of integrable systems [28]. (For a more complete description of the integrable structures of NLS, see, for example, [22], [27], [28], [32].) Under above perturbations, the circle C_ω of fixed points breaks and a saddle Q_ϵ appears in a neighborhood of C_ω. We shall prove that, for sufficiently small $\epsilon > 0$, there exists a solution homoclinic to Q_ϵ. As $\epsilon \to 0$, the homoclinic orbit converges to the union of two arcs of C_ω and an unperturbed heteoclinic orbit to C_ω in the form of (1.2). The speed of the motion along the homoclinic orbit has two different scales. When it is near the circle C_ω, the speed is $O(\sqrt{\epsilon})$, and when it is near an unperturbed heteoclinic orbit (1.2), the speed is $O(1)$.

The analysis for existence of homoclinic orbits for the perturbed NLS was initially carried out for finite dimensional versions, first, a four-dimensional Fourier truncation, in [16], [17], [18], [19], [29], and then a $(2N + 2)$-dimensional finite difference discretization, in [20], [21], [24]. In [23], the existence of homoclinic orbits was proved for the equation

$$iu_t = u_{xx} + 2(u\bar{u} - \omega^2)u + i\epsilon(\beta\hat{B}u - \alpha u - 1), \qquad (1.3)$$

where the operator $\hat{B}$ is a Fourier truncation of ∂_{xx}, i.e.

$$\hat{B}\Sigma_{k=0}^{\infty} \cos kx = \Sigma_{k=0}^{K} - k^2 \cos kx \qquad (1.4)$$

for some $K > 0$. The only difference between equation (1.3) and (1.1) is that the bounded operator $\hat{B}$ is replaced by the unbounded operator ∂_{xx}. The truncated diffusion is a bounded operator and equation (1.3) is a regular perturbation to the NLS. In these works, the constructions of homoclinic orbits are very geometric, in which invariant manifolds, foliations and Melnikov method play important roles. That analysis is very influential in this paper. In [23], the typical steps in proving the existence of homoclinic orbits in this setting were laid out:

1. Study the spatially uniform solutions of the perturbed equation;

2. In a neighborhood of the circle C_ω, construct its center-stable and center-unstable manifolds and the stable and unstable foliations of them, and the stable manifold of the saddle Q_ϵ;

3. By the Melnikov method, study the existence of orbits which start from the unstable manifold of Q_ϵ and enter the perturbed center-stable manifold of the circle C_ω;

4. Using the stable foliation of the center-stable manifold of the circle C_ω and the known phase shift of unperturbed homoclinic orbits, determine if the orbits mentioned in 3 enter the stable manifold of Q_ϵ.

In the estimate of the size of the local stable manifold of Q_ϵ, a normal form transformation is used in step 2. Recently, there have been studies on the existence of multi-pulse solutions of equation (1.3) and its discretized version, which are solutions repeatedly leaving and coming back to a neighborhood of the circle C_ω with two time scales, see [13] and [14].

We shall apply the similar idea. However, as the bounded perturbation in equation (1.3) is replaced by the unbounded one in (1.1), some difficulties arise. Let $S^t_\epsilon(u)$ and $\tilde{S}^t_\epsilon(u)$ denote the solution operators for equation (1.1) and (1.3), respectively, with initial value u. Then

- $\tilde{S}^t_\epsilon$ is defined for all $t \in (-\infty, +\infty)$ and for any t, $\tilde{S}^t_\epsilon$ is a diffeomorphism on the phase space, but S^t_ϵ is only defined for $t \geq 0$ and S^t_ϵ is a smooth compact map on the phase space for each $t > 0$.

- While $\tilde{S}^t_\epsilon(u)$ is smooth in ϵ and u, $S^t_\epsilon(u)$ is only continuous in ϵ and $D_u S^t_\epsilon$ is not continuous in ϵ. Therefore, S^t_ϵ is not a C^1 perturbation of S^t_0.

As the first consequence, the derivative of $S^t_\epsilon(u)$ in ϵ has to be interpreted carefully. More importantly, as S^t_ϵ is no longer a C^1 perturbation of S^t_0, the construction of invariant manifolds, foliations and their dependence on ϵ become very delicate issues. Though some results on these geometric structures can still be obtained, the center-unstable and center manifold of C_ω and stable foliation in the center-stable manifold of C_ω are destroyed.

It is worth mentioning that there is another approach to proving the persistence of homoclinic orbits by setting up the problem in space-time function spaces and using a Lyapunov-Schmidt reduction and the Fredholm Alternative, see, for example, [26]. The persistence of the 'breather' for a regularly perturbed sine-Gordon equation is proved in [30] and for a singularly perturbed one in [33] by this approach. Generally, this approach works well when the persistent homoclinic orbit has only a single time scale.

2 Sketch of the Proof

Step 1. We study equation (1.1) restricted to the plane of constant states

$$\Pi_c = \{u | u_x = 0\},$$

which is an invariant plane under both the perturbed and unperturbed NLS. For $\epsilon = 0$, C_r is a periodic orbit for $r \neq \omega$ or a circle of fixed points for $r = \omega$. These structures are destroyed for $\epsilon > 0$. In an $O(\sqrt{\epsilon})$ neighborhood of C_ω, a weak saddle Q_ϵ emerges. The linearized equation at Q_ϵ, restricted to Π_c, has a positive eigenvalue and a negative eigenvalue both of order $O(\sqrt{\epsilon})$. We refer to them as the weakly stable and weakly unstable eigenvalues. The stable and unstable curves of Q_ϵ inside Π_c are smooth curves P_ϵ^s and P_ϵ^u, respectively, which are $O(\sqrt{\epsilon})$ close to the circle C_ω. On a segment of the stable (unstable) curve of length of $O(1)$, the orbit moves towards Q_ϵ at an exponential rate $O(\sqrt{\epsilon})$ as $t \to \infty$ $(-\infty)$. Finally, we set up a form of equation (1.1) in the whole phase space, the space of even H^1 functions in x.

Step 2. Construct the stable and unstable manifolds of Q_ϵ, and the center stable manifold of C_ω. First, we consider the linearized equations. For $\epsilon = 0$, at any point on the circle of fixed points C_ω, the linearized unperturbed equation has a negative eigenvalue and a positive eigenvalue both with algebraic multiplicity 1. In addition, 0 is an eigenvalue with algebraic multiplicity 2 whose eigenspace is the plane of constants Π_c. All other spectrum of the linearized unperturbed NLS are purely imaginary and represent rotations at different frequencies. For small $\epsilon > 0$, we consider the linearized equation at the saddle Q_ϵ. As described in the above, the double eigenvalue 0 splits into the weakly stable and unstable eigenvalues of order of $O(\sqrt{\epsilon})$. In the directions normal to the plane Π_c, the original unstable and stable eigenvalues persist and we refer to these eigenvalues of $O(1)$ as strongly unstable and stable eigenvalues. All other originally purely imaginary eigenvalues are pushed to the left half of the complex plane by the perturbations. Though they are all stable eigenvalues, their real parts range from $O(\epsilon)$ to $-\infty$.

Based on the studies of the eigenvalues, we start to construct the unstable manifold of Q_ϵ for the perturbed equation. Since the weakly unstable eigenvalue is of order $O(\sqrt{\epsilon})$, the usual argument only yields a local unstable manifold of size $O(\sqrt{\epsilon})$, which is too small for our purpose. Fortunately, the invariance of the plane Π_c and the $O(1)$ gap between the strongly unstable eigenvalue and the other eigenvalues allows us to construct, based on each point in a δ neighborhood of C_ω inside Π_c, a strongly unstable fiber of length δ, where δ is a small number independent of ϵ. Here each fiber is a C^1 curve in the strongly unstable direction passing through its base point. The fibers are

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also C^1 in parameters α, β, and ϵ. The union of the strongly unstable fibers based on the unstable curve $P_\epsilon^u \subset \Pi_c$ is the 2-dimensional unstable manifold of Q_ϵ. Every point on the unstable manifold has a backward orbit which tends to Q_ϵ as $t \to -\infty$.

Next, we construct the codimension-1 local center-stable manifold of the circle C_ω. It is a locally invariant manifold transversal to the strongly unstable direction, which may be viewed as a codimension-1 stripe along the circle C_ω of width δ independent of ϵ. The C^1 center-stable manifold is also C^1 in parameters α and β. In addition, we show that its restriction to H^3 is Lipschitz in ϵ. One of the important properties is that orbits in the center-stable manifold do not separate at exponential rates larger than $O(\sqrt{\epsilon})$.

Finally, we construct the stable manifold of Q_ϵ. Recall that Q_ϵ has a stable curve P_ϵ^s inside the plane Π_c and all but 2 eigenvalues of the linearized equation at Q_ϵ have negative real parts. The stable manifold is a codimension-2 stripe along P_ϵ^s. The problem is the width of the stripe. Since the negative real parts of those stable eigenvalues may be as small as $O(\epsilon)$, the usual argument only yields a width of order $O(\epsilon)$, which is too small for our analysis. In[23], a normal form transformation is used to eliminate the quadratic terms in the equation and then the width improved to be of order $O(\epsilon^{\frac{3}{4}})$. Actually, that approach might be able to be used to improve the width to be of order $O(\epsilon^{\frac{1}{2}})$. Here, though the normal form transformation works, we take a different approach. In the first step, note that the gap between the weakly unstable eigenvalue and the weakest stable eigenvalue is of order $O(\sqrt{\epsilon})$. Taking advantage of this gap, we can construct a codimension-2 locally invariant manifold W, which may be viewed as a stripe of width $O(\sqrt{\epsilon})$ along the stable curve P_ϵ^s. The codimension-2 stable manifold of Q_ϵ is a stripe contained in W. Next, we study a modified Hamiltonian along orbits in W and show that the basin of attraction of Q_ϵ is a stripe along P_ϵ^s of width $O(\sqrt{\epsilon})$.

Step 3. With these geometric structures, we start to construct the homoclinic orbit for the perturbed equation (1.1). Keep in mind that if a solution is in the local unstable manifold of Q_ϵ at some time and in the local stable manifold at a later time, then it is homoclinic to Q_ϵ. Therefore, we take a solution with initial point in the unstable manifold of Q_ϵ and show that it actually enters the stable manifold of Q_ϵ if the parameters and the initial point are properly chosen. Since the stable manifold is codimension-2, we need 2 measurement to guarantee that it enters the stable manifold. The first measurement is to prove, by the Melnikov method, that the solution enters the codimension-1 center-stable manifold of the circle C_ω for appropriate parameters. Finally, we find conditions under which the solution enters the codimension-2 stable manifold of Q_ϵ, by the second measurement. This is done by tracking an un-

perturbed homoclinic orbit for a time period of order of $O(\log \frac{1}{\epsilon})$ and using its known phase shift.

Acknowledgments

References

1. P. W. Bates, K. Lu, and C, Zeng, *Existence and persistence of invariant manifolds for semiflows in Banach space*, Mem. Amer. Math. Soc., **135** (1998).

2. P. W. Bates, K. Lu, and C, Zeng, *Invariant foliations near a normally hyperbolic invariant manifold for Semiflows*, Trans. Amer. Math. Soc., To appear.

3. P. W. Bates, K. Lu, and C, Zeng, *Persistence of overflowing manifolds for semiflow*, Comm. Pure Appl. Math. **52**, 1999, pp. 983–1046.

4. G. Cruz-Pacheco, C. D. Levermore, and B. P. Luce, *Complex Ginzburg-Landau equations as perturbations of nonlinear Schrödinger equations: A Melnikov approach*, Submitted.

5. G. Cruz-Pacheco, C. D. Levermore, and B. P. Luce, *Complex Ginzburg-Landau equations as perturbations of nonlinear Schrödinger equations: Traveling wave persistence*, Submitted.

6. G. Cruz-Pacheco, C. D. Levermore, and B. P. Luce, *Complex Ginzburg-Landau equations as perturbations of nonlinear Schrödinger equations: Quasi-periodic solutions*, Preprint.

7. G. Cruz-Pacheco, C. D. Levermore, and B. P. Luce, *Complex Ginzburg-Landau equations as perturbations of nonlinear Schrödinger equations: A case study*, Submitted.

8. N. Ercolani, G. Forest, and D. McLaughlin, *Geometry of the modulational instability. I,II*, unpublished manuscript, 1987.

9. N. Ercolani, G. Forest, and D. McLaughlin, *Geometry of the modulational instability. III. Homoclinic orbits for the periodic sine-Gordon equation*, Phys. D **43**, 1990, pp. 349-384.

10. N. Ercolani and D. McLaughlin, *Toward a topological classification of integrable PDEs*, The geometry of Hamiltonian systems (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 22, Springer, New York, 1991, pp. 111–129.

11. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Applied Mathematical Sciences,

vol. 42, Springer-Verlag, New York, 1983.

12. J. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, **25**, American Mathematical Society, Providence, RI, 1988.

13. G. Haller, *Multi-dimensional homoclinic jumping and the discretized NLS equation*, Comm. Math. Phys., **193** (1998), pp. 1–46.

14. G. Haller, *Homoclinic jumping in the perturbed nonlinear Schrödinger equation*, Comm. Pure Appl. Math., **52** 1999, pp. 1–47.

15. P. Holmes and J. Marsden, *A partial differential equation with infinitely many periodic orbits: chaotic oscillations of a forced beam*, Arch. Rational Mech. Anal **76**, 1981, pp. 135-165.

16. G. Haller and S. Wiggins, *Orbits homoclinic to resonances: the Hamiltonian case*, Phys. D **66**, 1993, pp. 298–346.

17. G. Kovacic, *Orbits homoclinic to resonance: Chaos in a model of the forced and damped sine-Gordon equation*, Ph. D. thesis, California Institute of Technology, 1989.

18. G. Kovacic, *Singular perturbation theory for homoclinic orbits in a class of near integrable dissipative systems*, SIAM J. Math. Anal., **26** 1995, pp. 1611–1643.

19. G. Kovacic and S. Wiggins, *Orbits homoclinic to resonances, with an application to chaos in a model of the forced and damped sine-Gordon equation*, Phys. D, **57** (1992), pp. 185–225.

20. Y. Li, *Chaotic Behavior in PDEs*, Ph. D. thesis, Princeton University, 1993.

21. Y. Li and D. McLaughlin, *Homoclinic orbits and chaos in discretized NLS systems Part I Homoclinic orbits*, Journal of Nonlinear Sciences, **7** 1997, pp. 211–269.

22. Y. Li and D. McLaughlin, *Morse and Melnikov functions for NLS PDEs*, Comm. Math. Phys. **162**, 1994, pp. 175–214.

23. Y. Li, D. McLaughlin, J. Shatah, and S. Wiggins, *Persistent homoclinic orbits for a perturbed nonlinear Schrödinger equation*, Comm. Pure Appl. Math **49**, 1996, pp. 1175-1255.

24. Y. Li and S. Wiggins, *Homoclinic orbits and chaos in discretized NLS systems Part II Symbolic Dynamics*, Journal of Nonlinear Sciences, **7** 1997, pp. 315–370.

25. Y. Li and S. Wiggins, *Invariant manifolds and fibrations for perturbed nonlinear Schrödinger equations*, Applied Mathematical Sciences, **128**, Springer-Verlag, New York, 1997.

26. X.B. Lin, *Homoclinic bifurcations with weakly expanding center manifolds*, Dynam. Report, **5** (1996), 99–189.

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27. D. McLaughlin, *Whiskered tori for NLS equations*, Important Developments in Soliton Theory, Springer Ser. Nonlinear Dynam., Springer, Berlin, 1993, pp. 537–558.

28. D. McLaughlin and E. Overman, *Whiskered tori for integrable PDEs: chaotic behavior in near integrable pde's*, Surveys in Applied Mathematics, vol. 1, Plenum, New York, 1995, pp. 83-203.

29. D. McLaughlin, E. Overman, S. Wiggin, and C. Xiong *Homoclinic orbits in a four dimensional model of a perturbed NLS equation*, Dynamics Reported, Dynam. Report. Expositions Dynam. Systems (N.S.), **5** 1996, pp. 190–287.

30. D. W. McLaughlin and J. Shatah, *Homoclinic orbits for PDE's*, Recent advances in partial differential equations Venice 1996, Proc. Sympos. Appl. Math., **54**, 81–299.

31. J. Moser, *Stable and Random Motions in Dynamical Systems*, Princeton University Press, Princeton, NJ, 1973.

32. J. Poschel and E. Trubowitz, *Inverse Spectral Theory*, Pure and Applied Mathematics, vol. 130, Academic Press, Inc., Boston, MA, 1987.

33. J. Shatah and C. Zeng, *Homoclinic Orbits For Perturbed sine-Gordon Equation*, Comm. Pure Appl. Math., to appear.

34. W. Strauss, *Nonlinear Invariant Wave Equations*, In G. Velo and A.S. Wightman, editors, Invariant Wave Equations, Springer-Verlag, Berlin, 1978, pp. 78.

35. R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences, **68**, Springer-Verlag, New York-Berlin, 1988.

36. S. Wiggins, *Global Bifurcations and Chaos. Analytical methods*, Applied Mathematical Sciences, vol. 73, Springer-Verlag, New York-Berlin, 1988.

FORCED OSCILLATIONS OF A REGULARIZED LONG-WAVE EQUATION AND THIER GLOBAL STABILITY

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This paper studies a dynamic system described by a regularized long-wave equation posed on a finite domain. It shows that if the forcings of the system are time periodic with small amplitude, then the system admits a unique time periodic solution which, as a limit circle, forms an *inertial manifold* for the system.

1   Introduction

This paper is concerned with initial- and two-point boundary value problem (IVBP) for the regularized long-wave equation

$$u_t + u_x + uu_x - \alpha u_{xx} - u_{xxt} = f \qquad (1.1)$$

posed on a finite interval $x \in (0,1)$ for $t \geq 0$ with initial condition

$$u(x,0) = \phi(x), \qquad x \in (0,1), \qquad (1.2)$$

and the nonhomogeneous boundary conditions

$$u(0,t) = h_1(t), \qquad u(1,t) = h_2(t), \qquad (1.3)$$

where $\alpha \geq 0$ is constant, and $f \equiv f(x,t)$ is a given external forcing. When α is zero, the equation (1.1) is known as the BBM (Bejamin, Bona and Mahony) equation or alternative KdV equation, which is commonly used as a mathematical model for unidirectional propagation of small-amplitude long waves in nonlinear dispersive systems. The well-posedness of the IVBP (1.1)-(1.3) has been established earlier in the works of Bejamin et al. [1], Bona and Douglis [2], and Bona and Luo [3]. Our main concerns in this paper for the IVBP (1.1)-(1.3) are the existence of its time periodic solutions and thier stability. More precisely, we study the folloiwng problems.

Suppose that the forcings f, h_1 and h_2 are time periodic functions of period $\omega > 0$. (a) Does the equation (1.1) possess a time periodic solution $u(x,t)$ satisfying the boundary conditions (1.3)? (b) What is the long time behavior of the solution of (1.1)-(1.3)?

There have been many studies on time periodic solutions of partial differential equations. For early works on this subject, see Brézis[5], Vejvoda et al.[9], Keller and Ting[7], Rabinowitz[8] and the references therein. For recent works, see Bourgain[4], Craig and Wayne[6], Wayne.[11] In particular, see Wayne[10] for a recent review on periodic solutions of nonlinear partial differential equations.

In this paper we will show that with the time periodic forcings (of period ω) of small amplitude, the system (1.1)-(1.3) does possess a unique time periodic solution $u^*(x,t)$ of period ω. It will be shown that this unique time periodic solution $u^*(x,t)$, as a *limit circle* for the system (1.1)-(1.3), is also a *global attractor*, for any $\phi \in H^1(0,1)$ satisfying the compatibility conditions

$$\phi(0) = h_1(0), \qquad \phi(1) = h_2(0) \tag{1.4}$$

the corresponding solution $u(x,t)$ of (1.1)-(1.3) satisfies

$$\|u(\cdot,t) - u^*(\cdot,t)\|_{H^1(0,1)} \leq Ce^{-\mu t} \tag{1.5}$$

for any $t \geq 0$ where $\mu > 0$ is independent of ϕ.

The paper is organized as follows. In section 2 we discuss long time behavior of the system (1.1)-(1.3) without assuming time periodicity of the forcings. The results presented in this section is more or less similar to those in Bona and Luo[3] in nature. However, the estimates established in this section are crucial in discussing time periodic solutions of the system (1.1)-(1.3). The existence of the forced oscillation and its stability analysis will be discussed in section 3.

2 Asymptotic behaviour

In this section we present the following asymptotic stability results for the IVBP (1.1)-(1.3).

Theorem 2.1 *Let $\alpha > 0$ be given. Suppose that $f \in C_b(R^+; H^{-1}(0,1))$, $\phi \in H^1(0,1)$ and $h_1, h_2 \in C_b^1(0, +\infty)$ satisfy the compatibility condition (1.4) and*

$$\varlimsup_{t \to \infty} \left(\|f(\cdot,t)\|_{H^{-1}(0,1)} + |\vec{h}(t)| + |\vec{h}'(t)| \right) < \alpha^2/4$$

with $\vec{h} = (h_1, h_2)$ and $|\vec{h}(t)| = |h_1(t)| + |h_2(t)|$. Then, there exist $\gamma > 0$ depending only on α and $T > 0$ such that the corresponding solution u of (1.1)-(1.3) satisfies

$$\|u(\cdot,t)\|_{H^1(0,1)} \leq C_T e^{-\gamma t} + C_\alpha \left(\|f\|_{C_b(T,t;H^{-1}(0,1))} + \|\vec{h}\|_{C_b^1(T,t)} \right) \tag{2.1}$$

for any $t \geq T$ where $C_\alpha > 0$ only depends on α and $C_T > 0$ depends only on
$\|\phi\|_{H^1(0,1)}$ *and* $\|f\|_{C_b(0,T;H^{-1}(0,1)}$ *and* $\|\vec{h}\|_{C_b^1(0,T)}$. *In particular, if*

$$\lim_{t \to \infty} \left(\|f(\cdot,t)\|_{H^{-1}(0,1)} + |\vec{h}(t)| + |\vec{h}'(t)| \right) = 0,$$

then

$$\lim_{t \to \infty} \|u(\cdot,t)\|_{H^1(0,1)} = 0.$$

In addition, if there exists a $\beta > 0$ such that

$$\|f(\cdot,t)\|_{H^{-1}(0,1)} + |\vec{h}(t)| + |\vec{h}'(t)| \leq Ce^{-\beta t}$$

for some constant $C > 0$ and any $t \geq 0$, then there exists $\gamma > 0$ depending on α and β such that

$$\|u(\cdot,t)\|_{H^1(0,1)} \leq C^* e^{-\gamma t}$$

for any $t \geq 0$ where $C^* > 0$ depending on $\|\phi\|_{H^1(0,1)}$, $\|f\|_{C_b(R^+;H^{-1}(0,1))}$ and $\|h\|_{C_b^1(R^+)}$.

To prove Theorem 2.1, we consider first the linear problem

$$(2.2) \qquad \begin{cases} u_t + u_x - \alpha u_{xx} - u_{xxt} = 0, & x \in (0,1), \quad t \geq 0, \\ u(x,0) = \phi(x) \quad u(0,t) = 0, \quad u(1,t) = 0. \end{cases}$$

For any $\phi \in H_0^1(0,1)$, (2.2) admits a unique solution $u = W(t)\phi \in C_b(R^+; H_0^1(0,1))$ where $\{W(t)\}_{t=0}^\infty$ is an analytic semigroup generated by the bounded linear operator A in the space $H_0^1(0,1)$ defined by

$$Af = (I - \partial_x^2)^{-1} \left(\partial_x - \alpha \partial_x^2 \right)$$

for any $f \in H_0^1(0,1)$. Here $(I - \partial_x^2)^{-1}$ is the inverse of the elliptic operator $I - \partial_x^2$ with the domain $\{g \in H^2(0,1), \quad g(0) = g(1) = 0\}$. A direct computation shows that

$$\max\{\operatorname{Re} \lambda, \quad \lambda \in \sigma(A)\} = -\alpha.$$

As a result, there exists a constant $C > 0$ such that

$$\|W(t)\phi\|_{H_0^1(0,1)} \leq \|\phi\|_{H_0^1(0,1)} e^{-\alpha t} \qquad (2.3)$$

for any $t \geq 0$ and $\phi \in H_0^1(0,1)$.

We consider next the nonhomogenous problem

$$(2.4) \qquad \begin{cases} u_t + u_x - \alpha u_{xx} - u_{xxt} = f(x,t), \\ u(x,0) = \phi(x), \quad u(0,t) = h_1(t), \quad u(1,t) = h_2(t) \end{cases}$$

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for $x \in (0,1)$ and $t \geq 0$. Let $v(x,t) = xh_2(t) + (1-x)h_1(t)$ and $w(x,t) = u(x,t) - v(x,t)$. Then w solves

$$\begin{cases} w_t + w_x - \alpha w_{xx} - w_{xxt} = f(x,t) - v_t - v_x, \\ w(x,0) = \phi(x) - v(x,0) \equiv \phi^*(x), \quad w(0,t) = 0, \quad w(1,t) = 0 \end{cases}$$

for $x \in (0,1)$ and $t \geq 0$. If $f \in L^1_{loc}(R^+; H^{-1}(0,1))$, $\phi \in H^1(0,1)$ and h_1, $h_2 \in H^1_{loc}(0,+\infty)$ satisfying the compatibility conditions (1.4), then (2.4) admits a unique solution $u \in C(R^+; H^1(0,1))$ with

$$u(x,t) = v(x,t) + W(t)(\phi^*) + \int_0^t W(t-\tau)(I - \partial_x^2)^{-1} \left(f(\cdot, \tau) - v_t(\cdot, \tau) - v_x(\cdot, \tau) \right).$$

Applying the estimate (2.3) yields the following result.

Proposition 2.1 Let $f \in C_b(R^+; H^{-1}(0,1))$, $\phi \in H^1(0,1)$ and h_1, $h_2 \in C_b^1(0,+\infty)$ satisfying the compatibility condition (1.4). Then the corresponding solution u of (2.4) satisfies

$$\|u(\cdot, t)\|_{H^1(0,1)} \leq e^{-\alpha t} \|\phi\|_X + \frac{2}{\alpha} \left(\|f\|_{C_b(0,t;H^{-1}(0,1))} + \|\vec{h}\|_{C_b^1(0,t)} \right) \quad (2.5)$$

for any $t \geq 0$. If, in addition, there exists a $\beta > 0$ such that

$$\|f(\cdot, t)\|_{H^{-1}(0,1)} + \|\vec{h}(t)\| + |\vec{h}'(t)| \leq C_1 e^{-\beta t}$$

for any $t \geq 0$, then

$$\|u(\cdot, t)\|_{H^1(0,1)} \leq C e^{-\alpha t} \|\phi\|_{H^1(0,1)} + \frac{2C_1}{\alpha} e^{-\gamma t} \quad (2.6)$$

for any $t \geq 0$ with $\gamma = min\{\alpha, \beta\}$.

Proof of Theorem 2.1: The solution u of (1.1)-(1.3) may be written as

$$u(x,t) = v(x,t) + w(x,t)$$

where v solves the linear problem (2.4) and w solves the nonlinear problem

$$\begin{cases} w_t + w_x + ww_x + (vw)_x - \alpha w_{xx} - w_{xxt} = -vv_x, \\ w(x,0) = 0, \quad w(0,t) = 0, \quad w(1,t) = 0. \end{cases} \quad (2.7)$$

Multiply the both sides of the equation in (2.7) by $2w$ and integrate with respect to x over $(0,1)$. By integration by parts,

$$\frac{d}{dt} \int_0^1 (w^2 + w_x^2) dx + 2\alpha \int_0^1 w_x^2 dx + \int_0^1 v_x w^2 dx = -2 \int_0^1 vv_x w dx.$$

Since

$$\int_0^1 |v_x w^2| dx \leq \|w(\cdot,t)\|_{L^\infty(0,1)}^2 \|v(\cdot,t)\|_{H^1(0,1)} \leq \|w_x(\cdot,t)\|_{L^2(0,1)}^2 \|v(\cdot,t)\|_{H^1(0,1)}$$

and

$$2\int_0^1 |vv_x w| dx \leq \|w_x(\cdot,t)\|_{L^2(0,1)} \|v(\cdot,t)\|_{H^1(0,1)} \leq \alpha \|w_x(\cdot,t)\|_{L^2(0,1)}^2 + \frac{1}{\alpha}\|v(\cdot,t)\|_{H^1(0,1)}^2,$$

one arrives at

$$\frac{d}{dt}\int_0^1 (w^2 + w_x^2)dx + (\alpha - \|v(\cdot,t)\|_{H^1(0,1)})\int_0^1 w_x^2 dx \leq \frac{1}{\alpha}\|v(\cdot,t)\|_{H^1(0,1)}^2.$$

By Grownall's inequality, if thre is a $T > 0$ such that

$$\delta = \inf_{T \leq t < +\infty} \|v(\cdot,t)\|_{H^1(0,1)} < \alpha, \tag{2.8}$$

then

$$\|w(\cdot,t)\|_{H^1(0,1)}^2 \leq \|w(\cdot,T)\|_{H^1(0,1)}^2 e^{-\gamma_1 t} + \frac{1}{\alpha}\int_T^t e^{-\gamma_1(t-s)}\|v(\cdot,s)\|_{H^1(0,1)}^2 ds \tag{2.9}$$

with $\gamma_1 = \alpha - \delta$ for any $t \geq T$. As for v, by Proposition 2.1,

$$\|v(\cdot,t)\|_{H^1(0,1)} \leq e^{-\alpha t}\|v(\cdot,\tau)\|_{H^1(0,1)} + \frac{2}{\alpha}\left(\|f\|_{C_b(\tau,t;H^{-1}(0,1))} + \|\vec{h}\|_{C_b^1(\tau,t)}\right)$$

for any $t \geq \tau$. By the assumptions on the forcings, one my choose $\tau > 0$ such that

$$\|f\|_{C_b(\tau,t;H^{-1}(0,1))} + \|\vec{h}\|_{C_b^1(\tau,t)} \leq \frac{\delta\alpha}{4} \tag{2.10}$$

for any $t \geq \tau$. Thus (2.8) is satisfied if $T > 0$ is chosen such that

$$e^{-\alpha T}\|v(\cdot,\tau)\|_{H^1(0,1)} \leq \frac{\delta}{2}. \tag{2.11}$$

Then one may complete easily the proof by using the assumptions and the above estimates. □

3 Forced oscillations and global stability

In this section we assume that $\alpha > 0$ and the forcings f, h_1 and h_2 are all time-periodic functions of period $\omega > 0$.

Theorem 3.1 *Suppose that h_1, $h_2 \in C_b^1(R^+)$ and $f \in C_b(R^+; H^{-1}(0,1))$ and*

$$\|f\|_{C_b(0,\omega;H^{-1}(0,1))} + \|\vec{h}\|_{C_b^1(0,\omega)} < \frac{\alpha^2}{8C_\alpha} \tag{3.1}$$

where the constant $C_\alpha > 0$ is as given in Theorem 2.1. Then the equation (1.1) admits a unique time periodic solution $u^* \in C_b(R^+; H^2(0,1))$ of period ω satisfying the boundary conditions (1.3). Moreover, there exists a $\gamma > 0$ such that for any $\phi \in H^1(0,1)$ satisfying the compatibility conditions (1.4), the corresponding solution u of (1.1)-(1.3) satisfies

$$\|u(\cdot,t) - u^*(\cdot,t)\|_{H^1(0,1)} \leq Ce^{-\gamma t}$$

for any $t \geq 0$. In other words, the set $\{u^*(\cdot,t), \ 0 \leq t \leq \omega\}$, as a limit circle, forms an inertial manifold in the space $H^1(0,1)$ for the dynamic system (1.1)-(1.3).

Proof: For the given forcing functions f, h_1 and h_2 satisfying the inequality (3.1), choose $\phi \in H^2(0,1)$ such that the compatibility conditions (1.4) are satisfied. Let $u(x,t)$ be the corresponding solution of the IVBP (1.1)-(1.3). By Theorem 2.1,

$$\varlimsup_{t \to \infty} \|u(\cdot,t)\|_{H^1(0,1)} \leq C_\alpha \left(\|f\|_{C_b(0,\omega;H^{-1}(0,1)} + \|\vec{h}\|_{C_b^1(0,\omega)} \right). \tag{3.2}$$

Moreover, it can be furtherly shown that $u(\cdot,t) \in H^2(0,1)$ for any $t \geq 0$ and the set $\{\|u(\cdot,t)\|_{H^2(0,1)}\}_{t=0}^{+\infty}$ is uniformly bounded. Let t_k be a sequence with $t_k \to \infty$ as $k \to \infty$ such that $u(\cdot,t_k)$ converges to a function $\psi \in H^2(0,1)$ weakly in $H^2(0,1)$ and strongly in $H^1(0,1)$ as $k \to \infty$. Taking ψ as an initial data in the IBVP (1.1)-(1.3) with the given forcings f and $\vec{h}$, we prove that the corresponding solution, named as $u^*(x,t)$, is a time periodic function of period ω.

First, let $v(x,t) = u(x,t+\omega) - u(x,t)$. Then the following statement holds.
Claim: *There exist $\tau > 0$ and $\gamma > 0$ such that*

$$\|v(\cdot,t)\|_{H^1(0,1)} \leq \|v(\cdot,\tau)\|_{H^1(0,1)}e^{-\gamma(t-\tau)}$$

for any $t \geq \tau$.

Indeed, because of the periodicity of f and $\vec{h}$, one sees that $v(x,t)$ solves the following linear problem with the variable coefficient $b(x,t) = u(x,t+\omega) + u(x,t)$:

$$\begin{cases} v_t + v_x + (bv)_x - v_{xxt} - \alpha v_{xx} = 0, \\ v(x,0) = \phi^*(x), \quad v(0,t) = 0, \quad v(1,t) = 0, \end{cases} \tag{3.3}$$

where $\phi^*(x) = u(x, \omega) - u(x, 0)$. Multiply the both sides of th equation in (3.3) by $2v$ and integrate over $(0, 1)$ with respect to x. Integration by parts leads to

$$\frac{d}{dt} \int_0^1 (v^2 + v_x^2)dx + 2\alpha \int_0^1 v_x^2 dx + \int_0^1 b_x v^2 dx = 0,$$

or

$$\frac{d}{dt} \int_0^1 (v^2 + v_x^2)dx + (2\alpha - \|b(\cdot, t)\|_{H^1(0,1)}) \int_0^1 v_x^2 dx < 0$$

for any $t \geq 0$, which yields that

$$\|v(\cdot, t)\|_{H^1(0,1)} \leq \|v(\cdot, \tau)\|_{H^1(0,1)} e^{\int_\tau^t -(2\alpha - \|b(\cdot, s)\|_{H^1(0,1)})ds}$$

for any $t \geq \tau \geq 0$. By Theorem 2.1, one can choose a $\tau > 0$ such that

$$2\gamma = 2\alpha - \sup_{t \geq \tau} \|b(\cdot, t)\|_{H^1(0,1)} > 0.$$

The claim is therefore true.

In particular,

$$\|u(\cdot, t_k + \omega) - u(\cdot, t_k)\|_{H^1(0,1)} \leq \|v(\cdot, \tau)\|_{H^1(0,1)} e^{-\gamma_2(t_k - \tau)}$$

for any $t_k \geq \tau$. Note that while $u(\cdot, t_k)$ converges to $\psi = u^*(\cdot, 0)$ strongly in $H^1(0, 1)$, $u(\cdot, t_k + \omega)$ converges to $u^*(\cdot, \omega)$ strongly in $H^1(0, 1)$ as $k \to \infty$. Since

$$\|u^*(\cdot, \omega) - u^*(\cdot, 0)\|_{H^1(0,1)} \leq \|u^*(\cdot, \omega) - u(\cdot, t_k + \omega)\|_{H^1(0,1)} +$$

$$+ \|u(\cdot, t_k + \omega) - u(\cdot, t_k)\|_{H^1(0,1)} + \|u(\cdot, t_k) - u^*(\cdot, 0)\|_{H^1(0,1)}$$

for any $t_k \geq \tau$, we conlude that $u^*(x, \omega) = u^*(x, 0)$ for any $x \in (0, 1)$ and $u^*(x, t)$ is a time priodic function of period ω.

To show the uniqueness, let u_1 and u_2 be such two time periodic solutions with the given forcing funcitons f and h. Let $v = u_1 - u_2$. Then v solves the linear problem (3.3) with $b = u_1 + u_2$ and $\phi^*(x) = u_1(x, 0) - u_2(x, 0)$. By Theorem 2.1,

$$\sup_{t \geq 0} \|b(\cdot, t)\|_{H^1(0,1)} < 2\alpha$$

and consequently, v decays to zero exponentially in the space $H^1(0, 1)$. Therefore $u_1(x, t) \equiv u_2(x, t)$ for any $x \in (0, 1)$ and $t \geq 0$ because they are time periodic functions.

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Finally, we show that the time periodic solution $u^*(x,t)$, as a *limit circle* in the space $H^1(0,1)$ for the system (1.1)-(1.3) is globally exponentially stable. To this end, for any given $\phi \in H^1(0,1)$ satisfying the compatibility conditions (1.4), let $u(x,t)$ be the corresponding solution. Then $w(x,t) = u(x,t) - u^*((x,t)$ solves the linear system (3.3) with $b(x,t) = u(x,t) + u^*(x,t)$ and $\phi^*(x) = \phi(x) - u^*(x,0)$. Then the same argument as that used to prove the Claim shows that there exist $\tau > 0$ and $\gamma > 0$ such that

$$\|w(\cdot,t)\|_{H^1(0,1)} \leq \|w(\cdot,\tau)\|_{H^1(0,1)} e^{-\gamma(t-\tau)}$$

for any $t \geq \tau$. The proof is complete. $\square$.

References

1. T. B. Benjamin, J. L Bona and J. J. Mahony, *Philos. Trans. Roy. Soc. London Ser. A.*, **272** (1972), 47 – 78.
2. J. L. Bona and V. A. Dougalis, *J. Math. Anal. Appl.*, **75** (1980), 503 – 522.
3. J. L. Bona and L. Luo, *Evolution Equations* ed. by G. Ferreya et al. , Marcel Dekker, Inc., (1995), 65 – 94.
4. J. Bourgain, *Geom. Funct. Anal.*, **5** (1995), 105 – 140.
5. H. Brézis, *Bull. Amer. Math. ASoc.*, **8** (1983), 409–426.
6. W. Craig and C. E. Wayne, *Comm. Pure and Appl. Math.*, **46** (1993), 1409 – 1498.
7. J. B. Keller and L. Ting, *Comm. Pure and Appl. Math.* (1966), 371 – 420.
8. P. Rabinowitz, *Comm. Pure and Appl. Math.*, **30** (1977), 31 – 68.
9. O. Vejvoda, with L. Herrmann et al., *Partial differential equations: Time periodic soluitons*, Marttinus Nijhoff Publishers, 1982.
10. C. E. Wayne, *Notices of the Amer. Math. Soc.*, **44** (1997), 895 – 902.
11. C. E. Wayne, *Comm. Math. Phys.* **127** (1990), 479 – 528.

HOPF BIFURCATION OF NONLINEAR DIFFERENTIAL EQUATIONS WITH DELAYS

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In this paper, we deal with the Hopf bifurcation points and asymptotic periodic solutions of three classes of differential equations with delays.

We consider the Hopf bifurcation of the following non-linear retarded differential equations with time delay as a parameter or with others parameter

$$\dot{x}(t) = f(x(t), x(t-r), \mu), \tag{0.1}$$

$$\dot{x}(t) = ax(t) + bx(t-r) + f(x(t), x(t-r), t), \tag{0.2}$$

$$x''(t) + ax'(t) + b\sin x(t-r) + \beta x(t-r) = p(t). \tag{0.3}$$

We can suppose that $x = 0$ is a equilibrium point of system (2.11), (0.2) and (0.3).

1 Hopf bifurcation of system (2.11)

In system (2.11), μ is regarded as a bifurcation parameter and r is a positive constant. System (2.11) can be rewritten as:

$$\dot{x}(t) = a(\mu)x(t) + b(\mu)x(t-r) + g(x(t), x(t-r), \mu) \tag{1.4}$$

where $g(x(t), x(t-r), \mu)$ is a non-linear function. If there is a small $\varepsilon > 0$ such that $x(t) \to \varepsilon x(t)$, and $x(t-r) \to \varepsilon x(t-r)$, then (2.12)is rewritten as:

$$\dot{x}(t) = a(\mu)x(t) + b(\mu)x(t-r) + 0(\varepsilon). \tag{1.5}$$

When $\varepsilon = 0$, we have:

$$\dot{x}(t) = a(\mu)x(t) + b(\mu)x(t-r) \tag{1.6}$$

which is the linear part of (2.11) at zero point. If $x(t)$ is a solution of (2.14), then $x(t) + 0(\varepsilon)$ is a solution of (2.13). Moreover, $x(t) + 0(\varepsilon)$ is a solution of (2.11). The characteristic equation of (2.14) is:

$$h(\lambda, \mu) \triangleq a(\mu) + b(\mu)e^{-\lambda r} - \lambda = 0. \tag{1.7}$$

Now, we suppose that the following conditions are satisfied:

(H_1) There are μ_0 and μ_1 such that $a(\mu_0) = -|b(\mu_0)|$ and $a(\mu_1) = 0$.

(H_2) $a(\mu)$ and $|b(\mu)|$ are monotonically increasing.

Let $\lambda = \alpha + i\beta$, where $\alpha, \beta \in R$. From (2.15) we can get:

$$[a(\mu) + b(\mu)e^{\alpha r} \cos \beta r - \alpha] + i[-b(\mu)e^{-\alpha r} \sin \beta r - \beta] = 0$$

and

$$(\alpha - a(\mu))^2 = b^2(\mu)e^{-2\alpha r} - \beta^2. \tag{1.8}$$

It is obvious that the right of (2.16) is not negative.

Lemma 1-1: If $a(\mu) < -|b(\mu)|$, then all of the roots of (2.15) have negative real parts.

Lemma 1-2 If $a(\mu) \geq 0$, then (2.15) has a root with positive real part.

Lemma 1-3: If $-|b(\mu)| < a(\mu) < |b(\mu)|$, then there will be a pure imaginary root of (2.15).

Lemma 1-4 : If $\mu < \mu_0$, then any root of (2.15) has negative real part. If $\mu = \mu_0$, then (2.15) have only zero roots and the roots with negative real part. If $\mu_0 < \mu < \mu_1$, there will be a pure imaginary root of (2.15). If $\mu u \geq m u_1$, then (2.15) has a root with positive real part.

Theorem 1: For any $\mu \in [\mu_0, \mu_1]$, we conclude that μ is a Hopf bifurcation point of (2.11) and (2.11) has a asymptotic periodic solution as:

$$x(t) = ke^{i\sqrt{b^2(\mu)-a^2(\mu)}t} + 0(\varepsilon^2)$$

where k is any constant.

Proof : From Lemma 1-4, we know that the zero of (2.11) is asymptotic stable when $\mu < \mu_0$, the zero of (2.11) is unstable when $\mu \geq \mu_1$; and (2.11) has a periodic solution $x(t) = ke^{i\sqrt{b^2(\mu)-a^2(\mu)}t}$, where k is any constant, when $\mu \in [\mu_0, \mu_1]$. Using the definition of Hopf bifurcation, we can obtain the proof.

2 Hopf bifurcation of system (0.2) and (0.3)

In system (0.2), $a, b \in R$, $r \geq 0$ and r is regarded as bifurcation parameter. Suppose $0 < a < -b$ and $br \geq -1$.

Let $t = rs$ and $y(s) = x(rs)$. Then (0.2) can be rewritten as:

$$\dot{y}(t) = ary(t) + bry(t-1) + rf(ry(t), ry(t-1), rt) \tag{1.9}$$

By using the same way of studying system (2.11), we may only consider the linear part of (1.9):

$$\dot{y}(t) = ary(t) + bry(t-1). \tag{1.10}$$

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The characteristic equation of (1.10) is:

$$H(\lambda, r) \triangleq \lambda - ar - bre^{-\lambda} = 0. \tag{1.11}$$

Let $r_0 = -\omega_0/b \sin \omega_0$, where ω_0 is the unique solution of the equation

$$\cos \omega = -\frac{a}{b}, \quad \omega \in [0, \pi],$$

and denote

$$H(\lambda, \mu) = \lambda - a(r_0 + \mu) - b(r_0 + \mu)e^{-\lambda} = 0 \tag{1.12}$$

$$H(\lambda, 0) = \lambda - ar_0 - br_0 e^{-\lambda} = 0. \tag{1.13}$$

Using the same method of article [2], we can get the following results.

Lemma 2-1 The pure imaginary eigenvalue $i\omega$ of (1.13)is the solution of

$$\begin{cases} \cos \omega = -a/b \\ \sin \omega = -\omega/br_0. \end{cases} \tag{1.14}$$

And if r_0 satisfies (1.14), there is an unique pair of pure imaginary eigenvalues in (1.13).

Lemma 2-2 Let $\alpha(\mu) + i\omega(\mu)$ is the root of (1.12) and $\alpha(0) = 0$, $\omega(0) = \omega_0$, then we have

$$\frac{d\alpha(\mu)}{d\mu}\Big|_{\mu=0} > 0.$$

Lemma 2-3 (1.12) has a root $\lambda(\mu)$ with positive real part and $0 < \text{Im}\lambda(\mu) < \frac{3}{2}\pi$.

Lemma 2-4 (1.13) have roots of $\pm\omega_0$ and other roots with negative real parts.

Theorem 2: $r = r_0$ is a Hopf bifurcation point of (0.2) and (0.2) has a asymptotic periodic solution as:

$$x(t) = ke^{i\omega_0 t} + o(\varepsilon^2),$$

where k is a constant .

In system (0.3), $r = constant \geq 0$ and $a, b, \beta > 0$. a is regarded as bifurcation parameter. As the same way of (0.2),(0.3) can be rewritten as:

$$x''(t) + arx'(t) + br^2 \sin x(t-1) + \beta r^2 x(t-1) = r^2 p(rt). \tag{1.15}$$

Moreover we can only consider the linear part of (1.15):

$$\begin{cases} \dot{x}(t) = ry(t) \\ \dot{y}(t) = -ary(t) - (b+\beta)rx(t-1). \end{cases} \tag{1.16}$$

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Let $a_0 = [(b+\beta)r\sin\omega_0]/\omega_0$, where ω_0 is an unique solution of the equation

$$\cos\omega = \omega/[(b+\beta)r^2], \quad \omega \in (0, \pi/2).$$

We obtain

Theorem 3: $a = a_0$ is a Hopf bifurcation point of (0.3) and (0.3) has a asymptotic periodic solution as :

$$\begin{cases} x(t) = 2\varepsilon\cos\omega_0 t + o(\varepsilon^3) \\ y(t) = -\frac{2\varepsilon}{r}\omega_0\sin\omega_0 t + o(\varepsilon^3), \end{cases} \tag{1.17}$$

where $0 \le t \le \frac{2\pi}{\omega_0} + o(\varepsilon^3)$.

References

1. Z. X. Zheng, *Theory of Functional Differential Equations, Anhui education publishing house*, 1994.

ROTATION NUMBER, EIGENVALUES AND THEIR APPLICATIONS

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In this paper we use the rotation number approach to the periodic and anti-periodic eigenvalues to establish a relationship between the periodic, anti-periodic eigenvalues and certain two-point eigenvalues. Then we give some applications of such a relation to stability of Hill's equations, resonance regions of parameterized Hill's equations, and nonuniform nonresonance of semilinear nonautonomous differential equations.

1 Rotation number approach to eigenvalues

Let $q(t) \in L^1(0,T)$, where $T > 0$ is fixed. Consider the following eigenvalue problems

$$\ddot{x} + (\lambda + q(t))x = 0 \qquad t \in [0,T] \tag{1.1}$$

with the periodic boundary condition (P): $x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0$, or, with the anti-periodic boundary condition (A): $x(0) + x(T) = \dot{x}(0) + \dot{x}(T) = 0$,

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or, with the two-point boundary condition $(TP)_\alpha$: $x(0)\sin\alpha + \dot{x}(0)\cos\alpha = x(T)\sin\alpha + \dot{x}(T)\cos\alpha = 0$, where $0 \le \alpha < \pi$. Note that $(TP)_\alpha$ is the Dirichlet boundary condition when $\alpha = \pi/2$ and the Neumann boundary condition when $\alpha = 0$.

It is well-known that eigenvalue problem (1.1) has a sequence of periodic eigenvalues [1,8]

$$\lambda_0^P(q) < \lambda_1^P(q) \le \lambda_2^P(q) < \cdots < \lambda_{2k-1}^P(q) \le \lambda_{2k}^P(q) < \cdots$$

Meanwhile, (1.1) has also a sequence of anti-periodic eigenvalues

$$\lambda_1^A(q) \le \lambda_2^A(q) < \cdots < \lambda_{2k-1}^A(q) \le \lambda_{2k}^A(q) < \cdots$$

Furthermore, $(1.1)+(TP)_\alpha$ has a sequence of eigenvalues

$$\lambda_0^\alpha(q) < \lambda_1^\alpha(q) < \cdots < \lambda_k^\alpha(q) < \cdots$$

Note that for the Dirichlet case, i.e. $\alpha = \pi/2$, $\lambda_0^\alpha(q)$ means $-\infty$.

The periodic and the anti-periodic eigenvalues are called characteristic values as a whole. We rewrite characteristic values as

$$\underline{\lambda}_{2k-1}(q) = \lambda_{2k-1}^A(q), \quad \overline{\lambda}_{2k-1}(q) = \lambda_{2k}^A(q),$$

$$\underline{\lambda}_{2k}(q) = \lambda_{2k-1}^P(q), \quad \overline{\lambda}_{2k}(q) = \lambda_{2k}^P(q).$$

Then they have the following order:

$$\overline{\lambda}_0(q) < \underline{\lambda}_1(q) \le \overline{\lambda}_1(q) < \cdots < \underline{\lambda}_k(q) \le \overline{\lambda}_k(q) < \cdots$$

A classical approach to eigenvalue problems $(1.1)+(P)$ and $(1.1)+(A)$ is as follows [1,8].

Let $x = \varphi_1(t; \lambda)$ be the solution of (1.1) satisfying the initial value conditions: $x(0) = 1$, $\dot{x}(0) = 0$. Let $x = \varphi_2(t; \lambda)$ be the solution of (1.1) satisfying the initial value conditions: $x(0) = 0$, $\dot{x}(0) = 1$. Define the discriminate

$$\Delta(\lambda) = \varphi_1(T; \lambda) + \dot{\varphi}_2(T; \lambda).$$

Then all characteristic values are determined by solving the following equations:

$$\Delta(\lambda) = \pm 2. \tag{1.2}$$

The existence of solutions of (1.2) follows from the Value Distribution Theorem for entire functions with fractional orders, because the discriminate function $\Delta(\lambda)$ is an entire function with order 1/2.

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In this paper, we use the rotation number approach to eigenvalue problems (1.1)+(P) and (1.1)+(A). Such an idea is also used by Johnson, Moser, Pöschel to study the (complex) spectrum of one dimensional Schrödinger operators[7].

The idea for (real) characteristic values is as follows. We always understand that $q(t)$ is extended to the whole $\mathbf{R}$ by the T-periodicity. Using the Prüfer substitution $x = r\cos\theta$, $\dot{x} = -r\sin\theta$, we get from (1.1) that θ satisfies the following equation on circle $\mathbf{S} = \mathbf{R}/2\pi\mathbf{Z}$:

$$\dot{\theta} = (\lambda + q(t))\cos^2\theta + \sin^2\theta. \tag{1.3}$$

Let $\theta(t; \theta_0, q, \lambda)$ be the solution of (1.3) satisfying $\theta(0; \theta_0, q, \lambda) = \theta_0$. Then one can define the rotation number $\rho(\lambda)$ of Eq. (1.3) as

$$\rho(\lambda) = \lim_{t\to\infty} \frac{\theta(t; \theta_0, q, \lambda)}{t}$$

(independent of θ_0).

It is known that all characteristic values can be characterized using rotation number function $\rho(\lambda)$ in the following way.

Theorem 1.1 (Johnson and Moser [7]) $\underline{\lambda}_k(q) = \min\{\lambda : \rho(\lambda) = k\pi/T\}$ for $k \in \mathbf{N}$, and $\overline{\lambda}_k(q) = \max\{\lambda : \rho(\lambda) = k\pi/T\}$ for $k \in \mathbf{Z}^+$.

Now all characteristic values can be recovered from two-point eigenvalues.

Theorem 1.2 (Zhang [13]) *For any $k \in \mathbf{N}$ and $0 \le \alpha < \pi$, we have $\underline{\lambda}_k(q) = \min_{s\in\mathbf{R}} \lambda_k^\alpha(q_s)$ and $\overline{\lambda}_k(q) = \max_{s\in\mathbf{R}} \lambda_k^\alpha(q_s)$, where $q_s(t) \equiv q(t+s)$ are translations of the potential $q(t)$. Moreover, te zeroth characteristic value can be recovered from the zeroth Neumann eigenvalues: $\overline{\lambda}_0(q) = \max_{s\in\mathbf{R}} \lambda_k^0(q_s)$.*

We remark here that Theorems 1.1 and 1.2 also hold for weighted eigenvalues [15]

$$\ddot{x} + \lambda w(t)x = 0,$$

where the weights $w(t)$ are such that $w(t) \ge 0$ for a.e. t and $\int_0^T w(t)dt > 0$. (We will write this as $w \succ 0$.)

In the following sections, we present some applications of Theorem 1.2.

2 Stability of Hill's equations

Let $q(t)$ be T-periodic and $q \in L^1(0, T)$. We consider the stability (in the sense of Lyapunov) of the Hill's equation:

$$\ddot{x} + q(t)x = 0. \tag{2.1}$$

The classical Lyapunov stability condition for Eq. (2.1) is:

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Theorem 2.1 (Lyapunov, Borg[8]) *Assume that q satisfies $q \succ 0$. Then Eq. (2.1) is stable if*

$$\|q\|_1 = \int_0^T q(t)dt \le \frac{4}{T}. \qquad (2.2)$$

The result can be explained using the weighted eigenvalues of

$$\ddot{x} + \lambda q(t)x = 0 \qquad (2.3)$$

with respect to (A) and (P). If we use also $\underline{\lambda}_k(q)$ and $\overline{\lambda}_k(q)$ to denote the weighted characteristic values of (2.3), then condition (2.2) implies that $\underline{\lambda}_1(q) > 1$, which means that 1 is in the first stability interval $(0, \underline{\lambda}_1(q))$. Consequently, (2.1) is stable.

Based on the relations in Theorem 1.2, one can use certain two-point eigenvalues, say the Dirichlet eigenvalues, to estimate $\underline{\lambda}_1(q)$ of (2.3).

Theorem 2.2 (Zhang and Li[15]) *Assume that q satisfies $q \succ 0$ and $q \in L^\alpha(0,T)$ for some $1 \le \alpha \le \infty$. Then the first weighted characteristic value has the following lower bound:*

$$\underline{\lambda}_1(q) \ge K(2\alpha^*)/\|q\|_\alpha.$$

Moreover, the inequality is strict when $\alpha = 1$. Here $K(\beta)$ is the best Sobolev constant in the following inequality

$$C\|u\|_\beta \le \|\dot{u}\|_2^2 \qquad \forall u \in H_0^1(0,T)$$

and is explicitly given by

$$K(\beta) = \begin{cases} \frac{2\pi}{\beta T^{1+2/\beta}} \left(\frac{2}{2+\beta}\right)^{1-2/\beta} \left(\frac{\Gamma(1/\beta)}{\Gamma(1/2+1/\beta)}\right)^2, & \text{if } 1 \le \beta < \infty, \\ \frac{4}{T}, & \text{if } \beta = \infty. \end{cases}$$

As a result, we have the following stability result for Eq. (2.1), which generalizes (2.2) using L^α-norms of $q(t)$.

Theorem 2.3 (Zhang and Li[15]) *Assume that q satisfies $q \succ 0$ and $q \in L^\alpha(0,T)$ for some $1 \le \alpha \le \infty$. Then Eq. (2.1) is stable if*

$$\|q\|_\alpha < K(2\alpha^*), \qquad \text{if } 1 < \alpha \le \infty, \qquad (2.4)$$

$$\|q\|_1 \le K(\infty) = 4/T, \qquad \text{if } \alpha = 1. \qquad (2.5)$$

We remark here that the result (2.4) is best possible when the L^α-norms are used.

Let us compare (2.4) with (2.2) using the Mathieu equation:

$$\ddot{x} + \lambda(1 + \varepsilon \cos t)x = 0, \tag{2.6}$$

where the parameters $\lambda > 0$ and $\varepsilon \in [-1, 1]$. By the classical condition (2.2), one can only obtain the stability of (2.6) when the parameter λ satisfies

$$0 < \lambda \leq \frac{4}{T\|w_\varepsilon\|_1} = \frac{1}{\pi^2}$$

for all $\varepsilon \in [-1, 1]$, where $w_\varepsilon(t) = 1 + \varepsilon \cos t$. Such a result is not satisfactory because the exact first stability interval is $\lambda \in (0, 1/4)$ for $\varepsilon = 0$. However, using condition (2.4), one may choose different α for different $\varepsilon \in [-1, 1]$ to obtain satisfactory results. In fact, Eq. (2.6) is stable when

$$0 < \lambda < H_1(\varepsilon) := \max_{\alpha \in [1, \infty]} \frac{K(2\alpha^*)}{\|w_\varepsilon\|_\alpha}.$$

A numerical computation shows the curve $\lambda = H_1(\varepsilon)$ is almost close to the eigenvalue curve $\lambda = \underline{\lambda}_1(w_\varepsilon)$ of (2.6) for all $\varepsilon \in [-1, 1]$.

3 Resonance pockets of parameterized Hill's equations

Suppose that $p(t)$ is 2π-periodic and $p \in L^1(0, 2\pi)$. We consider resonance regions (instability regions) of one-parameter Hill's equations:

$$\ddot{x} + (\lambda + \varepsilon p(t))x = 0 \tag{3.1}$$

in the (λ, ε)-plane. A classical result [8] says that Eq. (3.1) is instable when λ is in the following intervals:

$$(-\infty, \overline{\lambda}_0(\varepsilon p)], \quad (\underline{\lambda}_1(\varepsilon p), \overline{\lambda}_1(\varepsilon p)), \quad \cdots, \quad (\underline{\lambda}_k(\varepsilon p), \overline{\lambda}_k(\varepsilon p)), \quad \cdots \tag{3.2}$$

In general, (3.2) gives a sequence of the so-called Arnold tongues

$$R_k = \{(\lambda, \varepsilon) : \underline{\lambda}_k(\varepsilon p) < \lambda < \overline{\lambda}_k(\varepsilon p)\}$$

in the (λ, ε)-plane. This occurs for the Mathieu case $p(t) = \cos t$. However, in some cases, the resonance regions R_k may contain some closed sub-regions which form the so-called resonance pockets [1].

When $p(t)$ are two-step potentials, some interesting phenomenon happens for resonance pockets. Actually we can use the rotation number approach to give a global description to the combinatorial structure of all resonance pockets. Let $p(t)$ be 2π-periodic potential given by

$$p(t) = \begin{cases} c_1, & t \in [0, t_1), \\ c_2, & t \in [t_1, 2\pi). \end{cases} \quad (c_1 \neq c_2) \tag{3.3}$$

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Theorem 3.1 (Gan and Zhang[4]) *Suppose that $p(t)$ is given by (3.3). Then the nth resonance region R_n of (3.1) contains exactly*

$$N_n = \begin{cases} n-2, & \textit{if } (nt_1)/(2\pi) \textit{ is an integer} \\ n-1, & \textit{if } (nt_1)/(2\pi) \textit{ is not an integer} \end{cases}$$

resonance pockets in the (λ, ε)-plane.

By Theorem 3.1, for "generic" 2-step potentials $p(t)$ (namely, t_1/π is irrational), then R_n contains exactly $n-1$ resonance pockets for each $n \in \mathbf{N}$. When the square wave potential $p(t)$ (i.e., $c_1 = -1$, $c_2 = +1$, and $t_1 = \pi$) is considered, R_n contains exactly $2[(n-1)/2]$ resonance pockets for each $n \in \mathbf{N}$, where $[x]$ denotes the largest integer less than or equal to x.

We remark here that if general families of 2-step potentials which are not linearly dependent on ε are considered, one may construct infinitely many resonance pockets inside one resonance region[14].

4 Nonresonance of semilinear nonautonomous differential equations

Consider the following nonautonomous differential equation

$$\ddot{x} + f(t,x) = 0, \qquad t \in [0,T], \tag{4.1}$$

where $f(t,x) : [0,T] \times \mathbf{R} \to \mathbf{R}$ is an L^1-Carathéodory function and is semilinear in the following sense: There exist $a(t)$, $b(t) \in L^1(0,T)$ such that

$$a(t) \leq \liminf_{|x| \to \infty} \frac{f(t,x)}{x} \leq \limsup_{|x| \to \infty} \frac{f(t,x)}{x} \leq b(t)$$

uniformly in a.e. $t \in [0,T]$. We are interested in the nonresonance (solvability) of Eq. (4.1) with any class of boundary conditions considered in Section 1.

There exist in the literature many nonresonance conditions so that the boundary value problems are solvable. For example, for the periodic problem of (4.1), a well-known one is that there is some $k \in \mathbf{N}$ such that

$$(2(k-1)\pi/T)^2 \prec a(\cdot) \leq b(\cdot) \prec (2k\pi/T)^2. \tag{4.2}$$

The other boundary value problems have also the similar nonresonance conditions[2,3,10]. The conditions (4.2) are also referred as the nonresonance intervals.

We propose from the view of point of nonautonomous differential equations the nonresonance intervals should be in the following form:

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Theorem 4.1 (Zhang[13]) *Assume that there is some $k \in \mathbf{N}$ such that*

$$\lambda^*_{k-1}(a) < 0 \qquad and \qquad \lambda^*_k(b) > 0 \tag{4.3}$$

then Eq. (4.1) has at least one solution satisfying (∗), where (∗) stands the boundary condition (P), (A), or (TP)$_\alpha$.

The proof is based on coincidence degree theory[9] and the following observations: 1. The concept of Property P of Habets et al.[5,11]; 2. A generalization of one property in the spectrum theory for completely continuous linear operators to parameterized completely continuous positively homogeneous operators[11]; 3. The comparison results for eigenvalues with respect to potentials.

Conditions (4.3) are also necessary in some sense. They generalize most of existing nonresonance conditions for semilinear nonautonomous equations. For example, condition (4.2) implies that

$$\lambda^P_{2k}(a) < 0 \qquad and \qquad \lambda^P_{2k+1}(b) > 0$$

by the comparison results of eigenvalues. Thus (4.2) is a special case of (4.3).

At last we mention here that the solvability of positive periodic solutions to certain differential equations with singularities[12] is also related with the eigenvalue results described in this paper.

References

1. H. W. Broer and C. Simó, Hill's equation with quasi-periodic forcing: resonance tongues, instability pockets and global phenomena, *Bol. Soc. Bras. Mat.* **29** (1998), 253–293.

2. M. del Pino, M. Elgueta, and R. Manásevich, A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$, *J. Differential Equations* **80** (1989), 1–13.

3. A. Fonda and J. Mawhin, Quadratic forms, weighted eigenfunctions and boundary value problems for nonlinear second order ordinary differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **112** (1989), 145–153.

4. S. Gan and M. Zhang, Resonance pockets of Hill's equations with two-step potentials, Preprint, Math/Tsinghua, 1999.

5. P. Habets and G. Metzen, Existence of periodic solutions of Duffing equations, *J. Differential Equations* **78** (1989), 1–32.

6. J. K. Hale, *Ordinary Differential Equations*, Second Edition, Robert E. Krieger Publishing Co., New York, 1980.

7. R. Johnson and J. Moser, The rotation number for almost periodic potentials, *Commun. Math. Phys.* **84** (1982), 403–438.

8. W. Magnus and S. Winkler, *Hill's Equations*, Dover, New York, 1979.

9. J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS no. **40**, Amer. Math. Soc., Providence, RI, 1979.

10. J. Mawhin and J. R. Ward, Nonuniform non-resonance conditions at the two first eigenvalues for periodic solutions of forced Liénard and Duffing equations, *Rocky Mountain J. Math.* **12** (1982), 643–654.

11. M. Zhang, Nonresonance conditions for asymptotically positively homogeneous differential systems: The Fučik spectrum and its generalization, *J. Differential Equations* **145** (1998), 332–366.

12. M. Zhang, A relationship between the periodic and the Dirichlet BVPs of singular differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), 1099–1114.

13. M. Zhang, Solvability of semilinear nonautonomous differential equations, Preprint, Math/Tsinghua, 1999.

14. M. Zhang and S. Gan, Constructing resonance calabashes of Hill's equations using step potentials, to appear in *Math. Proc. Camb. Phil. Soc.*

15. M. Zhang and W. Li, Weighted eigenvalues, nonuniform nonresonance and stability, Preprint, Math/Tsinghua, 1998.

ALMOST PERIODIC SOLUTIONS FOR DIFFERENCE SYSTEMS AND LYAPUNOV FUNCTIONS

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In this work, for the linear almost periodic difference systems, we introduce a criterion for the unique existence of uniform asymptotically stable almost periodic solution. Furthermore, for the general almost periodic difference systems we offer several criteria of existence for almost periodic solutions, among which two existence theorems are established in terms of discrete Liapunov functions.

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1 Introduction

As we have known, there have been quite a few results on the existence of almost periodic solutions for ordinary and delay differential systems. In this paper, we will deal with the existence of almost periodic solutions for (ordinary) difference systems.

To this end, we will define the notion of the so-called uniformly almost periodic sequences besides the almost sequences and asymptotically almost sequences. Then for the linear almost periodic difference systems we introduce a criterion for the unique existence of uniform asymptotically stable almost periodic solution. Next, we provide a criterion of the existence of almost periodic solutions for difference systems in the general form. Furthermore, by using the discrete Liapunov functions, we establish two existence criteria for almost periodic solutions.

For convenience, the letters n, m, l, h, i, j, k, τ in the sequel always denote integers, and the relevant intervals and inequalities are discrete ones.

Definition 1.1 A sequence $x : Z \to R^k$ is called an almost periodic sequence if for any given $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that each discrete interval of length $l(\epsilon)$ contains a $\tau = \tau(\epsilon)$ such that

$$|x(n + \tau) - x(n)| < \epsilon \quad \text{for all} \quad n \in Z.$$

Definition 1.2 A sequence $x : Z \to R^k$ is called an asymptotically almost periodic sequence if

$$x(n) = p(n) + q(n),$$

where $p(n)$ is an almost periodic sequence, and $q(n) \to 0$ as $n \to \infty$.

Remark 1.1 It is easy to see that the above decomposition of asymptotically almost periodic sequence is unique.

Definition 1.3 Let $f : Z \times D \to R^k$, where D is an open set in R^k. $f(n, x)$ is said to be almost periodic in n uniformly for $x \in D$, or uniformly almost periodic for short, if for any $\epsilon > 0$ and any compact set S in D, there exists a positive integer $l(\epsilon, S)$ such that any interval of length $l(\epsilon, S)$ contains a τ for which

$$|f(n + \tau, x) - f(n, x)| < \epsilon \quad \text{for all } n \in Z \text{ and all } x \in S.$$

Lemma 1.1 $\{x(n)\}$ is an almost periodic sequence if and only if for any sequence $\{h'_k\} \subset Z$ there exists a subsequence $\{h_k\} \subset \{h'_k\}$ such that $x(n + h_k)$ converges uniformly on $n \in Z$ as $k \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.

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Lemma 1.2 *$\{f(n,x)\}$ is a uniformly almost periodic sequence if and only if for any integer sequence $\{h'_k\}$, there exists a subsequence $\{h_k\}$ of $\{h'_k\}$ such that $f(n+h_k,x)$ converges uniformly on $Z \times S$ as $k \to \infty$, where S is any compact set in D. Furthermore, the limit sequence is also a uniformly almost periodic sequence.*

Lemma 1.3 *Let $f : Z \times D \to R^k$ be almost periodic in n uniformly for $x \in D$ and continuous in $x \in D$, and $p(n)$ be an almost periodic sequence such that $p(n) \in S$ for all $n \in Z$, where S is a compact set in D. Then $f(n,p(n))$ is almost periodic in n.*

Lemma 1.4 *$\{x(n)\}$ is an asymptotically almost periodic sequence if and only if for any sequence $\{h'_k\} \subset Z$ such that $h'_k > 0$ and $h'_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{h_k\} \subset \{h'_k\}$ such that $x(n+h_k)$ converges uniformly on $n \in Z^+$ as $k \to \infty$.*

2 Linear Difference Systems

Definition 2.1 *Let $p(n)$ be an almost periodic sequence. Then its mean value is defined as*

$$\bar{p} = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} p(j).$$

It is easy to see that the above-defined mean value is well-defined. Also, it can be shown (cf.[2]) that the following lemma is valid.

Lemma 2.1 *Let $p(n)$ be an almost periodic sequence. Then there hold*

(i) $\bar{p} = \lim_{n \to \infty} \frac{1}{n} \sum_{j=m+1}^{m+n} p(j) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} p(m+j),$

(ii) $\bar{p} = \lim_{n \to -\infty} \left\lceil -\frac{1}{n} \sum_{j=n}^{-1} p(j) \right\rceil = \lim_{n \to +\infty} \left\lceil \frac{1}{n} \sum_{j=-n}^{-1} p(j) \right\rceil.$

Now consider the following linear difference system:

$$x(n+1) = A(n)x(n) + b(n), \quad n \in Z, \tag{2.1}$$

where $x(n)$, $b(n) \in R^k$, $A(n)$ is a $k \times k$ almost periodic matrix sequence, and $b(n)$ is an almost periodic vector sequence. In the sequel, we denote by $|x|$, $|X|$ the norm of vector x, and the norm of matrix X, respectively.

Now we are in a position to show the following theorem.

Theorem 2.1 *Under the above assumption, if*

$$\bar{A} = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |A(j)| < 1, \tag{2.2}$$

then (1) has a unique uniformly asymptotically stable almost periodic solution.

Sketch of Proof First of all, we can verify that

$$x_0(n) = \sum_{j=-\infty}^{n-1} \left\lceil \prod_{i=j+1}^{n-1} A(i) \right\rceil b(j) \tag{2.3}$$

is well-defined for any $n \in Z$ by Lemma 2.1 and assumption (2).

Next, it can be easily shown that (3) is a solution of (1). Moreover, by Lemma 1.1 we can see that $x_0(n)$ is almost periodic since $A(n)$ and $b(n)$ are almost periodic. Furthermore, by letting $u(n) = x(n) - x_0(n)$ we arrive at the homogeneous system

$$u(n+1) = A(n)x(n).$$

It then from the expression $u(n) = [\prod_{j=n_0}^{n} A(j)]u(n_0)$ and assumption (2) we can claim that $x_0(n)$ is uniformly asymptotically stable, which also implies the uniqueness of almost periodic solution of (1). QED.

3   General Almost Periodic Difference Systems

Consider the following almost periodic difference systems:

$$x(n+1) = f(n, x(n)), \tag{3.4}$$

where $f : Z \times S_B \to R^k$, $S_B = \{x \in R^k : |x| < B\}$, and $f(n,x)$ is almost periodic in n uniformly for $x \in S_B$ and is continuous in x.

Theorem 3.1 *Suppose that system (4) has a bounded solution $\varphi(n)$ defined on Z^+ such that $|\varphi(n)| \leq B^*$, $B^* < B$, for all $n \in Z^+$. If the solution $\varphi(n)$ is asymptotically almost periodic, then (4) has an almost periodic solution.*

Proof Since $\varphi(n)$ is asymptotically almost periodic, it has the decomposition

$$\varphi(n) = p(n) + q(n),$$

where $p(n)$ is almost periodic in n and $q(n) \to 0$ as $n \to \infty$. Let $\{\tau_k\}$ be a sequence such that $\tau_k \to \infty$ as $k \to \infty$ and $p(n + \tau_k) \to p(n)$ as $k \to \infty$. Then we have

$$\varphi(n + \tau_k) = p(n + \tau_k) + q(n + \tau_k).$$

It follows that

$$|p(n + \tau_k)| \leq |\varphi(n + \tau_k)| + |q(n + \tau_k)| \leq B^* + |q(n + \tau_k)| \quad \text{for} \quad n + \tau_k \geq 0.$$

Letting $k \to \infty$, we have $|p(n)| \leq B^*$ for all $n \in Z$. By Lemma 1.3, $f(n, p(n))$ is almost periodic in n. Since $\varphi(n)$ is a solution of (4),

$$\varphi(n+1) = f(n, p(n)) + (f(n, \varphi(n)) - f(n, p(n))).$$

It is clear that $f(n, \varphi(n)) - f(n, p(n)) \to 0$ as $n \to \infty$. On the other hand, we have

$$\varphi(n+1) = p(n+1) + q(n+1).$$

By the uniqueness of the decomposition we conclude that

$$p(n+1) = f(n, p(n)) \quad \text{for} \quad n \in Z,$$

which shows that $p(n)$ is an almost periodic solution of (4). QED.

Now by using discrete Liapunov functions, we investigate the existence of an almost periodic solution of (4) which is uniformly asymptotically stable in the whole. To this end, related to system (4), we consider the coupled system:

$$x(n+1) = f(n, x(n)), \quad y(n+1) = f(n, y(n)). \tag{3.5}$$

Theorem 3.2 *Suppose that there exists a Liapunov function $V(n, x, y)$ defined for $n \in Z^+$, $|x| < B$, $|y| < B$ satisfying the following conditions:*

(i) $a(|x-y|) \leq V(n, x, y) \leq b(|x-y|)$, where a, $b \in \mathcal{K}$ with $\mathcal{K} = \{a \in C(R^+, R^+): a(0) = 0$ and a is increasing$\}$,

(ii) $|V(n, x_1, y_1) - V(n, x_2, y_2)| \leq L[|x_1 - x_2| + |y_1 - y_2|]$, where $L > 0$ is a constant,

(iii) $\Delta V_{(5)}(n, x, y) \leq -\alpha V(n, x, y)$, where $0 < \alpha < 1$ is a constant, and

$$\Delta V_{(5)}(n, x, y) \equiv V(n+1, f(n, x), f(n, y)) - V(n, x, y).$$

Moreover, if there exists a solution $\varphi(n)$ of (4) such that $|\varphi(n)| \leq B^* < B$ for $n \in Z^+$, then there exists a unique uniformly asymptotically stable almost periodic solution $p(n)$ of (4) which is bounded by B^*.

Proof Let $\{h'_k\}$ be a sequence such that $h'_k \to \infty$ as $k \to \infty$. Set $\varphi_k(n) = \varphi(n + h'_k)$. Then $\varphi_k(n)$ is a solution of $x(n+1) = f(n + h'_k, x(n))$ through $(0, \varphi(h'_k))$. Since $f(n, x)$ is uniformly almost periodic, by lemma 1.2, there exists a subsequence $\{h_k\}$ of $\{h'_k\}$ such that $f(n + h_k, x)$ converges uniformly on $Z \times S_{B^*}$ as $k \to \infty$. For a given $\epsilon > 0$, choose an integer $k_0(\epsilon)$ so large that if $m \geq k \geq k_0(\epsilon)$,

$$(1-\alpha)^{h_k} b(2B^*) < a(\epsilon)/2 \tag{3.6}$$

and

$$|f(n + h_k, x) - f(n + h_m, x)| < \alpha a(\epsilon)/(2L) \quad \text{on } Z \times S_{B^*}. \tag{3.7}$$

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It follows from (ii) and (iii) that
$$\Delta V(n, \varphi(n), \varphi(n + h_m - h_k))$$
$$\leq L|f(n + h_m - h_k, \varphi(n + h_m - h_k)) - f(n, \varphi(n + h_m - h_k))|$$
$$-\alpha V(n, \varphi(n), \varphi(n + h_m - h_k)).$$
In virtue of (7), we have

$$\Delta V(n, \varphi(n), \varphi(n + h_m - h_k)) \leq -\alpha V(n, \varphi(n), \varphi(n + h_m - h_k)) + \alpha a(\epsilon)/2,$$

which implies that

$$V(n + h_k, \varphi(n + h_k), \varphi(n + h_m)) \leq (a(\epsilon)/2 + (1 - \alpha)^{n + h_k} b(2B^*)).$$

Hence, if $m \geq k \geq k_0(\epsilon)$, by (6) and (i) we have

$$|\varphi(n + h_k) - \varphi(n + h_m)| < \epsilon \text{ for all } n \in Z^+ \text{ if } m \geq k \geq k_0.$$

By Lemma 1.4 this shows that $\varphi(n)$ is asymptotically almost periodic, and thus system (4) has an almost periodic solution $p(n)$ which is bounded by B^* in virtue of Theorem 3.1. By using the Liapunov function $V(n, x, y)$, with the standard arguments, it is easy to show that $p(n)$ is uniformly asymptotically stable and every solution remaining in S_B approaches $p(n)$ as $n \to \infty$, which also implies the uniqueness of $p(n)$. QED.

Next, consider the following system

$$x(n + 1) = f(n, x(n)) + g(n), \tag{3.8}$$

where $f(n, x)$ is as in (4), $g(n)$ is almost periodic in n and $|g(n)| \leq M$ for all $n \in Z^+$.

Theorem 3.3 *Suppose that there exists a Liapunov function $V(n, x, y)$ defined on $Z^+ \times S_B \times S_B$ satisfying the conditions (i), (iii) in Theorem 3.2 and*

(ii)' $|V(n, x_1, , y_1) - V(n, x_2, y_2)| \leq L|(x_1 - x_2) - (y_1 - y_2)|,$

where $L > 0$ is a constant. Moreover, suppose that system (4) has a solution $\varphi(n)$ such that $|\varphi(n)| \leq C$ for $n \in Z^+$ and some constant $C > 0$, $C < B$. Then, if $a^{-1}(2LM/\alpha) + C \leq B^* < B$, where $a^{-1}(r)$ is the inverse function of $a(r)$, the system (8) has a unique uniformly asymptotically stable almost periodic solution which is bounded by B^*.

To prove this theorem, we need to consider the following auxiliary system:

$$\begin{cases} x(n + 1) = f(n, x(n)) + g(n), \\ y(n + 1) = f(n, y(n)) + g(n). \end{cases} \tag{3.9}$$

Also, we need the following Lemma (cf.[1]).

Lemma 3.1 *Under the assumptions of Theorem 3.3, if $|x_0 - \varphi(n_0)| \leq b^{-1}(LM/\alpha)$, $n_0 \geq 0$, and $|x_0| \leq B^*$, then*

$$|x(n, n_0, x_0) - \varphi(n)| \leq a^{-1}(2LM/\alpha) \quad \text{for all } n \in Z^+,$$

where $x(n, n_0, x_0)$ is a solution of (8) through (n_0, x_0).

Now we are easy to prove Theorem 3.3. In fact, in virtue of (ii)' and (iii), we have

$$\Delta V_{(9)}(n, x(n), y(n)) \leq L|g(n) - g(n)| - \alpha V(n, x(n), y(n)) = -\alpha V(n, x(n), y(n)).$$

Moreover, by Lemma 3.1, we know that there does exist a solution $x(n)$ of system (8) such that $|x(n)| \leq B^* < B$ for all $n \in Z^+$. Therefore, by applying Theorem 3.2 to system (8) we can conclude the assertion of Theorem 3.3. QED.

1. Shunian Zhang, Existence of Almost Periodic Solutions for Difference Systems, *to appear.*
2. Shunian Zhang, *Chinese Science Bulletin,* **43**, 2041(1998).

THE SIMULTANEOUS COLLISION ORBITS FOR N-BODY PROBLEMS WITH QUASIHOMOGENEOUS POTENTIALS

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For N-body problems with weak-force type quasi-homogeneous potentials in $\mathbf{R}^k(k+1 \geq N)$ we prove that the minimizing critical points of the Lagrangian action from any given initial position to the simultaneous collision position must be such orbits that the configurations of the N-bodies have similar equilateral homographic shapes.

1. Introduction and Main Results

N-body problems with quasi-homogeneous potentials ([7], [12] and [13]) are related with the motion of N point masses $m_1, \cdots, m_N$ in $R^k(k \geq 1)$ under the action of the potential $-W(q)$ given by

$$W(q) = U(q) + V(q) \tag{1.1}$$

$$U(q) = \frac{a}{2} \sum_{1 \le i \ne j \le N} \frac{m_i m_j}{|q_i - q_j|^\alpha} \tag{1.2}$$

$$V(q) = \frac{b}{2} \sum_{1 \le i \ne j \le N} \frac{m_i m_j}{|q_i - q_j|^\beta} \tag{1.3}$$

where $a, b \ge 0, a^2 + b^2 \ne 0, \alpha, \beta \ge 0, \alpha^2 + \beta^2 \ne 0, q = (q_1, \cdots, q_N)$, $q_i \in \mathbf{R^k}$.

The equations of the motion for the N-body problems with a potential $-W(q)$ are given by

$$m_i \ddot{q}_i = \frac{\partial W(q)}{\partial q_i}, \quad i = 1, \cdots, N \tag{1.4}$$

Note that $-W(q)$ is the classical Newtonian potential when $a = 0, \beta = 1$, or $b = 0, \alpha = 1$ or $\alpha = \beta = 1$, and homogeneous potentials when $a = 0$ or $b = 0$ or $\alpha = \beta$.

Coti Zelati ([5]) used variational methods to study the existence of a non-simultaneous collision periodic solution for the classical Newtonian N-body problems under some restrictions on the masses $m_i(i = 1, \cdots, N)$.

Serra-Terracini ([14]) used variational methods to study the existence of a noncollision periodic solution for 3-body problems with the classical Newtonian potential and a radial perturbation potential in $\mathbf{R^3}$.

Long and Zhang ([11]) and Chenciner and Desolneux ([4]) proved independently that the shapes of the solutions from minimizing the Lagrangian action integral on the $T/2$-antiperiodic or zero mean loop space of class $W^{1,2}(\mathbf{R/TZ}, \mathbf{R^k})$ for 3-body problems with a homogeneous potential must be the planar equilateral triangle circulars.

In this paper, we study the variational properties for simultaneous collision solutions of N-body problems with quasihomogeneous potentials. We have the following theorem.

Theorem 1.1 For N-body problems (1.1)-(1.4) in $\mathbf{R^k}(k + 1 \ge \mathbf{N})$ with $0 < \alpha < 2$ and $0 < \beta < 2$, the $W^{1,2}$ orbits which minimize the Lagrangian action from any initial position to the simultaneous collision position with all bodies colliding at the center of masses:

$$f(q) = \frac{1}{2} \int_{t_1}^{t_2} \sum_{i=1}^{N} m_i |\dot{q}_i(t)|^2 dt + \int_{t_1}^{t_2} U(q)dt,$$
$$q_i \in W^{1,2}([t_1, t_2], \mathbf{R^k}), \mathbf{q_i(t_2)} = \mathbf{0},$$
$$-\infty < t_1 < t_2 < +\infty \tag{1.5}$$

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satisfy that the configurations of the N-bodies are similar equilateral geometric graphes, and the total energy h must satisfy

$$\frac{1}{2} \sum_{i=1}^{N} m_i |\dot{q}_i(t)|^2 - U(q) = h = 0 \qquad (1.6)$$

Specially the configurations of 3 and 4-body problems in $\mathbf{R}^k(\mathbf{k} \geq \mathbf{2}$ and $k \geq 3)$ are respectivly the Lagrangian equilateral triangle and the regular tetrahedron.

2. The Proof of Theorem 1.1

Lemma 2.1 $S = \{r \in W^{1,2}([0,T], \mathbf{R}), \mathbf{r}(0) = \mathbf{0}, \mathbf{r}(t) \geq \mathbf{0}\}$ is a weakly closed subset of $W^{1,2}([0,T], \mathbf{R})$.

Lemma 2.2 Let $\alpha, \beta, U_0, V_0 > 0$, then

$$\int_0^T l(r, \dot{r}) dt = \int_0^T \left(\frac{1}{2} \dot{r}^2 + U_0 r^{-\alpha} + V_0 r^{-\beta} \right) dt$$

has lower bound and is coercive on S.

Proof On S, we take the following norm:

$$\|r\| = \left(\int_0^T \|\dot{r}\|^2 dt \right)^{1/2} + r(0) = \|\dot{r}\|_2 \qquad (2.1)$$

Hence

$$\int_0^T l(r, \dot{r}) dt \geq \frac{1}{2} \|r\|^2 \qquad (2.2)$$

Lemma 2.3 If $0 < \alpha, \beta < 2$, then

$$g(r) = \int_0^T l(r, \dot{r}) dt = \int_0^T \left(\frac{1}{2} \dot{r}^2 + U_0 r^{-\alpha} + V_0 r^{-\beta} \right) dt \qquad (2.3)$$

is weakly lower semicontinuous.

Proof Firstly, we note that $\frac{1}{2} \int_0^T \dot{r}^2 dt = \frac{1}{2} \|r\|^2$ is weakly lower semicontinuous.

Secondly, the assumptions $0 < \alpha, \beta < 2$, $r \in W^{1,2}([0,T], \mathbf{R})$ and $r \geq 0$ and the Fatou's Lemma imply that $\int_0^T \left(U_0 r^{-\alpha} + V_0 r^{-\beta} \right) dt$ is also weakly lower semicontinuous. Now by Lemma 1-3, we have that

Lemma 2.4 $g(r) = \int_0^T \left(\frac{1}{2} \dot{r}^2 + U_0 r^{-\alpha} + V_0 r^{-\beta} \right) dt$ attains its global minimum value on S.

Lemma 2.5 1°.(Sundman's inequality [15] [16]) Let

$$I = I(t) = \sum_{i=1}^{N} m_i |q_i(t)|^2 \tag{2.4}$$

$$r = r(t) = (I(t))^{1/2} \tag{2.5}$$

$$K = K(t) = \sum_{i=1}^{N} m_i |\dot{q}_i(t)|^2 \tag{2.6}$$

Then

$$K(t) \geq |\dot{r}(t)|^2 \tag{2.7}$$

2°. The inequality (2.7) takes the equality if and only if

$$\dot{q}_i = \lambda q_i, \quad i = 1, 2, \cdots, N, \tag{2.8}$$

Hence the angular momentum of the system (1.1)-(1.4) is zero:

$$C = \sum_{i=1}^{N} m_i \dot{q}_i \times q_i = 0 \tag{2.9}$$

Proof By (2.4) we have

$$\dot{I}^2 = 4 \left(\sum_{i=1}^{N} m_i q_i \cdot \dot{q}_i \right)^2 \tag{2.10}$$

$$\leq 4 \left(\sum_{i=1}^{N} m_i |q_i|^2 \right) \left(\sum_{i=1}^{N} m_i |\dot{q}_i|^2 \right) \tag{2.11}$$

$$= 4I \cdot K \tag{2.12}$$

$$K \geq \frac{1}{4} I^{-1} \dot{I}^2 = |\dot{r}|^2 \tag{2.13}$$

Inequality (2.13) takes the equality if and only if (2.11) takes the equality, which is equivalent to conditions (2.8).

Now we prove Theorem 1.1:

Proof By the invariant property for $f(q)$ under inversing the time, without loss of the generality, we can assume that the initial moment $t_1 = 0$ and momentum of inertial $I(0) = 0$ and the final moment $t_2 = T$. Then Sundman's inequality ([15] [16]) implies that the kinetic energy:

$$\frac{1}{2} K \geq \frac{1}{2} \dot{r}^2 \tag{2.14}$$

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Since the well-known fact that the regular simplex with center of mass at o achieves the minimum U_0 and V_0 of the (homogeneous of degree 0) function $I^{\alpha/2}U$ and $I^{\beta/2}V$, we have that

$$U \geq U_0 I^{-\alpha/2} = U_0 r^{-\alpha}, \quad V \geq V_0 I^{-\beta/2} = V_0 r^{-\beta}, \tag{2.15}$$

By (2.14) and (2.15) we have

$$L \equiv L(q, \dot{q}) = \frac{1}{2}K + W \geq \frac{1}{2}\dot{r}^2 + U_0 r^{-\alpha} + V_0 r^{-\beta} = l(r, \dot{r}) = l \tag{2.16}$$

This says that at each instant, the Lagrangian $L(q, \dot{q})$ is greater or equal to the Lagrangian $l(r, \dot{r})$ of the Kepler problem on the line corresponding to the positive function $r(t)$.

Moreover, as L and l are positive, equality $\int_0^T L dt = \int_0^T l dt$ with the absolute minimum value (with fixed time T) can be achieved only in case at each time:

(1°) Sundman inequality is an equality (with zero angular momentum) and the path is "homothetic", ending in total collision at the center of mass.

(2°) the fixed configuration realises the minimum U_0 and V_0 of $I^{\alpha/2}U$ and $I^{\beta/2}V$, which implies it is a regular simplex.

That such a minimizing path must be a solution amounts to saying that Euler- Lagrange equations of the action functional for the problem on the line are the equations of motion

It remains to prove that among all keplerian motions on the line from $t_1 = 0$ to $t_2 = T$, the parabolic one, that is, the one with energy

$$h = \frac{1}{2}\dot{r}^2 - U_0 r^{-\alpha} - V_0 r^{-\beta} = 0 \tag{2.17}$$

minimizes the action $\int_0^T l(r.\dot{r})dt$. By Lemma 1-4, we know that there is a $r_0 \in S$ such that $\int_0^T l(r_0.\dot{r}_0)dt = \inf_{r \in S} \int_0^T l(r.\dot{r})dt.$

(i) $r(T) \neq 0$. We can restrict to motions where $r(t)$ increases from 0 to 1 when t goes from 0 to T, which allows to replace the time by r in the integration. Taking the energy h as a parameter, we have that

$$\begin{aligned}
g(h) &= \int_0^T l dt \\
&= \int_0^T (h + 2U_0 r^{-\alpha} + 2V_0 r^{-\beta})dt \\
&= \int_0^1 (h + 2U_0 r^{-\alpha} + 2V_0 r^{-\beta}) \\
&\quad \times (2h + 2U_0 r^{-\alpha} + 2V_0 r^{-\beta})^{-1/2}dt
\end{aligned} \tag{2.18}$$

Set $g'(h) = 0$, we have $h = 0$. Note $g''(0) > 0$, hence $\int_0^T l\,dt$ attains the minimum if and only if $h = 0$.

(ii) $r(T) = 0$ but $r(\widetilde{t}) \neq 0$ for some $\widetilde{t} \in [0, T]$, then

$$g(h) = \int_0^T l\,dt = \int_0^{\widetilde{t}} l\,dt + \int_{\widetilde{t}}^T l\,dt \tag{2.19}$$

Using the result of case (i), we can also prove the same result $h = 0$.

Comment: 1° If one drops the restriction $k \geq N - 1$ on the dimension, everything goes through except that the regular N-simplex does not embed anymore in the ambient space $\mathbf{R}^k$, so that the minimum U_0 and V_0 of the function $I^{\alpha/2}U$ and $I^{\beta/2}V$ are realized by central configurations which we are unable to compute explicitly.

2° For the planar four-body problem with four equal masses, by the above arguments and the results of Albouy ([1], [2]), we have that the minimizer of (1.5) must satisfy that the configuration of the four bodies are similar squares and the (1.6) holds.

references

1. A. Albouy, Symétrie des configurations centrales de quatre corps, *C. R. Acad. Sci. Paris Sér. I Math.* **320**(1995), 217-220.

2. A. Albouy, The symmetric central configurations of four equal masses, *Contemp. Math. AMS.* ,**198**(1996), 131-135.

3. V. Arnold, V. Kozlov and A. Neishtadt, *Dynamical Systems, III, Mathematical aspects of classical and celestial mechanics* (Springer, Berlin, 1988).

4. A. Chenciner and N. Desolneux, Minima de l'intégrale d'action et équilibres relatifs de n corps, *C. R. Acad. Sci. Paris, Sér. I Math.* **326**(1998), 1209-1212.

5. V. Coti Zelati, The periodic solutions of N-body type problems, *Ann. IHP. Anal. nonlinéaire* **7**(1990), 477-492.

6. M. Degiovanni and F. Giannoni, Dynamical systems with Newtonian type potentials, *Ann. Scuola Norm. Super. Pisa* **15**(1989), 467-494.

7. F. Diacu, Near-collision dynamics for particles systems with quasihomogeneous potentials, *J. Diff. Equ.* **128**(1996), 58-77.

8. L. Euler, De motu rectilineo trium corpörum se mutuo attrahentium, *Novi. Comm. Acad Sci. Imp. Petropll* (1767),145-151.

9. G. Hardy, J. Littlewood and G. Polya, *Inequalities*, (2nd ed. University Press, Cambridge, 1952).

10. J. Lagrange, Essai sur le probléme des trois corps, *Ouvres* **3**(1783), 229-331.

11. Y. Long and S. Q. Zhang, Geometric characterization for variational minimization solutions of the 3-body problm, *Acta Math. Sinica*, to appear.

12. G. Manev, La gravitation et l'énergie au zéro, *Comptes Rendus* **178**(1924), 2159-2161.

13. E. Pérez-Chavela and L. V. Vela-Arévalo, Triple Collision in the quasi-homogeneous collinear three-body problem, *J. Diff. Equ.* **148**(1998), 186-211.

14. E. Serra and S. Terracini, Collisionless periodic solutions to some three-body problems, *Arch. Rational Mech. Anal.* **120** (1992), 305-325.

15. C. Siegel and J. Moser, *Lectures on celestial mechanics*, (Springer, Berlin, 1971).

16. K. F. Sundman, Memoire sur le problem des trois corps, *Acta Math.* **36**(1913), 105-179.

17. A. Wintner, *Analytical foundations of celestial mechanics*, (Princeton University Press, Princeton, 1941).

18. S. Q. Zhang, Variational minimizing properties for N-body problems, in *dedicated to the memory of Professor Liao Shantao*, to appear

19. S. Q. Zhang, The periodic solutions for N-body problems, Submitted.

INVARIANT CURVES FOR A DIFFERENTIAL EQUATION WITH DELAY

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In order to understand the dynamics of a second order delay differential equation with piecewise constant argument, we study nonlinear invariant curves of its induced planar mapping by reducing to a functional equation.

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1 Introduction

A special form of functional differential equation with variable delays is the equation with a piecewise constant argument (EPCA for short)

$$\frac{d^k}{dt^k}x(t) + g(x([t])) = 0, \quad t \in \mathbf{R}, x \in \mathbf{R} \tag{EPCA$_k$}$$

where $[t]$ denotes the greatest integer less than or equal to t and $g : \mathbf{R} \to \mathbf{R}$ is piecewise continuous. Many nice results [1]-[3] on existence of solutions and on their periodicity and oscillation were given.

Like other delay differential equations, sometimes (EPCA$_k$) may display a complicated dynamics with orbits in an infinite-dimensional phase space. In fact, Li-Yorke chaos occurs in (EPCA$_1$) for $g(x) = x^2 + (1-\mu)x$ where $\mu > 3.75$, because letting $x_n := x(n) = x([t])$, $t \in [n, n+1)$, and integrating (EPCA$_1$) on $[n, n+1)$ deduce a logistic map $x_{n+1} = \mu x_n(1-x_n)$. In contrast with (EPCA$_1$) a modified (EPCA$_2$), where $[t]$ is replaced by t, is a planar Hamiltonian system displaying simple dynamical behaviors. This suggests researching nonlinear dynamics of at least (EPCA$_2$).

Similarly to (EPCA$_1$), let $x_n := x(n) = x([t])$, $x'_n := x'(n) = x'([t])$, $t \in [n, n+1)$. By integrating (EPCA$_2$) twice on $[n, n+1)$ we get

$$x'(t) = -g(x_n)(t - n) + x'_n, \tag{1.1}$$

$$x(t) = -\frac{1}{2}g(x_n)(t - n)^2 + x'_n(t - n) + x_n. \tag{1.2}$$

Letting $t \to (n+1)_-$ we have $x'_{n+1} = -g(x_n) + x'_n$, $x_{n+1} = -\frac{1}{2}g(x_n) + x'_n + x_n$. It follows that $x_{n+2} = 2x_{n+1} - x_n - \frac{1}{2}(g(x_{n+1}) + g(x_n))$, a difference equation, which induces a planar mapping $G : \mathbf{R}^2 \to \mathbf{R}^2$, $G(x, y) = (y, 2y - x - \frac{1}{2}(g(y) + g(x)))$. If G has an invariant curve $\Gamma : y = f(x)$, the system can be reduced further to a one-dimensional iteration $x_{n+1} = f(x_n)$ on Γ and the complexity of (EPCA$_2$) is linked to the nonlinearity of f. Clearly such invariant curves can be obtained by solving the functional equation

$$f(f(x)) = 2f(x) - x - \frac{1}{2}(g(f(x)) + g(x)), \quad x \in \mathbf{R}. \tag{FE}$$

This paper gives an exposition to joint works [4]-[5] with C.T.Ng, showing existence of nonlinear invariant curves for the induced mapping G of (EPCA$_2$) by discussing (FE), and give an answer to the question [4] of analytic extension.

2 Analytic Invariant Curves

By arranging terms and letting $h(x) := \frac{1}{4}(3x - f(x) - \frac{1}{2}g(x))$, equation (FE) is simplified to

$$h(f(x)) + h(x) = x. \tag{2.3}$$

Then for an analytic form of f we can answer what g guarantees the system has the invariant curve $\Gamma : y = f(x)$ by discussing existence of h in (3).

Theorem 2.1 For $f(x) = ax + b$ equation (3) on a domain in $\mathbf{R}$ with at least two points has a solution $h(x) = cx + d$ if and only if $c(a+1) = 1, \ cb + 2d = 0$.

The proof is simple. However, we are more concerned with nonlinear f. Unfortunately,

Theorem 2.2 If f is a polynomial function with degree $f \geq 2$, then equation (3) has no polynomial solution h on infinite domains in $\mathbf{R}$.

Proof Let $n = $ degree $f \geq 2$ and let h be a nonconstant polynomial of degree $m \geq 1$. Then $h(f(x)) + h(x) - x$ is a polynomial of degree mn, and cannot carry more than mn roots. Thus (3) can hold only for at most finitely many x values.

Despite of this, it is possible to find a solution of convergent power series.

Theorem 2.3 If $f(x) = x^2$, then equation (3) has a unique analytic solution $h(x)$ on the interval $(-1, 1)$ given by

$$h(x) = x + \sum_{k=1}^{\infty} (-1)^k x^{2^k}. \tag{2.4}$$

Proof Let $h(x) = \sum_{n=1}^{\infty} a_n x^n$ in (3) and solve for the coefficients a_n, and then we get (4) uniquely. Clearly the radius of convergence for the series is 1, so the equation holds for all $x \in (-1, 1)$.

A question [4] is whether the solution (4) on $(-1, 1)$ can be extended analytically to $\mathbf{R}$. Thanks to a private letter from Prof.Walter Rudin, we know $\lim_{x \to 1_-} x + \sum_{k=1}^{\infty} (-1)^k x^{2^k}$ does not exist. In fact, under the hypotheses that $\{\lambda_n\}$ is an increasing sequence of positive numbers which satisfies Hadamard's gap condition $\liminf_{n \to \infty} (\lambda_{n+1}/\lambda_n) > 1$ and that the series $\xi(x) = \sum_{n=1}^{\infty} a_n x^{\lambda_n}$, where all $a_n, \ n = 1, 2, ...$, are real numbers, converges for $0 \leq x < 1$, the "high indices theorem" [6] asserts that the series $\sum_{n=1}^{\infty} a_n$ converges if $\xi(x)$ tends to a finite limit as $x \to 1$. Hence the answer for the question[4] is negative.

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3 Quadratic Invariant Curves

We are specially interested in quadratic invariant curves, the simplest form of nonlinearity. We need solve (3) for given $f(x) = ax^2 + bx + c$, $a \neq 0$. Taking $z = a(x + \frac{b}{2a})$ and $k(z) := ah(\frac{1}{a}z - \frac{b}{2a}) + \frac{b}{4}$ in (3) we obtain an equivalent equation

$$k(p(z)) + k(z) = z, \quad p(z) = z^2 + C. \tag{3.5}$$

where $C = \frac{1}{4}(2b - b^2) + ac$.

For each $x \in \mathbf{R}$ and $n \in \mathbf{Z}_+$ we denote by $x^{(n)} = p^{(n)}(x)$ the image of x under the n-th iterate of p. Negative n may also be used, but where it is used we shall indicate which injective branch of p is taken. We have the following results [4]-[5].

Theorem 3.1 When $C > 1/4$, each initial function $k : [0, C) \to \mathbf{R}$ can be extended uniquely to a solution k for (5) on $[0, \infty)$ by the relation

$$k(x) = x - x^{(1)} + \dots + (-1)^{n-1}x^{(n-1)} + (-1)^n k(x^{(n)}), \quad \forall x \in [0, C), \ n \geq 0. \tag{3.6}$$

This can be further extended uniquely to a solution for (5) on $\mathbf{R}$ by the relation

$$k(t) = k(-t) + 2t, \quad \forall t < 0. \tag{3.7}$$

Moreover, k is continuous on $[0, \infty)$ iff the initial function k is continuous on $[0, C)$ and that $k(x) \to -k(0)$ as $x \to C$ from the left. Its extension to $\mathbf{R}$ is then continuous.

Theorem 3.2 When $C = 1/4$, (i) each initial function $k : [0, C) \to \mathbf{R}$ can be extended uniquely to a solution k on $[0, 1/2)$ by (6); (ii) each initial function $k : [c_1, c_2) \to \mathbf{R}$, where $c_1 > 1/2$ is arbitrarily fixed and $c_2 = p(c_1)$, can be extended uniquely to a solution k on $[1/2, \infty)$ by (6) and by that

$$k(x) = x^{(-1)} + \dots + (-1)^{n-1}x^{(n)} + (-1)^n k(x^{(n)}), \quad \forall x \in [c_1, c_2), \ n \leq -1. \tag{3.8}$$

(iii) $k : [0, \infty) \to \mathbf{R}$ satisfies (5) iff its restrictions to $[0, 1/2)$ and to $[1/2, \infty)$ are given by the above (i) and (ii) respectively and $k(1/2) = 1/4$; and (iv) each solution $k : [0, \infty) \to \mathbf{R}$ can be extended uniquely to a solution on $\mathbf{R}$ by (7). Moreover, (5) has many piecewise continuous solutions but only one continuous at $1/2$.

Theorem 3.3 When $0 < C < 1/4$, p has two fixed point x_1 and x_2 with $0 < x_1 < x_2 < \infty$. Each initial function on $[0, C)$ extends uniquely to a solution on $[0, x_1)$; each initial function on $[c_1, c_2)$, where $c_1 \in (x_1, x_2)$ is chosen arbitrarily and $c_2 = p(c_1)$, extends uniquely to a solution on (x_1, x_2); each initial function

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on $[c_3, c_4)$, where $c_3 \in (x_2, \infty)$ is chosen arbitrarily and $c_4 = p(c_3)$, extends uniquely to a solution on (x_2, ∞); $k(x_1) = x_1/2$ and $k(x_2) = x_2/2$ extend solutions on $[0, x_1) \cup (x_1, x_2) \cup (x_2, \infty)$ uniquely to solutions $[0, \infty)$, and relation (7) extends the latter to solutions on $\mathbf{R}$.

Theorem 3.4 When $C = 0$, the fixed points of p are $x_1 = 0$ and $x_2 = 1$. The construction of solutions for (5) is basically the same as in the case of $0 < C < 1/4$.

Theorem 3.5 When $-3/4 \leq C < 0$, p has two fixed points x_1 and x_2 with $x_1 < 0 < x_2$. (i) Each initial function $k : (p(C), 0] \to \mathbf{R}$ can be extended to $(x_1, 0]$ by that

$$k(x) = k(x^{(2n)}) + \sum_{i=0}^{n-1} (x^{(2i)} - x^{(2i+1)}), \quad \forall x \in (p(C), 0], \; n \geq 1, \qquad (3.9)$$

and to $[C, p^{(2)}(C))$ by the relation $k(x) = x - k(x^{(1)})$, $\forall x \in (p(C), 0]$, and subsequently to $[C, x_1)$ by (9); (ii) $k : [C, 0] \to \mathbf{R}$ satisfies (5) iff its restrictions to $(x_1, 0]$ and to $[C, x_1)$ are given by (i) and $k(x_1) = x_1/2$; (iii) each solution $k : [C, 0] \to \mathbf{R}$ can be extended uniquely to a solution on $(-x_2, x_2)$ by (9); (iv) each solution $k : [c_1, c_2) \to \mathbf{R}$ where $c_1 > x_2$ is arbitrarily fixed and $c_2 = p(c_1)$, can be extended uniquely to a solution k for (5) on (x_2, ∞) by (6) for $n \geq 1$ and by (2.10) for $n \leq -1$; (v) each solution $k : (x_2, \infty) \to \mathbf{R}$ can be extended uniquely to a solution on $(-\infty, -x_2)$ and to $(-\infty, -x_2) \cup (x_2, \infty)$ by (7) for all $t < -x_2$; (vi) $k : \mathbf{R} \to \mathbf{R}$ satisfies (5) iff its restrictions to $(-x_2, x_2)$ and to $(-\infty, -x_2) \cup (x_2, \infty)$ are given by above (iii), (iv) and (v), and $k(x_2) = x_2/2$ and $k(-x_2) = -(3/2)x_2$.

Theorem 3.6 When $C < -3/4$, equation (5) has no solution k on $\mathbf{R}$.

For continuity in the above cases we have more conclusions [4]-[5].

4 General Invariant Curves

In this section we are back to equation (FE) and give existence of general non-linear invariant curves. Let $C_b^1(\mathbf{R}) = \{f : \mathbf{R} \to \mathbf{R} | f \text{ is } C^1 \text{ smooth and } \|f\|_1 < \infty\}$ where $\|f\|_1 = \|f\| + \|f'\|$, $\|f\| = \sup_{x \in \mathbf{R}} |f(x)|$ and f' is the derivative of f. Let $\Phi(M, \Upsilon, \upsilon; I) = \{\phi \in C_b^1(\mathbf{R}) | \|\phi'\| \leq M, \; \mathrm{Lip}(\phi') \leq \Upsilon, \text{ and } |\phi'(x) - \phi'(y)| \geq \upsilon |x - y|, \; \forall x, y \in I\}$, where M, Υ, υ are nonnegative constants, $\upsilon \leq \Upsilon$, and I is a closed interval in $\mathbf{R}$.

Theorem 4.1 If $g : \mathbf{R} \to \mathbf{R}$ is C^1 smooth, and $g(x) = -2x + \phi(x)$, $\forall x \in \mathbf{R}$, where $\phi \in \Phi(M_1, \Upsilon_1, \upsilon_1; I)$, then within $\Phi(M_2, \Upsilon_2, \upsilon_2; I)$ equation (FE) has a unique solution f, provided that the nonnegative constants $M_1, \Upsilon_1, \upsilon_1, M_2, \Upsilon_2$

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and v_2 satisfy

$$2M_2^2 + (M_1 - 6)M_2 + M_1 \le 0, \tag{4.10}$$

$$(2M_2^2 + 2M_2 + M_1)\Upsilon_2 + M_2^2\Upsilon_1 \le \min\{6\Upsilon_2 - \Upsilon_1, v_1 - 6v_2\}, \tag{4.11}$$

$$\gamma := \frac{1}{6}\max\{2 + 2M_2 + M_1 + 2M_2\Upsilon_2 + M_2\Upsilon_1, 4M_2 + M_1\} < 1. \tag{4.12}$$

It is easy to prove that the conditions (4.10-12) are not self-contradictory. The structure of Φ_2 implies the solution f obtained in theorem 4.1 must be nonlinear on I when I is nondegenerated.

Proof of Theorem 4.1 Clearly $\Phi(M_1, \Upsilon_1, v_1; I)$ and $\Phi(M_2, \Upsilon_2, v_2; I)$, denoted by Φ_1 and Φ_2 respectively for short, are closed subsets of the Banach space $C_b^1(\mathbf{R})$. Let the mapping $T : \Phi_2 \to C_b^1(\mathbf{R})$ be defined by

$$Tf(x) = \frac{1}{3}f(f(x)) + \frac{1}{6}(\phi(f(x)) + \phi(x)), \quad f \in \Phi_2. \tag{4.13}$$

Let $(Tf)'(x)$ denote the derivative of $Tf(x)$. By (10) and (11) we have

$$\|(Tf)'\| \le \frac{1}{3}M_2^2 + \frac{1}{6}(M_1M_2 + M_1) \le M_2; \tag{4.14}$$

$$|(Tf)'(x) - (Tf)'(y)| \le \frac{1}{6}((2M_2^2 + 2M_2 + M_1)\Upsilon_2 + (M_2^2 + 1)\Upsilon_1)|x - y|$$

$$\le \Upsilon_2|x - y|, \quad \forall x, y \in \mathbf{R}; \tag{4.15}$$

$$|(Tf)'(x) - (Tf)'(y)| \ge \frac{1}{6}(v_1 - ((2M_2^2 + 2M_2 + M_1)\Upsilon_2 + M_2^2\Upsilon_1))|x - y|$$

$$\ge v_2|x - y|, \quad \forall x, y \in I. \tag{4.16}$$

This shows that T maps Φ_2 into itself. Furthermore, for any $f_1, f_2 \in \Phi_2$, by (12),

$$\|Tf_1 - Tf_2\|_1 = \|Tf_1 - Tf_2\| + \|(Tf_1)' - (Tf_2)'\|$$

$$\le \frac{1}{6}(2 + 2M_2 + M_1 + 2M_2\Upsilon_2 + M_2\Upsilon_1)\|f_1 - f_2\|$$

$$+ \frac{1}{6}(4M_2 + M_1)\|(Tf_1)' - (Tf_2)'\|$$

$$\le \gamma\|f_1 - f_2\|_1. \tag{4.17}$$

Hence T is a contraction on Φ_2 and has a unique fixed point $f \in \Phi_2$. By (13), f is a C^1 solution of equation (FE).

For example, take $M_1 = 1, M_2 = 1/4, \Upsilon_1 = 6, \Upsilon_2 = 102/70, v_1 = 5, v_2 = 1/6$. The conditions (10-12) are satisfied. Clearly, the function $\phi(x)$, defined

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by $\phi(x) = -x - 8/5$ as $x \in (-\infty, 4/5)$, $\phi(x) = \frac{5}{2}x^2 - 5x$ as $x \in [4/5, 6/5]$ and $\phi(x) = x - 18/5$ as $x \in (6/5, +\infty)$, belongs to $\Phi(M_1, \Upsilon_1, \upsilon_1; I)$, where $I := [4/5, 6/5]$. By theorem 4.1, equation (FE) with $g(x) = -2x + \phi(x)$ has a unique solution $f \in \Phi(M_2, \Upsilon_2, \upsilon_2; I)$.

Acknowledgments

References

1. K.L. Cooke and J. Wiener, J. Math. Anal. Appl. **99**, 256(1984).
2. A. R. Aftabizadeh and J.M. Xu, Proc.Am.Math.Soc. **99**, 673(1987).
3. I. Gyori and G. Ladas, Diff. and Integral Eqns. **2**, 123(1989).
4. C.T. Ng and Weinian Zhang, J. Difference Eqns. Appl. **3**(2), 147(1997).
5. C.T. Ng and Weinian Zhang, Quadratic Invariant Curves for a Planar Mapping, Math.Res.Report, University of Waterloo, Canada (1998).
6. G.H.Hardy and J.E.Littlewood, Proc.London Math.Soc. **25**, 219 (1926).

NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF SOME SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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In this paper, some necessary and sufficient conditions for a certain kind of second order nonlinear functional differential equations with various type of deviating arguments being oscillatory are given. These results improve and generalize some known results.

1 Introduction

In paper [1], by using convex cone method, we have studied the asymptotic behavior for second order nonlinear functional differential equation

$$[p(t, x(h_0(t)))g(x'(t - \tau))]' + f(t, x(h_1(t)), \cdots, x(h_n(t))) = 0 \qquad (1.1)$$

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and have given some results for classification and for necessary and sufficient conditions of existence of nonoscillatory solutions. This paper is a continuation of paper [1]. Some necessary and sufficient conditions for Eq.(1) being oscillatory will be given. These results improve and generalize some known results of L.Wen, T.Kusano, H.Onose and others (see [2-4]) for second order functional differential equations

$$[r(t)g(x^{'}(t))]^{'} + f(t, x(h_1(t)), \cdots, x(h_n(t))) = 0 \qquad (1.2)$$

or its special cases.

2 Some Hypotheses and Lemmas

In this paper, we assume that the following hypotheses hold for Eq.(1):

(H$_1$) $p(t,x) \in C[[t^*, +\infty) \times R, (0, +\infty)]$ and there exist

$$\phi, \quad \psi \in C[[t^*, +\infty), (0, +\infty)]$$

such that

$$\phi(t) \leq p(t,x) \leq \psi(t), \quad \text{for} \quad x \in R, \quad t \geq t^*$$

and

$$\lim_{t \to +\infty} \frac{\phi(t)}{\psi(t)} = a_0 > 0.$$

(H$_2$) τ is a real number, $h_i(t) \in C[t^*, +\infty)$, and $h_i(t) \to +\infty$ as $t \to +\infty$ $(i = 0, 1, 2, \cdots, n)$.

(H$_3$) function $g(u) \in C(-\infty, +\infty)$ is strictly monotone increasing and $g(0) = 0, g(u) \to \pm\infty$ as $u \to \pm\infty$.

(H$_4$) there exist positive numbers β and k, such that

$$g^{-1}(uv) \leq \beta g^{-1}(u)g^{-1}(v), \quad g^{-1}(-u) = -kg^{-1}(u),$$

for all $u > 0$ and $v > 0$, where $g^{-1}(u)$ is the inverse function of $g(u)$.

(H$_5$) $f(t, x_1, \cdots x_n) \in C[[t^*, +\infty) \times R^n, R]$ and $x_i f(t, x_1, \cdots, x_n) > 0$ if $x_1, x_2, \cdots, x_n$ have same signs $(i = 1, 2, \cdots, n)$.

(H$_6$) $\int^{+\infty} g^{-1}[\frac{1}{\phi(t)}]dt < +\infty$, where $\phi(t)$ is the same as in (H$_1$).

In addition, the following conditions for $g^{-1}(u)$ will be used:

α-**condition** $g^{-1}(u)$ is said to satisfy α-condition, if there exists a constant $\alpha > 0$ such that

$$g^{-1}(uv) = \alpha g^{-1}(u)g^{-1}(v), \quad \text{for} \quad u, v \in R.$$

γ-**condition** $g^{-1}(u)$ is said to satisfy γ-condition, if there exist constants $\gamma > 0$ and $u_0 > 0$, such that

$$g^{-1}(u) \geq \gamma u, \quad \text{for} \quad u \geq u_0.$$

σ-**condition** $g^{-1}(u)$ is said to satisfy σ-condition means that for given constant $\sigma > 0$, $1/g^{-1}(u^\sigma)$ has a positive primitive function $\Phi(u)$ for $u > 0$.

From paper [1], on the basis of the above hypotheses, we easily obtain some lemmas as follows.

Lemma 1 *Suppose that function $u(t) > 0$ is continuous, then there exist constants $m_1 > 0, m_2 > 0$ and sufficiently large $t_0 \geq t^*$, such that*

$$m_1 g^{-1}\left[\frac{u(t)}{\phi(t)}\right] \leq g^{-1}\left[\frac{u(t)}{\psi(t)}\right] \leq m_2 g^{-1}\left[\frac{u(t)}{\phi(t)}\right], \quad for \quad t \geq t_0 \qquad (2.3)$$

Lemma 2 *If $x(t)$ is an eventually positive solution (or an eventually negative solution) of Eq.(1), and $g^{-1}(u)$ satisfies α-condition, then there exist numbers $a_1 > 0, a_2 > 0$ and $t_0 \geq t^*$ such that*

$$a_1 \rho(t) \leq x(t) \leq a_2 \quad (or \quad -a_2 \leq x(t) \leq -a_1\rho(t)) \quad for \quad t \geq t_0, \qquad (2.4)$$

where

$$\rho(t) = \int_t^{+\infty} g^{-1}\left[\frac{1}{\phi(s+\tau)}\right]ds.$$

Lemma 3 *Suppose that f is sublinear or nondecreasing with respect to every $x_i > 0$ (or $x_i < 0$) ($i = 1, 2, \cdots, n$); $g^{-1}(u)$ that satisfies α-condition has continuous derivative $q(u)$ and there is a constant $r > 0$ such that*

$$p(t, x) \equiv \phi(t) \quad for \quad |x| \leq r \quad and \quad t \geq t^*.$$

Then Eq.(1) has nonoscillatory solutions $x(t) > 0$ (or $x(t) < 0$ eventually) if and only if there exists constant $c_0 > 0$ (or $c_0 < 0$) such that

$$\int^{+\infty} |f(t, c_0\rho(h_1(t)), c_0\rho(h_2(t)), \cdots, c_0\rho(h_n(t)))|dt < +\infty \qquad (2.5)$$

holds,where $\rho(t)$ is defined as Lemma 2.

Lemma 4 *Suppose that f is sublinear or nondecreasing with respect to every $x_i > 0$ (or $x_i < 0$) ($i = 1, 2, \cdots, n$), then for any $c > 0$ (or $c < 0$), Eq.(1) has a nonoscillatory solution $x(t)$ being eventually larger than c (or $x(t) \leq c$ eventually) if and only if there exists some $c_0, c_0 \geq c > 0$ (or $c_0 \leq c < 0$) such that*

$$\int^{+\infty} g^{-1}\left[\frac{1}{\phi(t)} \int_{t_0}^t |f(s, c_0, \cdots, c_0)| ds\right] dt < +\infty \qquad (2.6)$$

holds for sufficiently large $t_0 \geq t^*$.

3 Main Results

Theorem 1 *Suppose that $g^{-1}(u)$ satisfies α-condition, γ-condition and has continuous derivative; f is strongly superlinear; $h_i(t) \geq t - \tau$ ($i = 1, 2, \cdots, n$) and there exist constants $r > 0$ and $t_0 \geq t^*$ such that $p(t, x) \equiv \phi(t)$ for $|x| \leq r$ and $t \geq t_0$, and that $\phi(t)$ and $\phi(t + \tau)$ are infinity of same order as $t \to +\infty$. Then Eq.(1) is oscillatory if and only if*

$$\int^{+\infty} |f(t, c\rho(h_1(t)), \cdots, c\rho(h_n(t)))| dt = +\infty \qquad (3.7)$$

holds for all $c \neq 0$.

Theorem 2 *Suppose that f is strongly sublinear and its strongly sublinear constant is σ, $g^{-1}(u)$ satisfies α-condition and it also satisfies σ-condition for the σ; $h_i(t) \leq t - \tau$, ($i = 1, 2, \cdots, n$). Then a necessary and sufficient condition for Eq.(1) being oscillatory is*

$$\int^{+\infty} g^{-1}\left[\frac{1}{\phi(t)} \int_{t_0}^t |f(s, c, \cdots, c)| ds\right] dt = +\infty \qquad (3.8)$$

for every $c \neq 0$.

Proof. The necessity of Theorem 3.2 is obvious by Lemma 2.4. The sufficiency will be proved as follows.

If Eq.(1) has an eventually positive solution $x(t)$, then $x(t)$ is only A_+^+-type or A_0^--type according to formula (8) and Theorem 3.1 of paper [1]. If $x(t)$ is A_+^+-type, then there exist constants $c > c_1 > 0$, such that

$$0 < c_1 \leq x(h_i(t)) < c, \quad x'(t - \tau) > 0 \quad (i = 1, 2, \cdots, n) \quad \text{for} \quad t \geq t_0.$$

Integrating Eq.(1) from t_0 to $t > t_0$, we have

$$\int_{t_0}^t F(s, x(h(s))) ds = -p(t, x(h_0(t))) g(x'(t - \tau)) + p(t_0, x(h_0(t))) g(x'(t_0 - \tau))$$

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$$\leq p(t_0, x(h_0(t_0)))g(x'(t_0 - \tau)).$$

For the sake of convenience, here and in the following the function $f(t, x(h_1(t)), \cdots, x(h_n(t)))$ is often expressed as $F(t, x(h(t)))$. Thus

$$g[\frac{1}{\phi(t)} \int_{t_0}^t F(s, x(h(s)))ds] \leq g^{-1}[p(t_0, x(h_0(t_0)))g(x'(t_0 - \tau))/\phi(t)]$$

$$= \alpha g^{-1}[p(t_0, x(h_0(t_0)))g(x'(t_0 - \tau))]g^{-1}[\frac{1}{\phi(t)}],$$

because the strongly sublinear constant of f is σ and $0 < c_1 \leq x(h_i(t)) < c$, we get

$$g^{-1}[\frac{1}{\phi(t)} \int_{t_0}^t f(s, c, \cdots, c)ds]$$

$$\leq g^{-1}[\frac{1}{\phi(t)}]g^{-1}[p(t_0, x(h_0(t)))g(x'(t_0 - \tau))]/g^{-1}[(c_1/c)^\sigma].$$

Hence, we have

$$\int^{+\infty} g^{-1}[\frac{1}{\phi(t)} \int_{t_0}^t f(s, c, \cdots, c)ds]dt$$

$$\leq (\int^{+\infty} g^{-1}[\frac{1}{\phi(t)}]dt)g^{-1}[p(t_0, x(h_0(t_0)))g(x'(t_0 - \tau))]/g^{-1}[(c_1/c)^\sigma] < +\infty,$$

which contradicts formula (8).

If $x(t)$ is A_0^- − type, then there exist t_0 and c_2, such that

$$x'(t - \tau) < 0, \quad 0 < x(h_i(t)) < c_2 \quad (i = 1, 2, \cdots, n) \quad \text{for} \quad t \geq t_0.$$

From Eq.(1) and (H_4) as well ,we know

$$\int_{t_0}^t F(s, x(h(s)))ds \leq p(t, x(h_0(t)))g(-\frac{1}{k}x'(t - \tau)).$$

So

$$g^{-1}[\frac{1}{p(t, x(h_0(t)))} \int_{t_0}^t F(s, x(h(s)))ds] \leq -\frac{1}{k}x'(t - \tau). \tag{3.9}$$

Because of f being strongly sublinear, $x(t)$ being decreasing, $h_i(t) \leq t - \tau$ $(i = 1, 2, \cdots, n)$ and $g^{-1}(u)$ satisfying α−condition, we have

$$g^{-1}[\frac{1}{p(t, x(h_0(t)))} \int_{t_0}^t F(s, x(h(s)))ds]$$

$$\geq \alpha g^{-1}[(x(t-\tau)/c_2)^\sigma]g^{-1}[\frac{1}{\psi(t)} \int_{t_0}^t f(s,c_2,\cdots,c_2)ds].$$

And from formula (10), we have

$$g^{-1}[\frac{1}{\psi(t)} \int_{t_0}^t f(s,c_2,\cdots,c_2)ds] \leq \frac{-x'(t-\tau)}{\alpha^2 k g^{-1}(c_2^{-\sigma})g^{-1}[(x(t-\tau))^\sigma]}.$$

By integrating the above formula from $t_1 \geq t_0$ to $t \geq t_1$ and the $\sigma-$condition , we have

$$\int_{t_1}^t g^{-1}[\frac{1}{\psi(\lambda)} \int_{t_0}^\lambda f(s,c_2,\cdots,c_2)ds]d\lambda$$

$$< \frac{1}{\alpha^2 k g^{-1}(c_2^{-\sigma})}[\Phi(x(t_1-\tau))] < +\infty.$$

Hence, according to Lemma 1 and the above formula, there exists a constant $m_2 > 0$, such that

$$\int_{t_1}^t g^{-1}[\frac{1}{\phi(\lambda)} \int_{t_0}^\lambda f(s,c_2,\cdots,c_2)ds]d\lambda$$

$$< \frac{m_2}{\alpha^2 k g^{-1}(c_2^{-\sigma})}[\Phi(x(t_1-\tau))] < +\infty.$$

which contradicts formula (8). Therefore Eq.(1) has not eventually positive solutions.

Similarly, Eq.(1) has not eventually negative solutions either. Hence, Eq.(1) is oscillatory. The proof of Theorem 1 is similar to the above, and omitted.

Remark Theorem 3.1 and Theorem 3.2 in this paper include respectively Theorem 6 and Theorem 7 that are the main results for oscillation of Eq.(2) in paper [2]. They also improve and generalize Theorem 4.6.2–4.6.3 in [3]. See the following example.

Example Consider the equation

$$[(t^{2/3} + \sin^2(t-2\tau))x'^{1/3}(t-\tau)]'$$

$$+t^{-2/3}(x^{1/3}(t-3\tau) + x^{1/5}(t-4\tau) + x^{1/7}(t-5\tau)) = 0 \qquad (3.10)$$

where constant $\tau > 0$. It is obviously that

$$f(t,x_1,x_2,x_3) = t^{-2/3}(x_1^{1/3} + x_2^{1/5} + x_3^{1/7}),$$

$$\phi(t) = t^{2/3}, \quad \psi(t) = 1 + t^{2/3}, \quad g(u) = u^{1/3}, \quad g^{-1}(u) = u^3$$

satisfy all the conditions of Theorem 3.2. Therefore Eq.(11) is oscillatory. But other known results such as in paper [2-4] are not available to Eq.(11).

References

1. X.Wang and Z.Zhang, *Proceedings of International Conference on Dynamical Systems and Differential Equations*, Jiaotong University Press, Shanghai,158 (1998).
2. L.Wen, *Scientia Sinica*, **7**(Ser.A), 681(1986).
3. G.S.Ladde, V.Laksmikantham and B.G.Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, (Marcel Dekker, New York, 1987).
4. Z.Zheng, *Theory for Functional Differential Equations* (in Chinese), (Anhui Education Press, Hefei,1994).

THE INTEGRABILITY THEOREMS OF RICCATI DIFFERENTIAL EQUATION WITH APPLICATIONS

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In this paper,the new integrable theorems of Riccati equation are presented by invariant of Riccati equation.

1   Introduction

In 1841,J.Liouville[1] proved Riccati equation:

$$y' = P(x)y^2 + Q(x)y + R(x) \quad (PR \neq 0), \tag{1.1}$$

generally,has no elementary solving process.

In 1961,A.Klujagopal[2] obtained the integrable condition of eg.(1)

$$R = kPe^{2\int Q dx} \quad (k \text{ is a constant}). \tag{1.2}$$

In 1982, Hongxian Li [3] obtained the integrable condition:

$$R = kPe^{n\int Q dx} \quad (k \text{ is a constant}). \tag{1.3}$$

In 1998, Linlong Zhao [4] discussed the following Riccati equation:

$$y' = P(x)y^n + Q(x)y + R(x) \quad (PR \neq 0, n \neq 0, 1) \tag{1.4}$$

and obtained the integrable condition: .

$$R = kPe^{n\int(Q-\beta D)dx} \quad (k, \beta \text{ are constants}). \tag{1.5}$$

In 1999, Linlong Zhao [5] obtained the new integrable condition of eg.(1)

$$L[y_0] = kPe^{2\int(Q+2y_0P)dx}/(m\int Pe^{\int(Q+2y_0)dx}dx + n)^2, \tag{1.6}$$

$$L[y_0] = \frac{k(A/B)^2 L[y_0]e^{-2\int((2B/A)L[y_0]+Q+2y_0P)dx}}{(m\int L[y_0]e^{-\int((2B/A)L[y_0]+Q+2y_0P)dx}dx + n)^2}, \tag{1.7}$$

where k, m, n are constants, D, y_0, A, B are functions, and

$$L[y_0] = -y_0' + P(x)y_0^2 + Q(x)y + R(x) \quad (PR \neq 0). \tag{1.8}$$

2   The Invariant of Riccati Equation

Definition [5] The coefficient relations of eg.(1)

$$I_1 = PR, \tag{1.1a}$$

$$I_2 = P'/P + Q, \tag{1.1b}$$

$$I_2' = R'/R - Q, \tag{1.1b'}$$

is defined as invariant of eg.(1).

Let

$$y = 1/u. \tag{1.2}$$

By using (1.1b') Eq.(1) can be rewritten as follows:

$$u' = -Ru^2 - Qu - P \tag{1.3}$$

3   The Integrable Theorems of Riccati Equation

Lemma[4] If there are constants α, β, γ and function $D(x)(\neq 0)$ such that

$$I_1 = PR = \alpha\gamma D^2, \tag{2.1a}$$

$$I_2 = P'/P + Q = D'/D + \beta D, \tag{2.1b}$$

$$I_2' = R'/R - Q = D'/D - \beta D, \tag{2.1b'}$$

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then eg.(1) can be rewritten its integrable form:

$$\int dz/(\alpha z^2 + \beta z + \gamma) = \int .D dx, \qquad (2.2)$$

where

$$y = (\alpha D/P)Z \quad \text{or} \quad y = (R/\gamma D)Z. \qquad (2.3)$$

Theorem 1 If there are constants α, β, γ and functions $D(x)(\neq 0)$ such that:

$$I_1 = PL[y_0] = \alpha\gamma D^2, \qquad (2,4a)$$

$$I_2 = P'/P + Q + 2y_0 P = D'/D + \beta D, \qquad (2.4b)$$

$$I_2' = L'[y_0]/L[y_0] - Q - 2y_0 P = D'/D - \beta D, \qquad (2.4b')$$

then eq.(1) can be rewritten its integrable form (2.2), and

$$y = y_0 + (\alpha D/P)z \quad \text{or} \quad y = y_0 + (L[y_0]/\gamma D)z. \qquad (2.5)$$

Proof   Let

$$y = y_0 + A(x)u \quad (\text{function } A(x) \neq 0). \qquad (2.6)$$

Eq.(1) may be rewritten as follows

$$u' = Apu^2 + (-A'/A + Q + 2y_0 P)u + L[y_0]/A. \qquad (2.7)$$

From Lemma, (2.4a) and (2.4b)(or (2.4b')),the eq.(2.7) can be rewritten the integrable form (2.2), and

$$u = (\alpha D/AP)Z \quad \text{or} \quad u = (L[y_0]/\gamma AD)Z. \qquad (2.8)$$

Then (2.5) can be implied by (2.6).

Now, from (2.4b'),we obtain:

$$L[y_0] = kDe^{\int (Q+2y_0 P - \beta D)dx} \quad (k \text{ is const.}). \qquad (2.9)$$

From (2.4a),we have

$$L[y_0] = kPe^{2\int (Q+2y_0 P - \beta D)dx} \qquad (2.10).$$

Again from (2.4b), we obtain

$$mPe^{\int (Q+2y_0 P)dx} = De^{\int \beta D dx} \quad (m \text{ is const.}), \qquad (2.11)$$

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$$mPe^{\int(Q+2y_0P)dx} + n = \int De^{\int \beta D dx} = e^{\int \beta Dx} \quad (m, n \text{ is const.}). \qquad (2.12)$$

Then (6) is implied by (2.10).

Now, let $u \to A(x)/u$ (function $A(x) \neq 0$), we have theorem 2.

Theorem 2 If there are constants α, β, γ and functions $D(x)(\neq 0), y_o(x)$ such that:

$$I_1 = PL[y_0] = \alpha \gamma D^2, \qquad (2.13a)$$

$$I_2 = L'[y_0]/L[y_0] - Q - 2y_0P = D'/D + \beta D, \qquad (2.13b)$$

$$I_2' = P'/P + Q + 2y_0P = D'/D - \beta D. \qquad (2.13b')$$

Then eq.(1) may be written its integrable form (2.2), and

$$y = y_0 - L[y_0]/\alpha DZ \quad \text{or} \quad y = y_0 - \gamma D/PZ. \qquad (2.14)$$

Theorem 3 If there are constants α, β, γ and functions $D(x)(\neq 0), A(x)(\neq 0), B(x)(\neq 0)$ such that

$$I_1 = (B/A)^2 L[y_0]L[y_0 + A/B] = \alpha \gamma D^2, \qquad (2.15a)$$

$$I_2 = L'[y_0]/L[y_0] - (2B/A)L[y_0] - Q - 2y_0P = D'/D + \beta D, \qquad (2.15b)$$

$$I_2' = L'[y_0 + A/B]/L[y_0 + A/B] + (2B/A)L[y_0] + Q$$

$$+2y_0P - 2(A/B)'/(A/B) = D'/D - \beta D. \qquad (2.15b')$$

Then, eq.(1) may be written its integrable form(2.2), and

$$y = y_0 - \frac{L[y_0]}{\alpha DZ - (B/A)L[y_0]} \quad \text{or} \quad y = y_0 - \frac{\gamma A^2 D}{B^2 L[y_0 + A/B] - \gamma ABD}. \qquad (2.16)$$

The same proves as Theorem 1, let

$$y = y_0 + A(x)/(u + B(x)) \quad (\text{function} \quad A(x), B(x) \neq 0). \qquad (2.17)$$

For $(2.15b')$, we obtain

$$L[y_0+A/B] = k(A/B)^2 De^{-\int((2B/A)L[y_0]+Q+2y_0P+\beta D)dx} \quad (k \text{ is const.}). \quad (2.18)$$

That is, (7) is implied.

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4 Applications

Choosing special function in (6) and (7), then the eq.(1) has sufficient conditions which guarantee

1. In (7),let $k = n = 1, m = 0$, and

$$L[-k] = \alpha(A/B)^2 L[-A/B - k]e^{-2\int((2B/A)L[-A/B-k]+Q-2(A/B)P-2kP)dx}.$$

(3.1)

Now, let

$$(A/B)^2 L[-A/B - k] = Q - 2kP,$$
(3.2a)

$$-2(\frac{2B}{A}L[-\frac{A}{B} - k] + Q - \frac{2A}{B}P - 2kP) = \frac{Q^2 - 4PR}{2(Q - 2kP)}, \quad (Q \neq 2KP). \quad (3.2b)$$

(i) If $Q = 2kP$, from(3.2a), the result in [6] is implied

$$L[-k] = Pk^2 - Qk + P = 0.$$

(3.3)

(ii)If $Q \neq 2kP$, from (3.1),(3.2a),(3.2b), the result in [6] is implied

$$L'[-k]/L[-k] = (Q' - 2kP')/(Q - 2kP) + (Q^2 - 4PR)/2(Q - 2kP). \quad (3.4)$$

2. In (6),let $y_0 = (\psi - Q)/2P$, we have

$$L[\frac{\psi - Q}{2P}] = \frac{-1}{2}(\frac{\psi - Q}{P})' + \frac{(\psi - Q)^2}{4P} + \frac{Q}{2P}(\psi - Q) + R = \frac{kPe^{2\int \psi dx}}{(m\int Pe^{\int \psi dx} + n)^2}.$$

(3.5)

(i)If $\psi = Q$, the result in [6] is implied

$$L[0] = R = kPe^{2\int Qdx}/(m\int Pe^{\int Qdx}dx + n)^2.$$

(3.6)

(ii)If $\psi = 0$, the result in [6] is implied

$$L[-Q/2P] = (-1/2)(Q/P)' - (1/4)(Q/P)^2 + R = kP/(m\int Pdx + n)^2. \quad (3.7)$$

(iii)If $\psi = Q - 2PR/Q, k = 0$, the result in [7] is obtained

$$L[-R/Q] = (R/Q)' + P(R/Q)^2 = 0 \text{ iff } P = (Q/R)'. \quad (3.8)$$

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References

1. D.L.Ince, Ordinary Differential Equations (London,1927).
2. A.Klujagopal, Math. Manthly **68**, 777-779 (1962).
3. Hongxiang Li, Amer. Math. Monthly **89**, 198-208 (1982).
4. Linlong Zhao, Chinese Scieuce Bulletin **43**, 439-440 (1998).
5. Linlong Zhao, Chinese Quartevly Journal Math. **14**, to appear (1999)
6. Cebin Sun, Chinese Math. in Practice and Theory **22**, 27-35 (1992).
7. Yunlu Yang, China Math.Bulletin **33**, 40-41 (1991).
8. G.N. Waston, A Treaties on the theory of Bessel Functions (Cambrige, 1944)

ROBUST STABILIZATION OF INTEGRO-DIFFERENTIAL SYSTEMS

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In this paper, the problem of robust stability for uncertain parameters of integro-differential systems with infinite delay is studied, Applying a stabilizing local state feedback to each subsystem, making use of the method of Lyapunov function and combining the method of inequality analysis, the sufficient conditions of Robust global uniform asymptotic stability for uncertain parameters of integro-differential systems with infinite delay are obtained.

1 Introduction

The robust stability problem of large-scale dynamical system in the presence of uncertainty has been receiving considerable attention, because in many practical control problems uncertainty often occurs in dynamical systems due to modeling errors, measurement errors, linearization approximations, and so on. Many design techniques for uncertain dynamical systems have been developed to guarantee a required degree of robustness. Since time-delay is frequently

a source of instability and encountered in various engineering systems such as chemical processes, long transmission lines in pneumatic systems etc, the problem of the stability of time-delay systems has been explored over the last decade. Many methods to check the stability of time-delay systems were proposed by [1,3]. Additionally, the problem of the stabilization of uncertain time-delay systems has also been explored over the last years. Many approaches to solving the problem have been proposed by [2,4,5]. Hence, in this paper, the problem of robust stability for uncertain parameters of integro-differential systems with infinite delay is studied, Applying a stabilizing local state feedback to each subsystem, making use of the method of Lyapunov function and combining the method of inequality analysis, the sufficient conditions of Robust global uniform asymptotic stability for uncertain parameters of integrodifferential systems with infinite delay are obtained.

2   System Description and Preliminaries

We consider integro-differential large-scale systems with infinite delay

$$
\dot{x}_i(t) = (a_i + \Delta a_i)x_i + \sum_{j=1}^{N}[(a_{ij} + \Delta a_{ij})x_j(t - \tau_{ij}(t)) + (b_{ij} + \Delta b_{ij})
$$

$$
\times \int_{-\infty}^{t} T_{ij}(t - s)x_j(s)ds] + (b_i + \Delta b_i)u_i(t), \ t \geq 0, \ (i = 1, \cdots, N) (2.1)
$$

where $x_i \in R^{n_i}$ is the state vector, $\sum_{i=1}^{N} n_i = n$, $a_i \in R^{n_i \times n_i}$, $a_{ij}, b_{ij} \in R^{n_i \times n_j}$, $b_i \in R^{n_i \times m_i}$ are constant matrices, $u_i \in R^{m_i}$ is the input vector, $\sum_{i=1}^{N} m_i = m$; $\tau_{ij}, T_{ij} \colon R^+ \to R^+$ is continuous function and satisfies $\int_0^{+\infty} T_{ij}(s)ds = 1$, $0 \leq \tau_{ij}(t) \leq \tau, \tau \geq 0$ is constant; $\Delta a_i, \Delta b_i, \Delta a_{ij}, \Delta b_{ij}$ are parameter perturbation matrices with the following known upper norm-bounds

$$
\|\Delta a_i\| \leq \alpha_i, \|\Delta b_i\| \leq \beta_i, \|\Delta a_{ij}\| \leq \alpha_{ij}, \|\Delta b_{ij}\| \leq \beta_{ij}
$$

where $\alpha_i \geq 0$, $\beta_i \geq 0$, $\alpha_{ij} \geq 0$, $\beta_{ij} \geq 0$ are given constants.

For any $t_0 \geq 0$, we assume the initial condition is as follows

$$
x_i(t) = \phi_i(t), \quad -\infty < t \leq t_0, \quad (i = 1, 2, \cdots, N)
$$

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where $\phi_i(t)$ is a continuous n_i dimension vector-valued function defined on $(-\infty, t_0]$, and satisfies

$$\|\phi\| = \max_{1 \leq i \leq N} \{ \sup_{-\infty < t \leq t_0} \|\phi_i(t)\| \} < \infty$$

Let $u_i(t)$ be a linear local state feedback control law for system (1). We select that the decentralized local feedback control $u_i(t)$ is

$$u_i(t) = k_i x_i(t), \quad (i = 1, 2, \cdots, N) \tag{2.2}$$

where k_i is the feedback gain matrix of appropriate dimension. Applying the state feedback controller (2) to the system (1) yields a closed-loop system as follows

$$\dot{x}_i(t) = (a_i + b_i k_i) x_i(t) + \sum_{j=1}^{N} [(a_{ij} + \Delta a_{ij}) x_j(t - \tau_{ij}(t))$$

$$+ (b_{ij} + \Delta b_{ij}) \int_{-\infty}^{t} T_{ij}(t - s) x_j(s) ds]$$

$$+ (\Delta a_i + \Delta b_i k_i) x_i(t), \ t \geq 0, \ i = 1, \cdots, N. \tag{2.3}$$

3 Main Results

Theorem 3 *For the system (1), we suppose*

(1) There exists k_i, such that

$$\frac{1}{2}[(a_i + b_i k_i)^T + (a_i + b_i k_i)] = l_i I_i, \quad (i = 1, 2, \cdots, N)$$

where l_i is constant, I_i is the $n_i \times n_i$ identical matrix.

(2) $l_i + \alpha_i + \beta_i \|k_i\| = -r_i < 0, \quad (i = 1, 2, \cdots, N)$

(3) We select $w_{ij} = \frac{1}{r_i}(\|a_{ij}\| + \|b_{ij}\| + \alpha_{ij} + \beta_{ij}), (i, j = 1, 2, \cdots, N)$, If $\rho[(w_{ij})_{N \times N}] < 1$

Then, by using the local state feedback controller given by (2), the uncertain parameters of integro-differential systems (1) with infinite delay is Robust global uniform asymptotic stable.

If $n_i = 1, (i = 1, 2, \cdots, N)$, then $N = n$. We have the following corollary
Corollary 3.1 For the system (1), we suppose

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(1) There exists k_i, such that

$$a_i + b_i k_i + \alpha_i + \beta_i |k_i| = -r_i < 0, \quad (i = 1, 2, \cdots, n)$$

where r_i is constant.

(2) We select $w_{ij} = \frac{1}{r_i}(|a_{ij}| + |b_{ij}| + \alpha_{ij} + \beta_{ij})$, $(i, j = 1, 2, \cdots, n)$,
If $\rho[(w_{ij})_{n \times n}] < 1$

Then, by using the local state feedback controller given by (2), the uncertain parameters of integro-differential systems (1) with infinite delay is Robust global uniform asymptotic stable.

Theorem 4 *For the system (1), we suppose*

(1) There exists k_i such that

$$\frac{1}{2}[(a_i + b_i k_i)^T + (a_i + b_i k_i)] = -r_i I_i, \quad (i = 1, 2, \cdots, N)$$

where $r_i > 0$ is constant, I_i is the $n_i \times n_i$ identical matrix.

(2) $\rho[(w_{ij})_{N \times N}] < 1$, where

$$w_{ij} = \begin{cases} \frac{1}{r_i}[\alpha_i + \beta_i \|k_i\| + \|a_{ij}\| + \|b_{ij}\| + \alpha_{ij} + \beta_{ij}], & if \ i = j, \\ \frac{1}{r_i}[\|a_{ij}\| + \|b_{ij}\| + \alpha_{ij} + \beta_{ij}], & if \ i \neq j. \end{cases}$$

Then, by using the local state feedback controller given by (2), the uncertain parameters of integro-differential systems (1) with infinite delay is Robust global uniform asymptotic stable.

Acknowledgments

References

1. A. Hmamed, *Int. J. Control.*, **43**, 321-324 (1986).
2. H . Wu and K. Mizukami, *Int. J. Systems Sci.*, **24**, 265-276 (1993).
3. D. Y. Xu, *Automatica*, **30**, 703-706 (1994).
4. S. M. Zhong and T. Z. Huang, *ACTA Automatica SINICA.*, **24** , 837–839 (1998).

5. S. M. Zhong, Z. L. Li and T. Z. Huang, *Control Theory and Applications* ,**12** , 477-481 (1995).

ON THE RETARDED LIÉNARD-TYPE EQUATION BLOW-UP ON SOME PARABOLIC EQUATIONS

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In this paper, the retarded Liénard-type equation

$$\ddot{x} + f_1(x)\,\dot{x} + f_2(x)\,\dot{x}^2 + g(x(t-h)) = e(t),$$

is considered. The sufficient conditions on the stability, boundedness and existence of periodic solutions are given by means of the method of Liapunov functional and Razumikhin- type theorems. The main results of this paper generalzie or extend the well-known results in the literature.

1 Introduction

In this paper, we consider the retarded Liénard-type equation

$$\ddot{x} + f_1(x)\,\dot{x} + f_2(x)\,\dot{x}^2 + g(x(t-h)) = 0, \qquad (1.1)$$

and the retarded Liénard-type equation with forced form

$$\ddot{x} + f_1(x)\,\dot{x} + f_2(x)\,\dot{x}^2 + g(x(t-h)) = e(t), \qquad (1.2)$$

where h is a nonnegative constant, f_1, f_2 and g are continuous functions on R, $e(t)$ is a continuous bounded on R^+.

Recently, the equation (1.1) without delay, i.e

$$\ddot{x} + f_1(x)\,\dot{x} + f_2(x)\,\dot{x}^2 + g(x) = 0, \qquad (1.3)$$

has been widely investigated concerning the boundedness, stability, oscillation and periodicity of solutions of (1.3).(see [1])

Clearly, if $f_1(x) = f(x), f_2(x) = 0$, the equation (1.1) is reduced to the well-known retarded Liénard equation

$$\ddot{x} + f(x)\,\dot{x} + g(x(t-h)) = 0. \qquad (1.4)$$

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The problems on the stability, boundedness of solutions for (1.4) have been extensively studied by many authors since 1960s. Surveys of the results regarding these properties may be seen in the papers of Krasovskii [2], Burton [3]. Some practical problems concerning physics, mechanics et,al applied science and the engineering technique fields associated with the equation (1.4) may be found in [2-3,5]. The following theorem is the well-known and can be found in most bibliographies listed above.

Theorem A. *Suppose that*

(1) $xg(x) > 0$ *for all* $x \neq 0$, *and* $G(x) = \int_0^x g(\xi)\,d\xi \to +\infty$ *as* $|x| \to +\infty$.

(2) *there is a constant* $L > 0$ *such that*

$$|g'(x)| \leq L, \quad f(x) > hL, \quad \text{for all } x \in R.$$

Then all solutions of (1.4) and their derivatives are bounded.

Recently, Zhang [4] extended and generalized Theorem A, and also investigated the oscillation of solutions for the equation (1.4).

This paper is to devoted to the study of the stability and boundedness of solutions of the equation (1.1) and (1.2). The primary goal here is to extend Theorem A to the equation (1.1) and (1.2). Similary to [1],a suitable transformation is our technicl tool.By using Liapunov functional method and Razumikhin-type theorems, we give the sufficient conditions to ensure the stability, boundedness of solutions, and the existence of periodic solutions for the equation (1.1) or (1.2). Our results extend and generalize Theorem A.

Let R^-, R^+, R denote this intervals $(-\infty, 0]$, $[0, +\infty)$ and $(-\infty, +\infty)$, respectively. $C([-h, 0], R^2)$ denotes the space of continuous functions (ϕ, ψ); $[-h, 0] \to R^2$ with the supremum norm $|| \cdot ||$, When a function is written without its argument, then that argument is t.

For the sake of convenience, we set $a(x) = e^{\int_0^x f_2(\xi)\,d\xi}$. By introducing a transformation $y = a(x)\,\dot{x}$, the equation (1.2) can be transformed into the following system;

$$\dot{x} = \frac{1}{a(x)}y, \quad \dot{y} = -f_1(x)y - a(x)g(x(t-h)) + a(x)e(t). \tag{1.5}$$

It is easy to known [4] that for a, f_1 and g continuous, given a continuous initial function $(\phi, \psi) \in C([-h, 0], R^2)$ and a number σ, then there exists a solution of (1.5) on an interval $[0, \alpha)$ satisfying the initial condition; if the solution remains bounded then $\alpha = +\infty$. We denote such a solution by $x(t) = x(t, \phi, \psi, \sigma), y(t) = y(t, \phi, \psi, \sigma)$.

2 Main Results

First, we consider the stability and boundedness of solutions for the equation (1.1). The following theorem is the first main result of this paper.

Theorem 2.1 *Suppose that g is differentiable on R, then the following statements are true;*

(i) If the following conditions hold;

(1) $xg(x) > 0$ for all $x \neq 0$.

(2) there are constants $L > 0, M > 0$ such that

$$a(x) \leq M, \ |g'(x)| \leq La(x), \ f_1(x) > hLM, \ \text{for all } x \in R.$$

then the zero solution of the equation (1.1) is uniformly stable.

(ii) If, in addition to (i),

$$\lim_{|x| \to +\infty} G(x) = \lim_{|x| \to +\infty} \int_0^x a^2(\xi)g(\xi)\, d\xi = +\infty,$$

then all solutions of (1.1) and their derivatives are uniformly bounded.

(iii) If, the condition (2) in (i) is strengthened to; there are constants $L > 0, M > 0$ and $N > 1$ such that

$$a(x) \leq M, \ |g'(x)| \leq La(x), \ f_1(x) > hNML, \ \text{for all } x \in R.$$

then the zero solution of the equation (1.1) is uniformly asymptotically stable.

Proof (i). Obviously, we consider only the system (1.5) equivalent to the equation (1.1) with $e(t) = 0$. For our purpose, by

$$d(x(t+s)) = \frac{y(t+s)}{a(x(t+s))}dt,$$

it is more convenient to rewrite the system (1.5) in the following form

$$\begin{cases} \dot{x} = \dfrac{1}{a(x)}y, \\ \dot{y} = -f_1(x)y - a(x)g(x) + a(x)\displaystyle\int_{-h}^0 \dfrac{g'_x(x(t+s))}{a(x(t+s))}y(t+s)\, ds. \end{cases} \tag{2.1}$$

Let $x(t) = x(t, \phi, \psi, \sigma), y(t) = y(t, \phi, \psi, \sigma)$ be solution of (2.1) defined on $[0, \alpha)$. We may assume that $\alpha = +\infty$, since the estimates which follow give an a priori bound on $(x(t), y(t))$. Let $D > 0$ be such that $\|(\phi, \psi)\| + |\sigma| \leq D$.

Define Liapunov functional

$$V = V(t) = \frac{1}{2}y^2 + \int_0^x a^2(\xi)g(\xi)\, d\xi + \frac{LM}{2}\int_{-h}^0 \int_{t+s}^t y^2(u)\, du\, ds. \tag{2.2}$$

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we have

$$\dot{V}|_{(2.1)} \le -[f_1(x) - hLM]y^2 \le 0.$$

Clearly,This implies that the zero solution of the system (2.1) is uniformly stable.

(ii) By (2.2) and (2.3), there exists a positive constant $D_0 = D_0(D)$ such that

$$V(t) \le V(0) \le D_0 \quad \text{for all } t \in R^+. \tag{2.4}$$

It is easy to see that $y(t)$ is bounded, there exists a positive number $B_1 > 0$ such that $|y(t)| \le B_1$ for all $t \in R^+$.

Since

$$\lim_{|x| \to +\infty} G(x) = \lim_{|x| \to +\infty} \int_0^x a^2(\xi)g(\xi)\, d\xi = +\infty,$$

then there exists a positive number $B_2 > 0$ such that $G(B_2) > D_0$. Thus $x(t) \le B_2$ for all $t \in R^+$ by (2.4). Hence, all solutions of the system (2.1) are uniformly bounded.

(iii) From the proof of (i), we have

$$\dot{V}|_{(2.1)} \le -[f_1(x) - hLM]y^2 < -(N-1)hLMy^2.$$

According to [3] or [5] (see [5, Theorem 5.1, Chapter 2,Part I]), we conclude that the zero solution of (2.1) is uniformly asymptotically stable.

This completes the proof of Theorem 2.1.

Remark 2.1 If $f_1(x) = f(x), f_2(x) = 0$, The part (ii) of Theorem 1 is Theorem A. Clearly, Theorem 2.1 substantially extend and generalize Theorem A.

Next, we consider the boundedness of solutions and the existence of periodic solutions for the equation (1.2). The following theorem is the second main result of this paper.

Theorem 2.2 *Suppose that g is differentiable on R, then the following statements are true;*

(i) If the following conditions hold;

(1) $xg(x) > 0$ for all $x \ne 0, a(x)g(x)\mathrm{sgn}x \to +\infty$ as $|x| \to +\infty$, and

$$\lim_{|x| \to +\infty} G(x) = \lim_{|x| \to +\infty} \int_0^x a^2(\xi)g(\xi)\, d\xi = +\infty.$$

(2)there are constants $L > 0, M > 0$ and $N > 1$ such that

$$a(x) \le M, \quad |g'(x)| \le La(x), \quad f_1(x) > hNLM, \text{for all } x \in R.$$

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then all solutions of the equation (1.2) and their derivatives are uniformly ultimately bounded.

(ii)If, in addition to (i), $e(t)$ is a T-periodic function,i,e;$e(t+T) = e(t)$, then the equation (1.2) has a T-periodic solution.

proof (i) By the same the proof as Theorem 2.1, we consider the system (1.5) equivalent to the equation (1.2), which is to be rewritten the following form

$$\begin{cases} \dot{x} = \dfrac{1}{a(x)}y, \\ \dot{y} = -f(x)y - a(x)g(x) + a(x)e(t) + a(x)\int_{-h}^{0} \dfrac{g'(x(t+s))}{a(x(t+s))}y(t+s)\,ds. \end{cases} \tag{2.5}$$

Let $x(t) = x(t,\phi,\psi,\sigma), y(t) = y(t,\phi,\psi,\sigma)$ be still a solution of (2.5). We define Liapunov funvtion

$$V(x,y) = \frac{1}{2}y^2 + \int_{0}^{x} a^2(\xi)g(\xi)\,d\xi$$

and choose $p(s) = q^2 s$, where $q > 1$ satisfying $q < N$. Now assume that H is a enough large number. If the following conditions

$$|y(t+\theta)| > H, \quad V(x,y(t+\theta)) < p(V(x,y(t))), \quad \text{for } \theta \in [-h,0].$$

hold, we may choose a suitable number $q > 1$ such that $|y(t+\theta)| \le q|y(t)|$. Thus, we have

$$\dot{V}\mid_{(2.5)} \le -f_1(x)y^2 + M|e(t)||y| + hLMqy^2$$
$$\le -[(N-q)hLM - \frac{M|e(t)|}{|y|}]y^2.$$

Since $e(t)$ is a bouuded, then exists a constant $\mu > 0$, such that the following inequality

$$[(N-q)hLM - \frac{M|e(t)|}{|y|}] \ge \mu \quad \text{for } |y| > H$$

holds. This yields

$$\dot{V}\mid_{(2.5)} \le -\mu y^2, \quad \text{for } |y| > H.$$

By Razumikhin-type theorem [3] or [5](see [5, Theorem 6.4, Chapter 2, Part I]), we can claim that $x(t)$ is uniformly ultimately bounded, there exists a constant β such that $|y(t)| \le \beta$ for all $t \in R^+$.

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Let $V_1 = V + y$, we have

$$\dot{V_1}\mid_{(2.5)} = -f_1(x)y^2 + a(x)e(t)y + a(x)y \int_{-h}^{0} \frac{g'(x(t+s))}{a(x(t+s))} y(t+s)\, ds$$

$$+ [-f_1(x)y - a(x)g(x) + a(x)e(t) + a(x) \int_{-h}^{0} \frac{g'(x(t+s))}{a(x(t+s))} y(t+s)\, ds]$$

This implies that there exists a constant $K_1 > 0$ such that

$$\dot{V}_{1(2.5)} \leq -a(x)g(x) + K_1, \quad \text{for all } |y| \leq \beta.$$

Since $a(x)g(x)\operatorname{sgn} x \to +\infty$ as $|x| \to +\infty$, we can choose a constant B_1 such that $\dot{V_1}\mid_{(2.5)} \leq -\mu$ for all $x \geq B_1$. Therefore, we claim that there exists a constant $\alpha_1 > 0$ such that $x(t) \leq \alpha_1$ for all $|y| \leq \beta$.

Similarly, let $V_2 = V - y$, we can show that there exists a constant $\alpha_2 > 0$ such that $x(t) \geq \alpha_2$ for $|y| \leq \beta$. It follows immediately that $x(t)$ is uniformly ultimately bounded.

(ii)From the proof of (i), we have known that the solutions of the system (2.5) are uniformly ultimately bounded. Thus, by Brouwer fixed point theorem, we conclude that the system (2.5) has a T-periodic solution whenever $e(t)$ is T-preiodic function.

This completes the proof of Theorem 2.1.

Remark 2.2 Clearly, if $f_1(x) = f(x), f_2(x) = 0$, The part (i) of Theorem 2.2 is a generalization of Theorem A for the retarded Liénard equation (1.4) with forced form $e(t)$. In particular, if $e(t) = 0$, we can obtain that all solutions of (1.4) are uniformly utimately bounded.

References

1. Jian Ji-Fa,*J.Math. Analysis Applic*, **194**, 597(1995).
2. N.N.Krasovskii, *Stability of Motion*, (Stanford Univ, Press. Stanford, CA, 1963).
3. T.A.Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, (Academic Press, Orlando, FL. 1985).
4. B. Zhang,*Proc. Amer. Math. Soc*, **115**, 779(1992).
5. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, (Academic Press, New York, 1993).

DIMENSION OF THE GLOBAL ATTRACTOR FOR SEMILINEAR WAVE EQUATIONS WITH DAMPING

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The upper bound of the Hausdorff dimension of the global attractor is given for semilinear wave equations with damping under homogeneous Dirichlet boundary condition.

1 Preliminaries

In this section, we present a method to estimate the upper bound of the Hausdorff dimension of the global attractor for a semigroup associated with a evolution equation. (see ref. [1] or [2] for details)

Let H be a complete metric space and $\{S(t), t \geq 0\}$ be a continuous semigroup on H.

Definition 1. A set X of H is called a global attractor for the semigroup $\{S(t), t \geq 0\}$ if (i) X is an invariant set, i.e., $S(t)X = X$, $\forall t \geq 0$. (ii) X is a compact set. (iii) X attracts any bounded set of H, i.e., for any bounded set $B \subset H$, $d(S(t)B, X) = \sup_{x \in S(t)B} \inf_{y \in X} d(x, y) \to 0$ as $t \to \infty$.

Definition 2. Let H be a metric space and Y be a subset of H. The quantity

$$d_H(Y) = \inf\{d | \sup_{s \geq 0} \inf_{\{B_i\}_{i \in I}} \{\sum_{i \in I} r_i^d : Y \subset \cup_{i \in I} B_i,$$
$$|B_i| = 2r_i \leq 2\varepsilon, \forall i\} = 0, d \in R_+\}$$

is called the Hausdorff dimension of Y, where $|B_i| = \sup\{|x - y| : x, y \in B_i\}$, $\{B_i\}_{i \in I}$ is a family of balls of H of radii $\leq \varepsilon$ covering Y.

Let H be a Hilbert space with inner product $(\cdot, \cdot)_H$ and W be a Hilbert subspace of H, the injection of W in H being continuous. Let F be a function from W into H. We consider the initial-boundary value problem

$$\begin{cases} \frac{du}{dt}(t) = F(u(t)), t > 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

and we assume that

(H_1) the problem (1) is well posed for every $u_0 \in H$ with $u(t) \in W, \forall t \geq 0$ and the mapping $S(t) : u_0 \in H \to u(t) \in H, \forall t \geq 0$ forms a continuous semigroup on H.

(H_2) F is Fréchet differentiable from W into H with differential F' and the (linear) initial-value problem

$$\begin{cases} \frac{dU}{dt}(t) = F'(S(t)u_0)U(t), t > 0, \\ U(0) = \xi, \end{cases}$$
(1.2)

is well posed for every $u_0, \xi \in H$.

(H_3) $S(t)$ is differentiable in H with the differential $L(u_0, t)$ defined by $L(u_0, t)\xi = U(t), \forall \xi \in H$ and $U(t)$ the solution of (2).

Suppose above assumptions (H_1)-(H_3) hold and there exists a global attractor X for the semigroup $\{S(t), t \geq 0\}$ associated with the initial-value problem (1). Let Φ denote a set of m vectors $\{\Phi_1, \Phi_2, \cdots, \Phi_m\}$ which are orthonormal in H. If

$$q_m = \lim_{t \to +\infty} \sup_{\Phi \subset H} \sup_{\varphi \in X} \frac{1}{t} \int_0^t \sum_{j=1}^m (F'(S(\tau)\varphi)\Phi_j(\tau), \Phi_j(\tau))_H \; d\tau < 0,$$

then the Hausdorff dimension of the global attractor X is less than or equal to $m : d_H(X) \leq m$.

2 Dimension of the global attractor for semilinear wave equations with damping

In this section, we will give the upper bounds of the Hausdorff dimension of the global attractor for semilinear wave equation with Dirichlet boundary condition when there exist linear and nonlinear dampings and the nonlinearity satisfies the noncritical and critical growth conditions. The problems considered here all satisfy the assumptions (H_1)-(H_3) and their solutions determine a continuous semigroup $\{S(t), t \geq 0\}$ on $E = H_0^1(\Omega) \times H^2(\Omega)$, here $S(t) : \{u_0, u_1\} \to \{u(t), u_t(t)\}$ from E into itself for all $t \geq 0$. The existence of the global attractor for the semigroup $\{S(t), t \geq 0\}$ has been studied by many authors[1-4].

2.1 Linear damping

We consider the semilinear wave equation

$$u_{tt} + \alpha u_t - \Delta u + f(u) = g, x \in \Omega, t > 0, \tag{2.3}$$

with Dirichlet boundary condition

$$u(x,t)|_{x \in \partial \Omega} = 0, t > 0, \tag{2.4}$$

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and the initial-value conditions

$$u(x,0) = u_0(x), \frac{\partial u}{\partial t}(x,0) = u_1(x), x \in \Omega, \tag{2.5}$$

where $u = u(x,t)$ is a real-valued function on $\Omega \times [0, +\infty)$, Ω is an open bounded set of R^n with a smooth boundary $\partial\Omega$, $g \in L^2(\Omega)$, $f(u) \in C^1(R; R)$, $\alpha > 0$.

Noncritical exponent

Let $G(s) = \int_0^s f(r)dr$. We make the following assumptions on the functions $G(s)$ and $f(s)$:

(i)

$$\lim_{|s| \to \infty} \inf \frac{G(s)}{s^2} \geq 0. \tag{2.6}$$

(ii) There exist two positive constants $c_1 > 0$, $c_2 > 0$ such that

$$\lim_{|s| \to \infty} \inf \frac{sf(s) - c_1 G(s)}{s^2} \geq 0, \tag{2.7}$$

and

$$|f'(s)| \leq c_2(1 + |s|^p) \text{ with } \begin{cases} 0 \leq p < \infty, \text{ when } n = 1, 2, \\ 0 \leq p < 2, \text{ when } n = 3, \quad \forall s \in R, \\ p = 0, \quad \text{ when } n \geq 4, \end{cases} \tag{2.8}$$

(iii) There exists $\delta_1 > 0$ and for every $M > 0$ there exists $c' = c'(M)$ such that

$$\|f'(u_1) - f'(u_2)\|_{L(H_0^1(\Omega), L^2(\Omega))} \leq c'\|u_1 - u_2\|^{\delta_1}, \tag{2.9}$$

for all $u_1, u_2 \in H_0^1(\Omega), \|u_1\| \leq M, \|u_2\| \leq M$, where $\|\cdot\|$ and $\|\cdot\|_{L(H_0^1(\Omega), L^2(\Omega))}$ denote the norms of $L^2(\Omega)$ and $L(H_0^1(\Omega), L^2(\Omega))$ (the space of linear continuous operators from $H_0^1(\Omega)$ into $L^2(\Omega)$, respectively.

Témam[1] obtained an upper bound of the Hausdorff dimension of the global attractor for system (3)-(5) as

$$d_H \leq \min \left\{ m \left| m \in N, \frac{1}{m} \sum_{j=1}^m \lambda_j^{-1} < \begin{cases} \frac{\lambda_1^2}{32\gamma^2\alpha^2}, \alpha \geq \sqrt{2\lambda_1} \\ \frac{\alpha^2}{128\gamma^2}, \alpha \leq \sqrt{2\lambda_1} \end{cases} \right. \right\}, \tag{2.10}$$

where

$$\gamma = \sup_{w \in H_0^1(\Omega) \cap H^2(\Omega), \|\triangle w\| \le R} \|f'(w)\|_{L(H_0^1(\Omega), L^2(\Omega))} < \infty,$$

$$R = \sup_{\{\xi, \eta\} \in X} \|\triangle \xi\| < \infty,$$

$\{\lambda_j\}_{j \in N} : 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \cdots$, are the eigenvalues of operator $-\triangle$ with the homogeneous Dirichlet boundary condition on Ω. It is easy to see that this upper bound in (10) is directly proportional to the coefficient α of damping for $\alpha \ge \sqrt{2\lambda_1}$ and tends to infinity as $\alpha \to +\infty$, which is obviously not precise in the physical sense.

Under the same conditions (6)-(9), by carefully estimating and splitting the positivity of the linear operator in the corresponding evolution equation of the first order in time, Zhou[5] obtain a more strict upper bound of the Hausdorff dimension for the global attractor X as follows: for any $\alpha \ge \alpha_0 > 0$,

$$d_H(X) \le \min \left\{ m \left| \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{-1} < \frac{2\lambda_1 \alpha^2}{k^2 \sqrt{\alpha^2 + 4\lambda_1}(\alpha + \sqrt{\alpha^2 + 4\lambda_1})} \right. \right\}. \quad (2.11)$$

where $k = k(\alpha_0)$ is a positive constant which is independent of α. This estimate shows that the Hausdorff dimension decreases as the damping α grows and is uniformly bounded for large α, which conforms to physical intuition.

Critical exponent

Let Ω be an open bounded set of R^3 with a smooth boundary $\partial \Omega$, we consider the initial-boundary value problem (3)-(5) where $f(u) = f_1(u) + f_2(u) \in C^1(R; R)$. Let $G_i(s) = \int_0^s f_i(r) dr$, $i = 1, 2$. We assume $f(u)$ satisfies (9) and make the following assumptions on functions $G_i(s)$, $f_i(s)$, $i = 1, 2$:

(iv)

$$f_1(s)s \ge 0, \quad \lim_{|s| \to +\infty} \inf \frac{G_2(s)}{s^2} \ge 0, \quad \forall s \in R. \quad (2.12)$$

(v) There exist constants $c_{1i} > 0$, $c_{2i} > 0$, $i = 1, 2$ such that

$$\lim_{|s| \to +\infty} \inf \frac{s f_i(s) - c_{1i} G_i(s)}{s^2} \ge 0, \quad i = 1, 2, \quad \forall s \in R, \quad (2.13)$$

and

$$|f_1'(s)| \le c_{21}(1 + |s|^2), \quad |f_2'(s)| \le c_{22}(1 + |s|^p) \text{ with } 0 \le p < 2, \quad \forall s \in R. \quad (2.14)$$

By using the similar technique in [5], Zhou[6] obtained that for any $\alpha \geq \alpha_0 > 0$,

$$d_H(X) \leq \min\left\{ m \; \middle| \; \frac{1}{m}\sum_{j=1}^{m}\lambda_j^{-4\nu_0} < \frac{2\lambda_1\alpha^2}{k\sqrt{\alpha^2 + 4\lambda_1}(\alpha + \sqrt{\alpha^2 + 4\lambda_1})} \right\} \quad (2.15)$$

where $k = k(\alpha_0)$ is a positive constant, $0 < \nu_0 < \min\{\frac{2-p}{4}, \frac{1}{4}\}$, $p \in [0, 2)$ is as in (14).

2.2 Nonlinear damping and Critical Exponent

We consider the semilinear wave equation with nonlinear damping:

$$u_{tt} + h(u_t) - \Delta u + f(u) = g, \quad x \in \Omega, \quad t > 0, \quad (2.16)$$

with the homogeneous Dirichlet boundary condition (4) and the initial value conditions (5).

We assume that the function $h(v)$ satisfies:

(vi) There exist two positive constants $0 < \alpha \leq \beta$ such that

$$h(0) = 0, \quad 0 < \alpha \leq h'(s) \leq \beta < +\infty, \quad \forall s \in R. \quad (2.17)$$

(vii) For every $M' > 0$, there exists $c_4 = c_4(M')$ such that

$$\|h'(v_1) - h'(v_2)\|_{L(L^2(\Omega), L^2(\Omega))} \leq c_4 |v_1 - v_2|^{\delta_2} \quad (2.18)$$

for any $v_1, v_2 \in L^2(\Omega)$, $|u_1| \leq M'$, $|u_2| \leq M'$, where $\delta_2 > 0$, $|\cdot|$ and $\|\cdot\|_{L(L^2(\Omega), L^2(\Omega))}$ denote the norms of $L^2(\Omega)$ and $L(L^2(\Omega), L^2(\Omega))$ (the space of linear continuous operators from $L^2(\Omega)$ into $L^2(\Omega)$), respectively.

Let Ω be an open bounded set of R^3 with a smooth boundary $\partial\Omega$, we consider the initial-boundary value problem (16), (3), (5) where $f(u) = f_1(u) + f_2(u) \in C^1(R; R)$ satisfies the conditions (9), (12)-(14), and $h(v)$ satisfies (17)-(18). Zhou and Fan[7] obtained that

$$d_H(X) \leq \min\left\{ m \; \middle| \; \frac{1}{m}\sum_{j=1}^{m}\lambda_j^{-4\nu_0} < \frac{2\lambda_1\alpha^2}{k\left(\beta^2 + 4\lambda_1 + \beta\sqrt{\beta^2 + 4\lambda_1}\right)} \right\}, \quad (2.19)$$

where $0 < \nu_0 < \min\{\frac{2-p}{4}, \frac{1}{4}\}$, p is as in (14) and k is a positive constant.

Remark: For the upper bound of the global attractor for sine-Gordon equation and strongly damped nonlinear wave equations, we can see papers [8-10].

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References

1. R. Témam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, (Appl. Math. Sciences 68, Springer-Verlag, New York, 1988).
2. O. Ladyzhenskaya, *Cambridge University Press, Cambridge, New York Port Chester, Melbourhe Sydney*, 1991.
3. J. Arrieta, A. N. Hale, *Commun. Partial Differential Equations*, **17**, 841 (1992).
4. E. Feireisl and E. Zuazua, *Commun. Partial Differential Equations*, **18**, 1539 (1993).
5. Zhou Shengfan, *J. Math. Phys.*, **40**, 1432 (1999).
6. Zhou Shengfan, *J. Math. Phys.*, **40**, (1999). (to be published)
7. Shengfan Zhou and Xiaoming Fan, Dimension of the Global Attractor for Semilinear Wave Equations with Nonlinear Damping and Critical Exponent, *Preprint* (1999).
8. Zhou Shengfan, *Acta Math. Appl. Sinica*, **21**, 445 (1998).
9. Zhu Shu and Zhou Shengfan, *Nonlinear Analysis*, **37**, 389 (1999).
10. Shengfan Zhou, *J. Math. Anal. Appl.*, **233**, 102 (1999).